NOTES

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Double Integrals for Euler's Constant and $\ln \frac{4}{\pi}$ and an Analog of Hadjicostas's Formula

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It seems to be expected of every pilgrim up the slope of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock.

——J. J. Sylvester (1814–1897) [**19**, p. 214].

1. INTRODUCTION. Euler's constant γ is defined as the limit

$$\gamma = \lim_{N\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \ln N\right). \tag{1}$$

(For a proof that the limit exists, see [**3**, sec. 10.1], [**7**, chap. 2], or [**11**, sec. 9.1].) In this note we prove the formulas

$$\gamma = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \ln\frac{n+1}{n}\right) = \iint_{[0,1]^2} \frac{1-x}{(1-xy)(-\ln xy)}\,dx\,dy, \tag{2}$$

$$\ln\frac{4}{\pi} = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \ln\frac{n+1}{n}\right) = \iint_{[0,1]^2} \frac{1-x}{(1+xy)(-\ln xy)}\,dx\,dy. \tag{3}$$

In view of series (2), which is due to Euler, series (3) reveals $\ln(4/\pi)$ to be an "alternating Euler constant."

The constants γ and $\ln(4/\pi)$ are related by Euler's formula

$$\gamma = \ln\frac{4}{\pi} + 2\sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{2^n n}, \tag{4}$$

where

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad (\Re(s) > 1) \tag{5}$$

is the Riemann zeta function. Relation (4), a special case of a formula for the gamma function [**1**, eq. 6.1.33], can be proved using series (2), (3), and (5).

We constructed the double integral (2) for γ by analogy with Beukers's integrals [**2**]

$$\zeta(2) = \iint_{[0,1]^2} \frac{1}{1-xy}\,dx\,dy \tag{6}$$

and

$$\zeta(3) = \frac{1}{2} \iint_{[0,1]^2} \frac{-\ln xy}{1-xy}\, dx\, dy. \tag{7}$$

Since going "up" from $\zeta(2)$ to $\zeta(3)$ involves multiplying the integrand of (6) by the derivative

$$-\ln xy = -\left. \frac{\partial}{\partial t}(xy)^t \right|_{t=0},$$

in order to go "down" from $\zeta(2)$ to γ (which by comparing (1) and (5) one may think of as "$\zeta(1)$") we multiplied by the integral

$$-\frac{1}{\ln xy} = \int_0^\infty (xy)^t dt. \tag{8}$$

Because of the asymmetry in definition (1) (and the preference for a convergent improper integral), we also multiplied by $1-x$. It then turned out that the resulting integral (2) does indeed represent γ.

We noticed (3) in a different way. We first made series (2) alternate and correspondingly changed a sign in the denominator of integral (2). We then found the value $\ln(4/\pi)$ of the modified series and integral. Related formulas for $\ln(\pi/2)$ are given in [**17**].

Beukers used "damped" versions of integrals (6) and (7) to simplify Apéry's famous proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ (see [**2**], [**4**, sec. 11.3], [**12**], and [**20**]). "Damping" integrals (2) and (3), however, only leads to *criteria* for irrationality of γ [**16**] and $\ln(4/\pi)$; their arithmetic nature remains undetermined.

We prove the series formulas (2) and (3) in section 2, and the integral ones in section 3. We state a recent generalization of the integral formula (2) in section 4, and prove an analogous generalization of (3) in the final section. A special case is the evaluation

$$\iint_{[0,1]^2} \frac{1-x}{(1+xy)(\ln xy)^2}\, dx\, dy = \ln \frac{\pi^{1/2} A^6}{2^{7/6} e}, \tag{9}$$

where A is the Glaisher-Kinkelin constant, defined as the limit (see [**8**, sec. 2.15])

$$A = \lim_{n\to\infty} \frac{1^1 2^2 3^3 \cdots n^n}{n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}\, e^{-\frac{1}{4}n^2}}. \tag{10}$$

The constant A plays the same role in the approximation (10) to $1^1 2^2 3^3 \cdots n^n$ as the constant $\sqrt{2\pi}$ plays in Stirling's approximation to $n!$, which can be written

$$\sqrt{2\pi} = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{n^{n+1/2} e^{-n}}.$$

Euler's constant plays a similar role in the approximation (1) to $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ by $\ln n$.

2. SUMMING THE SERIES. To see that Euler's series (2) sums to γ, write the series as the limit

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n}-\ln\frac{n+1}{n}\right)=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\ln N-\ln\frac{N+1}{N}\right)$$

and use definition (1) .

To sum series (3), write it as the difference between the alternating harmonic series,

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}=1-\frac{1}{2}+\frac{1}{3}-\cdots=\ln 2, \tag{11}$$

and the logarithm of Wallis's product,

$$\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{(-1)^{n-1}}=\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\cdots=\frac{\pi}{2},$$

obtaining $\ln 2-\ln(\pi/2)=\ln(4/\pi)$.

3. EVALUATING THE INTEGRALS. Denote integral (2) by I and expand the expression $1/(1-xy)$ in a geometric series:

$$I=\iint\limits_{[0,1]^2}\sum_{n=0}^{\infty}\frac{1-x}{-\ln xy}(xy)^n\,dx\,dy=\sum_{n=0}^{\infty}\iint\limits_{[0,1]^2}(1-x)\frac{(xy)^n}{-\ln xy}\,dx\,dy.$$

(To justify interchanging summation and integration, replace the series on the left with a finite sum plus remainder and estimate the remainder—see [**16**, sec. 2].) Substitute the generalization of (8)

$$-\frac{(xy)^n}{\ln xy}=\int_n^{\infty}(xy)^t dt$$

and reverse the order of integration (permitted because the integrand is nonnegative) to obtain

$$I=\sum_{n=0}^{\infty}\iint\limits_{[0,1]^2}\int_n^{\infty}(1-x)(xy)^t\,dt\,dx\,dy=\sum_{n=0}^{\infty}\int_n^{\infty}\iint\limits_{[0,1]^2}(x^t-x^{t+1})y^t\,dx\,dy\,dt.$$

Replace n with $n-1$ and integrate, first with respect to x and y, then with respect to t, to arrive at series (2) for γ. (For a different proof, see the remark at the end of [**18**].)

The evaluation of integral (3) is similar. Since the expansion of $1/(1+xy)$ is an alternating series, the result is series (3) for $\ln(4/\pi)$.

4. HADJICOSTAS'S FORMULA. After seeing the integral formula (2) for γ, Hadjicostas conjectured a generalization [**9**], which Chapman then proved [**5**]:

$$\iint\limits_{[0,1]^2}\frac{1-x}{1-xy}(-\ln xy)^s\,dx\,dy=\Gamma(s+2)\left[\zeta(s+2)-\frac{1}{s+1}\right]\qquad(\Re(s)>-2). \tag{12}$$

Here Γ is defined by Euler's integral [**6**],

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \qquad (\Re(s) > 0), \tag{13}$$

and ζ is the analytic continuation of (5) (see, for example, [**13**] or [**17**]).

Taking the limit as $s \to -1$ in (12) and using the fact that (see [**14**])

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma, \tag{14}$$

we obtain (2). Conversely, (2) and (12) imply (14).

For $s = 0$ and 1, formula (12) is related to the integral representations (6) and (7) of $\zeta(2)$ and $\zeta(3)$, respectively.

5. AN ANALOG. Hadjicostas asked if there is an analog of (12) for the integral formula (3) for $\ln(4/\pi)$ [**10**]. The following result fills the bill and, as a bonus, yields (9).

Theorem 1. *For complex s with* $\Re(s) > -3$,

$$\iint\limits_{[0,1]^2} \frac{1-x}{1+xy}(-\ln xy)^s\, dx\, dy = \Gamma(s+2)\left[\zeta^*(s+2) + \frac{1-2\zeta^*(s+1)}{s+1}\right], \tag{15}$$

where $\zeta^*(s)$, *the alternating zeta function, is the analytic continuation of the convergent Dirichlet series*

$$\zeta^*(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \qquad (\Re(s) > 0).$$

Here $\Gamma(s)$ is analytically continued for $\Re(s) > -1$ and $s \neq 0$ by the functional equation (established by integrating by parts in (13))

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \tag{16}$$

and $\zeta^*(s)$ is an entire function related to $\zeta(s)$ by the product (see [**13**] or [**15**])

$$\zeta^*(s) = (1 - 2^{1-s})\zeta(s). \tag{17}$$

Proof of Theorem 1. The integral in (15) defines a holomorphic function $I(s)$ on the half-plane $\Re(s) > -3$. We prove (15) for $\Re(s) > 0$ and the theorem then follows by analytic continuation.

We make the change of variables $u = xy$, $v = 1 - x$ and integrate with respect to v. We then substitute $t = -\ln u$, obtaining

$$I(s) = \int_0^\infty \left(\frac{t^{s+1} - 2t^s}{e^t + 1} + e^{-t} t^s\right) dt.$$

Using the evaluation

$$\int_0^\infty \frac{t^{s-1}}{e^t + 1}\, dt = \Gamma(s)\zeta^*(s) \tag{18}$$

(to prove (18), write the denominator as $e^t(1+e^{-t})$ and expand $1/(1+e^{-t})$ in a geometric series, then integrate termwise via (13)), we get

$$I(s) = \Gamma(s+2)\zeta^*(s+2) + \Gamma(s+1)\left[1 - 2\zeta^*(s+1)\right].$$

Now (16) (with $s+1$ in place of s) gives (15) and the theorem follows. ∎

In order to show that Theorem 1 implies the integral formulas (3) and (9), we need the following values of the alternating zeta function and its derivative: $\zeta^*(1) = \ln 2$ (from (11)), $\zeta^*(0) = 1/2$, $\zeta^*(-1) = 1/4$, and $\zeta^{*\prime}(0) = \frac{1}{2}\ln\frac{\pi}{2}$ (see [**13**]). It follows that $I(-1) = \zeta^*(1) - 2\zeta^{*\prime}(0)$, which gives (3). Using (16) (with $s+2$ in place of s), we deduce that $I(-2) = \zeta^{*\prime}(0) + 2\zeta^*(-1) - 1 + 2\zeta^{*\prime}(-1)$. Employing (17) and the formula $\ln A = \frac{1}{12} - \zeta'(-1)$ for the Glaisher-Kinkelin constant (see [**8**, sec. 2.15]), we arrive at (9).

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