A Bug's Shortest Path on a Cube

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In any introductory course on combinatorics, the problem of finding the number of shortest paths from one corner of a city to the opposite corner is almost always introduced. As is well known, if the city is viewed as a grid of $m+1$ streets going N–S and $n+1$ streets going E–W (i.e., m blocks by n blocks), then the number of shortest paths is

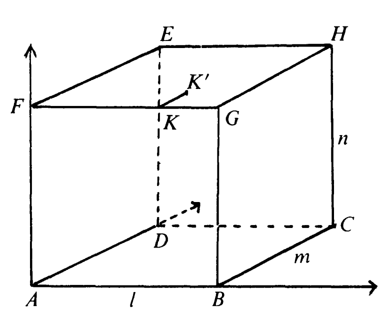
$$f(m,n)=\binom{m+n}{m}=\frac{(m+n)!}{m!n!}.$$

This can be readily obtained by identifying each shortest path either as a selection of m objects from $m+n$ objects or as a permutation of $m+n$ objects consisting of exactly m of the symbols E (east) and n of the symbols N (north).

Now one naturally wonders about the 3-dimensional analogue of this problem. Imagine for example, that a bug situated at one corner of an $n\times n\times n$ Rubik's cube wants to reach the opposite corner of the opposite face in such a way that its path consists of line segments of integer length each of which is parallel to one of the axes. What is the number of such shortest paths? If the bug can "chew" its way through the interior of the cube, then the same argument used in the 2-dimensional version immediately gives the answer $(3n)!/(n!)^3$. But if the bug can only crawl on the surface of the cube and because the surface is slippery, must crawl along the edges or the grooves between the small cubes, to determine the number, $f(n)$, of such shortest paths turns out to be a much harder problem than the 2-dimensional version. Though it is obvious that $f(1)=6$, to compute the value of $f(2)$ by brute force might be quite a challenging job. (Any one with $2\times 2\times 2$ Rubik's cube is encouraged to give it a try.) To determine the value of $f(3)$, with or without the use of an ordinary Rubik's cube, seems to be a formidable task. For the interest of the readers, we will not reveal these two values until the end of the article. To answer this question, we consider the more general problem in which the cube is replaced by a rectangular box.

Let Ω be the $l\times m\times n$ rectangular prism in space with vertices at $A=(0,0,0)$, $B=(l,0,0)$, $C=(l,m,0)$, $D=(0,m,0)$, $E=(0,m,n)$, $F=(0,0,n)$, $G=(l,0,n)$ and $H=(l,m,n)$. See FIGURE 1, where l, m, and n are positive integers. A bug wants to go from A to H on the surface of Ω in such a way that its path consists of line segments of integer lengths each of which is parallel to one of the axes. What is the number $f(l,m,n)$ of different shortest paths? (It is clear that the length of such a path is $l+m+n$.) This is question (1) asked in [**1**].

FIGURE 1



THEOREM.

$$f(l,m,n)=2\left\{\binom{l+m+n}{l}+\binom{l+m+n}{m}+\binom{l+m+n}{n}-\binom{l+m}{l}-\binom{m+n}{m}-\binom{n+l}{n}\right\}.$$

First Proof. We partition the set $\mathscr{F}$ of all shortest paths T into families $\mathscr{F}_i$ as follows: define $T\in\mathscr{F}_i$ if T passes through the interior of exactly i faces. It is evident that $0\leqslant i\leqslant 2$. Let $c_i=|\mathscr{F}_i|$. Then clearly $c_0=6$. If $T\in\mathscr{F}_1$ and goes through the interior of the face $ABCD$, then it clearly must go from A to C and then along the edge CH. The number of such paths is $\binom{l+m}{l}-2$ since we must exclude the two paths $A\to B\to C\to H$ and $A\to D\to C\to H$. Taking each one of the three faces containing A into account, we see that

$$c_1=2\left\{\binom{l+m}{l}+\binom{m+n}{m}+\binom{n+l}{n}-6\right\}.$$

Consider now $T\in\mathscr{F}_2$. Note that $c_2=c(l;m,n)+c(m;n,l)+c(n;l,m)$ where $c(l;m,n)$ is the number of paths of $\mathscr{F}_2$ passing through the interior of two faces of dimension $l\times m$ and $l\times n$ respectively. Suppose T goes through the interior of both faces $ABGF$ and $EFGH$. Let $K=(i,0,n)$ be the point on FG where the bug exits the face $ABGF$, $1\leqslant i\leqslant l-1$. Then the second "part" of T must start from $K'=(i,1,n)$. The number of shortest partial paths from A to K is $\binom{n+i}{n}-1$ (excluding $A\to F\to K$) and the number of shortest partial paths from K' to H is $\binom{l+m-i-1}{m-1}$. Therefore,

$$\begin{aligned}c(l;m,n)&=2\sum_{i=1}^{l-1}\left\{\binom{n+i}{n}-1\right\}\binom{l+m-i-1}{m-1}\\&=2\left\{\sum_{i=1}^{l-1}\binom{n+i}{n}\binom{l+m-i-1}{m-1}-\sum_{i=1}^{l-1}\binom{l+m-i-1}{m-1}\right\}.\end{aligned}\tag{1}$$

Now, by a known formula ([**2**], p. 64, formula (1)),

$$\sum_{k=0}^{a}\binom{a-k}{r}\binom{b+k}{s}=\binom{a+b+1}{r+s+1}.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{l-1}\binom{n+i}{n}\binom{l+m-i-1}{m-1}&=\sum_{i=0}^{l}\binom{n+i}{n}\binom{l+m-i-1}{m-1}-\binom{l+m-1}{m-1}-\binom{n+l}{n}\\&=\sum_{i=0}^{l+m-1}\binom{l+m-1-i}{m-1}\binom{n+i}{n}-\binom{l+m-1}{l}-\binom{n+l}{l}\\&=\binom{l+m+n}{m+n}-\binom{l+m-1}{l}-\binom{n+l}{l}.\end{aligned}\tag{2}$$

(In the second equality above, we used the fact that $\binom{l+m-1-i}{m-1}=0$ if $i>l$.) Also, from ([**2**], p. 64, formula (7)),

$$\begin{aligned}\sum_{i=1}^{l-1}\binom{l+m-i-1}{m-1}&=\sum_{i=1}^{l-1}\binom{l+m-i-1}{l-i}=\sum_{j=1}^{l-1}\binom{m+j-1}{j}\\&=\sum_{j=0}^{l-1}\binom{m+j-1}{j}-1=\binom{m+l-1}{l-1}-1.\end{aligned}\tag{3}$$

From (1), (2), and (3), and using the identity

$$\binom{l+m-1}{l}+\binom{l+m-1}{l-1}=\binom{l+m}{l}$$

we obtain

$$c(l;m,n)=2\left\{\binom{l+m+n}{l}-\binom{l+m}{l}-\binom{n+l}{l}+1\right\}.$$

Adding up similar expressions for $c(m;n,l)$ and $c(n;l,m)$, we get

$$c_2=2\left\{\binom{l+m+n}{l}+\binom{m+n+l}{m}+\binom{n+l+m}{n}-2\binom{l+m}{l}-2\binom{m+n}{m}-2\binom{n+l}{n}+3\right\}.$$

Therefore,

$$\begin{aligned}f(l,m,n)&=c_0+c_1+c_2\\&=2\left\{\binom{l+m+n}{l}+\binom{l+m+n}{m}+\binom{l+m+n}{n}-\binom{l+m}{l}-\binom{m+n}{m}-\binom{n+l}{n}\right\}.\end{aligned}$$

Now we give a second proof which is more "combinatorial" in the sense that it does not involve those tedious computations in the above proof.

Second Proof. Using the same notations as in the first proof, the number of shortest paths through the faces $ABGF$ and $EFGH$ is $\binom{l+m+n}{l}$ since we could view the two faces as constituting a planar grid of dimension $l\times(m+n)$. Summing over six pair of faces (i.e., $(ABGF, EFGH)$, $(ABGF, BCHG)$, $(ABCD, BCHG)$, $(ABCD, CHED)$, $(ADEF, EFGH)$, $(ADEF, CHED)$), we have

$$2\left\{\binom{l+m+n}{l}+\binom{l+m+n}{m}+\binom{l+m+n}{n}\right\}$$

shortest paths, some of which are counted more than once. Indeed, paths in $\mathscr{F}_0$ have been counted three times and those in $\mathscr{F}_1$, twice; e.g., the path $A\to F\to G\to H$ is counted by the three pairs of faces $(ABGF, EFGH)$, $(ABGF, BCHG)$, and $(ADEF, EFGH)$; and a path via the interior of the face $ABGF$ followed by the edge GH is counted by the two pairs of faces $(ABGF, EFGH)$, and $(ABGF, BCHG)$. Thus we have

$$3c_0+2c_1+c_2=2\left\{\binom{l+m+n}{l}+\binom{l+m+n}{m}+\binom{l+m+n}{n}\right\}. \tag{4}$$

The number of shortest paths through the faces $ABGF$ and along the edge GH is $\binom{n+l}{n}$. Summing over the six faces we get $2\left\{\binom{l+m}{l}+\binom{m+n}{m}+\binom{n+l}{n}\right\}$ shortest paths, some of which are counted more than once. Indeed, each path in $\mathscr{F}_0$ has been counted twice. Thus we have

$$2c_0+c_1=2\left\{\binom{l+m}{l}+\binom{m+n}{m}+\binom{n+l}{n}\right\}. \tag{5}$$

Our result now follows by subtracting (5) from (4).

In particular, if our prism is a cube (i.e., $l=m=n$), then we get the answer to the question that was raised in the beginning of this note:

$$f(n)=f(n,n,n)=6\left\{\binom{3n}{n}-\binom{2n}{n}\right\}.$$

Thus, for example, $f(1)=6$, $f(2)=54$ and $f(3)=384$.

This joint article was written when the first author was visiting Wilfrid Laurier University in July, 1984. The hospitality of this Institute is greatly appreciated.

Research supported by the Natural Sciences and Engineering Research Council of Canada under grant A9121.

References

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[2] Alan Tucker, Applied Combinatorics, John Wiley and Sons, Inc., 1980.