Similarities in Irrationality Proofs for π, ln2, ζ(2), and ζ (3)

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Similarities in Irrationality Proofs for π, ln2, $\zeta(2)$, and $\zeta(3)$

Dirk Huylebrouck

1. FOUR REMARKABLE NUMBERS. The first two numbers, π and ln2, are familiar to high school graduates. Their expressions as series are standard:

$$\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{i \geq 0} (-1)^i \cdot \frac{1}{2i+1} \qquad \text{(Leibniz' series)}$$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{i \geq 1} (-1)^{i-1} \cdot \frac{1}{i} = 0.693147\ldots.$$

Similar expressions define the less familiar $\zeta(2)$ and $\zeta(3)$:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{i^2} + \cdots = \sum_{i \geq 1} \frac{1}{i^2} = 1.64493\ldots$$

$$\text{and} \quad \zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{i^3} + \cdots = \sum_{i \geq 1} \frac{1}{i^3} = 1.20205\ldots.$$

The irrationality of π dominated a good 2000 years of mathematical history, starting with the closely related circle-squaring problem of the ancient Greeks. In 1761 Lambert proved the irrationality of π (Lindemann would complete the transcendence proof in 1882 [**2**, pp. 52 and 172]). The interest in π's younger brother $\zeta(3)$ started only a few centuries ago, but the number resisted until 1978, when R. Apéry presented his 'miraculous' proof [**14**]. Even after Apéry's lecture, scepticism remained general, until Beukers' simplified version confirmed it [**3**]. The character of the ζ-numbers still fascinates the mathematical community, and even very recently it was upset by results of Tanguy Rivoal (communication J. Van Geel, University of Ghent).

Essential in the simplified proofs are the representations of $\zeta(2)$ and $\zeta(3)$ as integrals. Since

$$\begin{aligned}\iint_0^1 \frac{1}{1-xy}\,dx\,dy &= \iint_0^1 \sum_{i \geq 0} x^i y^i \,dx\,dy \\ &= \sum_{i \geq 0} \int_0^1 x^i dx \int_0^1 y^i dy \\ &= \sum_{i \geq 0} \left[\frac{x^{i+1}}{i+1}\right]_0^1 \cdot \left[\frac{y^{i+1}}{i+1}\right]_0^1 = \sum_{i \geq 0} \frac{1}{(i+1)^2},\end{aligned}$$

we have

$$\zeta(2) = \iint_0^1 \frac{1}{1-xy}\,dx\,dy.$$

Similarly for $\zeta(3)$:

$$\iint_0^1 \frac{1}{1-xy}\ln xy\,dx\,dy = \iint_0^1 \sum_{i\geq 0} x^i y^i \ln xy\,dx\,dy = 2\iint_0^1 \sum_{i\geq 0} x^i y^i \ln x\,dx\,dy$$
$$= 2\sum_{i\geq 0}\int_0^1 x^i \ln x\,dx \int_0^1 y^i\,dy$$
$$= 2\sum_{i\geq 0}\frac{1}{i+1}\left(\left[x^{i+1}\ln x\right]_0^1 - \int_0^1 x^{i+1}\frac{1}{x}dx\right)\frac{1}{i+1}$$
$$= 2\sum_{i\geq 0}\frac{1}{i+1}\left(-\frac{1}{i+1}\right)\frac{1}{i+1} = -2\sum_{i\geq 0}\frac{1}{(i+1)^3} = -2\zeta(3).$$

Thus,

$$\zeta(3) = -\frac{1}{2}\iint_0^1 \frac{1}{1-xy}\ln xy\,dx\,dy.$$

Incidentally, the integral for $\zeta(2)$ can be evaluated very easily. This computation is not really needed here, but it provides an illustration of a calculation involving a zeta expression. The easiest components of Apostol's, Beuker's, and Kalman's results are combined, and except for some elementary knowledge about double integrals, no other prerequisites seem needed: see [**1**], [**3**], [**10**]. The combination provides a proof at graduate level for Euler's $\zeta(2)$ result; see [**4**] for a more rigorous and general proof.

First, rewrite the difference of two integrals using the substitution $X = x^2$ and $Y = y^2$:

$$\iint_0^1 \left(\frac{1}{1-xy} - \frac{1}{1+xy}\right)dx\,dy = \iint_0^1 \left(\frac{2xy}{1-x^2y^2}\right)dx\,dy$$
$$= \frac{1}{2}\iint_0^1 \left(\frac{1}{1-XY}\right)dX\,dY. \qquad (1)$$

Next, obtain their sum:

$$\iint_0^1 \left(\frac{1}{1-xy} + \frac{1}{1+xy}\right)dx\,dy = 2\iint_0^1 \frac{1}{1-x^2y^2}dx\,dy. \qquad (2)$$

Adding (1) and (2) gives

$$2\iint_0^1 \frac{1}{1-xy}dx\,dy = \frac{1}{2}\iint_0^1 \frac{1}{1-XY}dX\,dY + 2\iint_0^1 \frac{1}{1-x^2y^2}dx\,dy,$$

or

$$2\zeta(2) = \frac{1}{2}\zeta(2) + 2\iint_0^1 \frac{1}{1-x^2y^2}dx\,dy.$$

Thus,

$$\frac{3}{4}\zeta(2) = \iint_0^1 \frac{1}{1-x^2y^2}dx\,dy.$$

Substituting $x = \sin\theta/\cos\phi$ and $y = \sin\phi/\cos\theta$ yields a Jacobian equal to $1 - \tan^2\theta\tan^2\phi$ and this is the denominator of the integrand.

$$\zeta(2) = \frac{4}{3}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}-\theta} \frac{1}{1 - \left(\frac{\sin\theta}{\cos\varphi}\right)^2\left(\frac{\sin\varphi}{\cos\theta}\right)^2}\left(1 - \tan^2\theta\tan^2\varphi\right) d\varphi$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}-\theta} d\varphi = \frac{4}{3}\frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

This is Euler's well-known result: $\zeta(2) = \pi^2/6$.

2. FOUR PROOFS IN ONE. The Borweins collected irrationality proofs for these four numbers [**5**, pp. 353, 366, 369, and 370], in a very rigorous treatise. In order to make their similarity more evident, we first summarise the highlights of the demonstrations in general terms.

Suppose the irrationality of a number ξ must be shown. In the four cases we present here, a *family of integrals* ($j \in \mathbb{N}$) concerning that number is proposed:

$$\int_0^1 x^j f(x)dx = R_j + S_j\xi,$$

where R_j and $S_j \in \mathbb{Q}$; f is an unknown function.

Now if ξ were a fraction a/b, this family of integrals would yield rational expressions $\int_0^1 x^j f(x)dx = C_j/D_j$, where C_j and $D_j \in \mathbb{Z}$. Integer multiples of these integrals and their sums would again exhibit this property. Thus, if $p_{nj} \in \mathbb{Z}$ and $n \in \mathbb{N}$:

$$\int_0^1 \sum_{j=0}^n p_{nj}x^j f(x)dx = \sum_{j=0}^n p_{nj}\int_0^1 x^j f(x)dx = \sum_{j=0}^n p_{nj}\frac{C_j}{D_j} = \frac{E_n}{F_n},$$

with again E_n and $F_n \in \mathbb{Z}$.

Apply this property to the Legendre polynomials:

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n) = \sum_{j=0}^n p_{nj}x^j.$$

For example, $P_0(x) = 0$, $P_1(x) = 1 - 2x$, $P^2(x) = 2 - 12x - 12x^2$. Note that $p_{nj} \in \mathbb{Z}$.

Thus, for the given family of integrals,

$$\int_0^1 P_n(x)f(x)dx = \frac{A_n}{B_n},$$

with $A_n, B_n \in \mathbb{Z}$.

The choice of Legendre polynomials is inspired by the possibility of performing integrations by parts easily. Indeed, since $P_n(0) = 0 = P_n(1)$, and similarly for the derivatives of $x^n(1-x)^n$ up to the order n, many terms can be simplified:

$$\int_0^1 P_n(x)f(x)dx = \int_0^1 \frac{1}{n!}\frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}}(x^n(1-x)^n)\right)f(x)dx$$

$$= \left[\frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} (x^n(1-x)^n) f(x) \right]_0^1$$

$$- \int_0^1 \frac{1}{n!} \frac{d}{dx} \frac{d^{n-1}}{dx^{n-1}} (x^n(1-x)^n) \frac{df(x)}{dx} dx$$

$$= 0 - \int_0^1 \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} (x^n(1-x)^n) \frac{df(x)}{dx} dx.$$

Integrating by parts n times leads to

$$\int_0^1 \frac{1}{n!} x^n (1-x)^n \frac{d^n f(x)}{dx^n} dx.$$

In the four encountered cases the function $f(x)$ happens to be such that

$$\left| \int_0^1 \frac{1}{n!} x^n (1-x)^n \frac{d^n f(x)}{dx^n} dx \right| = \left| \int_0^1 \frac{1}{n!} (g(x))^n h(x) dx \right|,$$

where the maximum value M of $g(x)$ is small enough to ensure that

$$\left| B_n M^n \int_0^1 h(x) dx \right| \to 0.$$

In addition, all the integrals $\int_0^1 P_n(x) f(x) dx$ are non-zero, and this immediately implies the irrationality of ξ:

$$0 < |A_n| = \left| B_n \int_0^1 P_n(x) f(x) dx \right| \leq \left| B_n M^n \int_0^1 h(x) dx \right| \to 0.$$

Indeed, for any $n \in \mathbb{N}$, $|A_n|$ is a positive integer so this is impossible; ξ cannot be rational.

Of course, the difficulty in the proofs lies in the appropriate choice of $f(x)$. It must yield a family of non-zero integrals whose members are easily expressed as a combination rational number and ξ. In addition, it should simultaneously be possible to maximize their product with an integer that becomes larger and larger, and still ensure the indicated convergence to 0.

3–I. Irrationality of π Take $f(x) = \sin(\pi x)$. It is a standard calculus exercise to show that the members of family of integrals of the form $\int_0^1 x^j \sin(\pi x) dx$ are polynomials in π of degree at most j, divided by π^j. The linear combinations $\int_0^1 P_n(x) \sin(\pi x) dx$ are non-zero, and thus, if π were the rational number a/b:

$$0 < |A_n| = \left| a^n \int_0^1 P_n(x) \sin(\pi x) dx \right|$$

$$= \left| a^n \int_0^1 \frac{1}{n!} x^n (1-x)^n \frac{d^n}{dx^n} (\sin \pi x) dx \right| \leq \left| a^n \int_0^1 \frac{1}{n!} x^n (1-x)^n \pi^n dx \right|,$$

since the n-th order derivative of $\sin(\pi x)$ is $\pm\pi^n \sin(\pi x)$ or $\pm\pi^n \cos(\pi x)$. The maximum value of $x(1-x)$ is $1/4$, attained at $x = 1/2$. Thus, the final expression is less than

$$\left| a^n \frac{1}{n!} \left(\frac{1}{4}\right)^n \pi^n 1 \right|,$$

which is arbitrary small for large values of n; see [**13**].

3–II. Irrationality of ln2 The choice is $f(x) = 1/(1+x)$. An Euclidian division of x^j by $1+x$ allows us to compute the family of integrals

$$\int_0^1 \frac{x^j}{1+x} dx = \frac{1}{j} - \frac{1}{j-1} + \cdots \mp 1 \pm \ln 2.$$

If ln2 were a/b, then

$$\begin{aligned} 0 < |A_n| &= \left| (bd_n) \int_0^1 P_n(x) \frac{1}{1+x} dx \right| \qquad \text{(where } d_n = \text{LCM}\{1, 2, 3, \ldots, n\}) \\ &= \left| (bd_n) \int_0^1 \frac{1}{n!} x^n (1-x)^n \left[\frac{d^n}{dx^n} \left(\frac{1}{1+x} \right) \right] dx \right| \\ &\leq \left| (bd_n) \int_0^1 \left(\frac{x(1-x)}{1+x} \right)^n \frac{1}{1+x} dx \right|. \end{aligned}$$

since the n-th order derivative of $1/(1+x)$ is $(-1)\ldots(-n)(1/(1+x))^{n+1}$. Now on $[0, 1]$, the maximum value of $x(1-x)/(1+x)$ is $3 - 2\sqrt{2}$, achieved at $x = -1 + \sqrt{2}$. A rough inequality from number theory is $d_n = \text{LCM}\{1, \ldots, n\} \leq 3^n$. Finally, since $(3(3 - 2\sqrt{2}))^n < 1$, the irrationality of ln2 is established; see [**5**, p. 370].

3–III. Irrationality of $\zeta(2)$ The choice is

$$f(x) = \int_0^1 \frac{(1-y)^n}{1-xy} dy.$$

Each member of the family of integrals

$$\int_0^1 x^j \left[\int_0^1 \frac{(1-y)^n}{1-xy} dy \right] dx$$

is a sum of integrals of the form

$$\iint_0^1 \frac{x^r y^s}{1-xy} dy dx, \text{ with } r,\ s \in \mathbb{N}.$$

These can again be computed through an Euclidian division of $x^j y^k$ by $1 - xy$, which gives a sum of integrals of the form

$$\iint_0^1 x^p y^q dy dx, \iint_0^1 \frac{x^p}{1-xy} dy dx, \iint_0^1 \frac{y^q}{1-xy} dy dx \text{ or } \iint_0^1 \frac{1}{1-xy} dy dx.$$

The latter is $\zeta(2)$, while the others are sums of fractions, which can be computed using partial integration for the integral $\int x^m \ln x dx$. Thus when $r \neq s$,

$$\iint_0^1 \frac{x^r y^s}{1-xy} dy dx$$

is a sum of fractions whose common denominator is the square of least common multiple (LCM) of the first $n+1$ integers. When $r = s$, the integral equals

$$\sum_{i>r} \frac{1}{i^2} = \zeta(2) - \left(1 + \cdots + \frac{1}{r^2}\right).$$

Thus,

$$\left|\int_0^1 P_n(x) f(x) dx\right| = \frac{|A_n|}{d_{n+1}^2},$$

where d_{n+1} is the LCM of the first $n+1$ natural numbers and $A_n \in \mathbb{Z}_0$.

Now

$$\begin{aligned} 0 < |A_n| &= \left| d_{n+1}^2 \int_0^1 P_n(x) f(x) dx \right| \\ &= \left| d_{n+1}^2 \int_0^1 \frac{1}{n!} x^n (1-x)^n \frac{d^n}{dx^n} \left(\int_0^1 \frac{(1-y)^n}{1-xy} dy \right) dx \right| \\ &= \left| d_{n+1}^2 \int_0^1 \frac{1}{n!} x^n (1-x)^n (-1) \cdots (-n) \int_0^1 \frac{y^n (1-y)^n}{(1-xy)^{n+1}} dy dx \right| \\ &= \left| d_{n+1}^2 \int_0^1 \left(\frac{x(1-x)y(1-y)}{(1-xy)} \right)^n \frac{1}{1-xy} dy \right| \end{aligned} \tag{3}$$

On $[0, 1]$, the maximum value of $x(1-x)y(1-y)/(1-xy)$ is $((-1+\sqrt{5})/2)^5$, and is attained for $x = y = (-1+\sqrt{5})/2$. Together with $d_{n+1} \leq 3^{n+1}$, this shows that the expression (3) is less than

$$\left| \left(3^{n+1}\right)^2 \int_0^1 \left(\left(\frac{-1+\sqrt{5}}{2} \right)^5 \right)^n \frac{1}{1-xy} dx dy \right| = \left| 9 \left(\frac{9\left(-1+\sqrt{5}\right)^5}{2^5} \right)^n \zeta(2) \right| < 1,$$

which establishes the irrationality of $\zeta(2)$; see [**3**].

3–IV. Irrationality of $\zeta(3)$ Take

$$f(x) = \int_0^1 \frac{P_n(y)}{1-xy} \ln xy \, dx dy.$$

The members of the family of integrals

$$\int_0^1 x^j \left[\iint_0^1 \frac{P_n(y)}{1-xy} \ln xy \, dx dy \right] dx$$

are computed through the derivative of

$$\iint_0^1 \frac{x^{r+t}y^{s+t}}{1-xy}dxdy$$

with respect to t, which is

$$\iint_0^1 \frac{x^{r+t}y^{s+t}}{1-xy}\ln xy\,dxdy.$$

If $r \neq s$, this is a sum of fractions since $d(r+t)^{-m}/dt = -m/(r+t)^{-m-1}$, and the LCM of the denominators is $(d_{r+t})^3$.

When $r = s$, the result is

$$\sum_{i>r+t} \frac{d}{dt}\frac{1}{(r+t)^2} = \sum_{i>r+t} \frac{-2}{(r+t)^3}$$
$$= -2\left(\zeta(3) - \left(1 + \cdots + \frac{1}{r^3} + \frac{1}{(r+1)^3} + \cdots + \frac{1}{(r+t)^3}\right)\right).$$

Thus,

$$\left|\int_0^1 P_n(x)f(x)dx\right| = \frac{|A_n|}{|d_{n+1}^3|},$$

where $A_n \in \mathbb{Z}$ and d_{n+1} is the LCM of the first $n+1$ natural numbers.

Now

$$|A_n| = \left|d_{n+1}^3 \int_0^1 P_n(x)\left[\int_0^1 \frac{P_n(y)}{1-xy}\ln xy\,dy\right]dx\right|$$
$$= \left|d_{n+1}^3 \iiint_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z}dxdydz\right| \qquad \text{(integration by parts)}$$
$$= \left|d_{n+1}^3 \iiint_0^1 \frac{x^n(1-x)^nP_n(y)y^nz^n}{(1-(1-xy)z)^{n+1}}dxdydz\right| \qquad \left(\text{put } w = \frac{1-z}{1-(1-xy)z}\right)$$
$$= \left|d_{n+1}^3 \iiint_0^1 \frac{(1-x)^nP_n(y)(1-w)^n}{1-(1-xy)w}dxdydw\right| \qquad \text{(integration by parts)}$$
$$= \left|d_{n+1}^3 \iiint_0^1 \frac{x^n(1-x)^ny^n(1-y)^nw^n(1-w)^n}{(1-(1-xy)w)^{n+1}}dxdydw\right|$$
$$= \left|d_{n+1}^3 \iiint_0^1 \left(\frac{w(1-x)y(1-y)w(1-w)}{1-(1-xy)w}\right)^n \frac{1}{1-(1-xy)w}dxdydw\right| \quad (4)$$

The maximum value of $x(1-x)y(1-y)w(1-w)/(1-(1-xy)w)$ on $[0,1]$ is $(\sqrt{2}-1)^4$, and is attained for $x = y = -1+\sqrt{2}$ and $z = 1/\sqrt{2}$. Together with $d_{n+1} \leq 3^{n+1}$, this shows that the expression (4) is less than

$$\left|\left(3^{n+1}\right)^3 \iiint_0^1 \left(\sqrt{2}-1\right)^{4n} \frac{1}{1-(1-xy)w}dxdydw\right| = \left|27\left(27\left(\sqrt{2}-1\right)^4\right)^n \zeta(3)\right|$$
$$< 1,$$

which establishes the irrationality of $\zeta(3)$, see **[3]**.

4. THE GOLDEN SECTION. In our proofs, the maximum values of certain functions play an important role. They were attained at the points $1/2$, $(-1+\sqrt{5})/2$, and $-1+\sqrt{2}$. There is a coincidental link since they are all related to the golden section.

Classically, the *golden section* ϕ arises when a line segment of length x greater than 1 is divided into two parts. This could be done by cutting it into two halves (recall that $1/2$ popped up in the irrationality proof of π). Or, if unequal segments are desired, one could look for two pieces of lengths 1 and $x-1$, such that the ratio $x/1$ equals the ratio $1/(x-1)$. This equality produces the quadratic equation $x^2-x-1=0$, of which $1.6180\ldots=\phi$ is the positive solution.

More generally, the positive roots of $x^2-nx-1=0$ yield the family of *metallic means*, for various values of $n\in\mathbb{N}$. For $n=2$, we get the *silver mean* $\sigma_{Ag}=1+\sqrt{2}$, for $n=3$ the *bronze mean* $\sigma_{Br}=(3+\sqrt{13})/2$, etc. The properties of these numbers have been described in numerous publications; for a comprehensive survey, see [**6**], while [**8**] and [**12**] pointed out that some authors often had too much enthusiasm. A common misconception is that a rectangle of width 1 and length ϕ would be the "most elegant" one and thus is used in various designs. However, no reliable statistical studies confirm this statement about the optimal choice provided by the golden number [**7**].

A statement that comes close is the fact that ϕ would be "the most irrational of all irrational numbers" because its representation as a continued fraction contains only 1s:

$$\phi=1+\frac{1}{1+\frac{1}{1+\cdots}}=[1,1,\ldots].$$

The silver mean would be "the second most irrational number" since $\sigma_{Ag}=[2,2,\ldots]$, etc.; see [**6**]. Yet, this again does not provide an interpretation of the golden section an optimal solution, in the standard mathematical sense.

However, the various irrationality proofs lead to a property of these metallic means where the expression "optimal solution" has its common mathematical meaning. Table 1 illustrates some interesting facts:

TABLE 1.

Proof	Function used	Maximum	Attained at	Name
π	$x(1-x)$	$x=\frac{1}{4}$	$x=\frac{1}{2}$	2^{-1}
ln2	$\frac{x(1-x)}{(1+x)}$	$3-2\sqrt{2}$	$x=-1+\sqrt{2}$	$-\sigma_{Ag}{}^{-1}$
$\zeta(2)$	$\frac{x(1-x)y(1-y)}{(1-xy)}$	$\left(\frac{\left(-1+\sqrt{5}\right)}{2}\right)^5$	$x=y=\frac{\left(-1+\sqrt{5}\right)}{2}$	$-\phi^{-1}$
$\zeta(3)$	$\frac{x(1-x)y(1-y)w(1-w)}{(1-(1-xy)w)}$	$\left(-1+\sqrt{2}\right)^4$	$x=y=-1+\sqrt{2};\ z=\frac{1}{\sqrt{2}}$	$-\sigma_{Ag}{}^{-1}$

A substitution of $X=-1/x$ and $Y=-1/y$ in $x(1-x)y(1-y)/(1-xy)$ changes the expression into $(1+x)(1+y)/(xy(xy-1))$. Its extremum is obtained at $X=Y=\phi$. Similarly, σ_{Ag} provides the optimal solution to $(X-1)/((X+1)X)$. In these cases, the word "optimal" is used in the usual mathematical way, in contrast to the loose terms often used in golden section papers. The geometric interpretation of these facts is developed in a forthcoming text [**9**].

There are other links between $\zeta(2)$ and $\zeta(3)$, and the metallic means. For example, in the easy proof for Euler's $\zeta(2)$ result, hyperbolic sines and cosines ($x = \sinh\theta/\cosh\phi$ and $y = \sinh\phi/\cosh$) can be substituted instead of the similar expressions with circular sines and cosines. In that case,

$$\zeta(2) = \left(\ln\left(1+\sqrt{2}\right)\right)^2 + 2\int_{\ln(1+\sqrt{2})}^{+\infty} (t - \arg\cosh(\sinh t))\, dt.$$

Now the silver section $1+\sqrt{2}$ appears, while in Table 1, $\zeta(2)$ was already linked to the golden section. More involved computations relate $\zeta(3)$ to the golden section, too [**11**, p. 156]:

$$\zeta(3) = 10\int_0^{\ln(\frac{1+\sqrt{5}}{2})} t^2 \cosh t dt.$$

These relations did not inform us about the "optimal" properties of the metallic numbers, and we give them here for sake of completeness.

Incidentally, since

$$\zeta(4) = \iiiint_0^1 \frac{(1-xy)}{(1-(1-xy)w)(1-(1-xy)v)} dxdydwdv = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots,$$

a natural generalization of the functions used in the study of the irrationality of $\zeta(2)$ and $\zeta(3)$ would be

$$\frac{x(1-x)y(1-y)w(1-w)v(1-v)(1-xy)}{(1-(1-xy)w)(1-(1-xy)v)}.$$

The maximum of this function is

$$\frac{\left(5-\sqrt{13}\right)^4\left(-7+2\sqrt{13}\right)^2}{54(-3+\sqrt{13})^4},$$

obtained for $z = w = (1-\sqrt{(xy)})/(1-xy)$ and $x = y = (-3+\sqrt{13})/2$. Here again a metallic mean is found, and it is the next one, $-{\sigma_{Br}}^{-1}$. Unfortunately, it does not provide a proof for the irrationality of $\zeta(4)$ (and by extension for $\zeta(5)$) since the members of the family of integrals are not combinations of $\zeta(4)$ with rational numbers. The quest for these proofs remains open.

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Another stay in Africa again came to an abrupt end after a coup in Burundi (by someone else), and so he settled at the Sint-Lucas Institute for Architecture in Brussels, Belgium. He still has an African dream: to let its oldest mathematical artifact one day reach space.
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Solution to Sherwood Forest Puzzle on p. 143 of the February issue:

The idea of 'singer of a person's song to another' can be taken as an operation $*$, that is, "$x * y = z$" denotes "x's song is sung to y by z". Thus:

1) we have closure of $*$ (everyone's song is sung to everyone by a singer),
2) we have an 'identity' (the priest),
3) we have 'inverse' for everyone (mates),
4) the (unavoidably) cryptic third paragraph is the associativity of the operation $*$.

Thus we have a group structure. But, $10{,}201 = 101 \times 101$ and 101 is a prime, and any group of order of square of a prime is abelian.

Thus, it was Marian who sang Little John's song to Robin. For the second question, it suffices that we have a group structure since we have Marian $=$ Robin $*$ Little John. We should recall Marian and Robin are mates and so we can left multiply the equation by Robin's inverse to get Marian $*$ Marian $=$ Little John. And so Little John sang to Marian her song.