Proof: Consider the identity $\sqrt{2}^{(\sqrt{2}+1)} = (\sqrt{2}^{\sqrt{2}})\sqrt{2}$.

If $\sqrt{2}^{\sqrt{2}}$ is irrational then we are finished. If not, then $\sqrt{2}^{\sqrt{2}}$ is rational. Hence $(\sqrt{2}^{\sqrt{2}})\sqrt{2}$ is irrational, and $\sqrt{2}^{(\sqrt{2}+1)}$ is the example in this case.

There is also a simple identity by means of which it can be proved that a rational number raised to an irrational power may be irrational. But perhaps the reader would enjoy finding this one himself.

References

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A SIMPLE PROOF OF THE FORMULA $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$

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Start with the inequality $\sin x < x < \tan x$ for $0 < x < \pi/2$, take reciprocals, and square each member to obtain

$$\cot^2 x < 1/x^2 < 1 + \cot^2 x.$$

Now put $x = k\pi/(2m+1)$ where k and m are integers, $1 \leqq k \leqq m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}. \tag{1}$$

But since we have

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} = \frac{m(2m-1)}{3}, \tag{2}$$

(a proof of (2) is given below) relation (1) gives us

$$\frac{m(2m-1)}{3} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \frac{m(2m-1)}{3}.$$

Multiply this relation by $\pi^2/(4m^2)$ and let $m \to \infty$ to obtain

$$\lim_{m\to\infty} \sum_{k=1}^{m} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

Proof of (2). By equating imaginary parts in the formula

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \sin^n\theta(\cot\theta + i)^n$$

$$= \sin^n\theta \sum_{k=0}^{n} \binom{n}{k} i^k \cot^{n-k}\theta,$$

we obtain the trigonometric identity

$$\sin n\theta = \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - + \cdots \right\}.$$

Take $n = 2m + 1$ and write this in the form

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m(\cot^2\theta) \text{ with } 0 < \theta < \frac{\pi}{2}, \tag{3}$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - + \cdots .$$

Since $\sin\theta \neq 0$ for $0 < \theta < \pi/2$, equation (3) shows that $P_m(\cot^2\theta) = 0$ if and only if $(2m+1)\theta = k\pi$ for some integer k. Therefore $P_m(x)$ vanishes at the m distinct points $x_k = \cot^2 \pi k/(2m+1)$ for $k = 1, 2, \cdots, m$. These are all the zeros of $P_m(x)$ and their sum is

$$\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = \binom{2m+1}{3} \Big/ \binom{2m+1}{1} = \frac{m(2m-1)}{3},$$

which proves (2).

NOTE. This paper was translated from a Greek manuscript and communicated to the MONTHLY on behalf of the author by Tom M. Apostol, California Institute of Technology. After this paper was written it was learned that the same proof was discovered independently and published in Norwegian by Finn Holme in Nordisk Matematisk Tidskrift, vol. 18 (1970), pp. 91–92. See also A. M. Yaglom and I. M. Yaglom, *Challenging mathematical problems with elementary solutions*, vol. II, Holden-Day, San Francisco, 1967, problem 145.

ANOTHER ELEMENTARY PROOF OF EULER'S FORMULA FOR $\zeta(2n)$

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1. Introduction. The classic formula

$$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = (-1)^{n-1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} \tag{1}$$

which expresses $\zeta(2n)$ as a rational multiple of π^{2n} was discovered by Euler [**2**]. The numbers B_n are Bernoulli numbers and can be defined by the recursion formula

$$B_0 = 1, \; B_n = \sum_{s=0}^{n} \binom{n}{s} B_s \text{ for } n \geqq 2,$$

or equivalently, as the coefficients in the power series expansion