NOTES

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A Simple Formula for π

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Dedicated to the memory of Tom Tymoczko (1943–1996), an innovative investigator into the nature of computer proofs

1. THE RADICAL BBP IDEA. In 1995 David Bailey, Peter Borwein, and Simon Plouffe [**2**] discovered the following shocking formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

This result is shocking because it can be used to generate the nth base-16 digit of π without having to look at any prior digits. And, so long as n is less than a billion or so, the entire computation can be carried out with 16-digit numbers. This is a radical idea, since all previous algorithms for the nth digit of π required the computation of all previous digits, and the use of d-digit arithmetic in the computation. For more details of the fairly easy argument that leads from the BBP formula to an algorithm for far-out hex digits of π see [**1**] or [**2**].

Proving the BBP formula is not difficult. But that misses the main point: How did they find it? In short, they had a hunch that such a formula might exist and they searched for it using high-precision approximate reals, a high-performance SGI workstation, and the PSLQ algorithm [**3**], [**4**]. In this note we show how a simpler formula of this type can be discovered in such a way that a proof accompanies the discovery. We will present only a single result. Several more formulas of this type can be found [**1**].

Before leaving the BBP formula, here is a proof that it is correct using *Mathematica* to perform the summation.

```
FullSimplify[TrigToExp[FullSimplify[
```

$$\sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)]]\,/.$$

```
a_ Log[b_] + a_ Log[c_] :> a Log[b c]].
```

π

This "proof" is of very little value, for it gives us no insight whatsoever. Some might even say that it is not truly a proof! But in principle, such a computation *can* be viewed as a proof. There are some subtleties. Some types of computations come along with certificates that allow verification; for example, if a computer churns out an indefinite integral, the result can be differentiated to see if it agrees with the integrand. *Mathematica* does not provide such certificates for sums, but recent work of Wilf and Zeilberger has shown that certain sums, such as the ones that occur here, do carry certificates, and implementing the production and verification of such certificates has in fact been done (see **[5]**). So it is true that, provided one uses the latest work on symbolic summation algorithms, a computation such as the above can be taken to be a proof.

But such a proof is not very helpful. The real power of sophisticated symbolic software is that this first computation provides the starting point for an investigation that yields both deeper understanding of the formula and, with luck, some new formulas. That sort of investigation is what we carry out here. It turns out that the sums that arise in this note can be transformed to integrals, and then antiderivatives can be checked so that a proper standard of proof is maintained. We show how to do that at the end of Section 2, but we start our work with the reasonable working assumption that the results of `Sum` are correct.

2. DISCOVERY AND PROOF. Suppose we wish to see if π can be expressed in the following form (we examined several such forms and are presenting here the simplest one that worked).

$$\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\left(\frac{a_1}{4k+1} + \frac{a_2}{4k+2} + \frac{a_3}{4k+3} + \frac{a_4}{4k+4}\right).$$

We just feed the general sum to *Mathematica* (we used version 3.0.0; other versions may yield slightly different forms).

`Simplify[FunctionExpand[`

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\left(\frac{a_1}{4k+1} + \frac{a_2}{4k+2} + \frac{a_3}{4k+3} + \frac{a_4}{4k+4}\right)]]$$

$$\frac{1}{8}(2(4(a_2 \text{ ArcCot}[2] - a_4 \text{ Log}[4] + a_4 \text{ Log}[5] + a_3(\pi/4 + \text{ArcCot}[3] - \text{Log}[25]/4))) + a_1(\pi + 4 \text{ ArcCot}[3] + \text{Log}[25]))$$

Now we make some simplifications, the last one based on the identity arctan 1 + arctan 2 + arctan 3 = π.

```
Expand[%/.{Log[25] → 2 Log[5], Log[4] → 2 Log[2],
    ArcCot[x_]:>(π/2-ArcTan[x])}/.
    ArcTan[3] → (3 π/4-ArcTan[2])]
```

$$\frac{a_2\pi}{2} + \frac{1}{2}a_1 \text{ ArcTan}[2] - a_2 \text{ ArcTan}[2] + a_3 \text{ ArcTan}[2] - 2\ a_4 \text{ Log}[2] + \frac{1}{4}a_1 \text{ Log}[5] - \frac{1}{2}a_3 \text{ Log}[5] + a_4 \text{ Log}[5]$$

```
Collect[%, {π, ArcTan[2], Log[5], Log[2]}]
```

$$\frac{a_2\pi}{2} + \left(\frac{a_1}{2} - a_2 + a_3\right)\text{ArcTan}[2] - 2a_4\text{Log}[2] + \left(\frac{a_1}{4} - \frac{a_3}{2} + a_4\right)\text{Log}[5]$$

Now we simply search for *a*-values that cause all but the first summand to vanish, and the first to equal π. This is easily done by hand, but since *Mathematica* is running:

```
Solve[{a2/2 == 1, a1/2 - a2 + a3 == 0, a4 == 0, a1/4 - a3/2 + a4 == 0}]
{{a2 → 2, a1 → 2, a3 → 1, a4 → 0}}.
```

And so we have a new formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\left(\frac{2}{4k+1} + \frac{2}{4k+2} + \frac{1}{4k+3}\right).$$

We reiterate that the proof comes for free along with the discovery (though for a rigorous proof one might prefer to use integrals instead of sums, as we discuss in a moment). As with BBP, our formula can be used in a digit-extraction scheme in base 4. Of course, digit extraction in base 4 is fully equivalent to the base-16 case. This method of undetermined coefficients can also be used to generate the BBP formula; we leave that task to the reader who wishes to exercise a computer algebra system.

Further explorations along these lines seem to be easier if one uses integrals instead of series. This also eases the task of producing a verifiable proof. Such a transformation, focussing on the base-4 case under discussion, is carried out as follows.

1. Define $g(i) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k}\left(\frac{1}{4k+i}\right)$.

2. $g(i) = \sum_{k=0}^{\infty} \sqrt{2^i} \int_0^{1/\sqrt{2}} (-1)^k z^{4k+i-1}\, dz$ (easy integration).

3. $g(i) = \sqrt{2^i} \int_0^{1/\sqrt{2}} \sum_{k=0}^{\infty} ((-1)^k z^{4k+i-1})\, dz$ (interchange).

4. $g(i) = \sqrt{2^i} \int_0^{1/\sqrt{2}} \frac{z^{i-1}}{1+z^4}\, dz$ (geometric series).

5. Use undetermined coefficients a_i with (4) and call on either a computer or an integration expert to get a closed-form expression for $\sum_{i=1}^{4} a_i g(i)$. It will agree with the four-transcendental expression we obtained at the beginning of this section.

Of course, many other forms can be investigated in the hope of getting more formulas for π or other constants. Sadly, it seems as if these ideas may not lead to a formula that allows extraction of base-10 digits. From one point of view, the crucial miracle that makes the above formulas work is that certain arctangents that arise are rational multiples of π. J. Buhler has shown that this can happen only in situations that are essentially equivalent to the ones above; in particular, no base-10 formula relying on this particular phenomenon exists. Still, there might be other numerical miracles that could give a base-10 formula or, more generally, other kinds of formulas or techniques for rapid extraction of base-10 digits.

REFERENCES

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Borsuk-Ulam Implies Brouwer: A Direct Construction

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1. INTRODUCTION. The Borsuk-Ulam theorem and the Brouwer fixed point theorem are well-known theorems of topology with a very similar flavor. Both are non-constructive existence results with somewhat surprising conclusions. Most topology textbooks that cover these theorems (e.g., [**4**], [**5**], [**6**]) do not mention the two are related—although, in fact, the Borsuk-Ulam theorem implies the Brouwer Fixed Point Theorem.

The theorems themselves are often proved using the machinery of algebraic topology or the concept of degree of a map. That one theorem implies the other can therefore be established once one understands this machinery, but this requires background. Moreover, such proofs tend to be indirect, relying on the equivalence of these existence theorems with corresponding *non-existence* theorems. For instance, Dugundji and Granas [**3**] show that the Borsuk-Ulam theorem is equivalent to the statement that no antipode-preserving, continuous map f: $S^n \to S^n$ can be homotopic to a constant map. From this one can see that the Brouwer fixed point theorem is a special case, because it can be shown equivalent to the statement that the identity map id: $S^n \to S^n$ (which is antipode-preserving) is not homotopic to a constant map.

However, such an indirect approach is not really necessary, and perhaps a more direct proof would give insight as to how the two theorems are related. The purpose of this note is to provide a completely elementary proof that the Borsuk-Ulam theorem implies the Brouwer theorem by a *direct* construction, in which the existence of antipodal points in one theorem yields the asserted fixed point in the other.

2. THE THEOREMS. Let S^n denote the unit n-sphere in $\mathbb{R}^{n+1}$, i.e., all points at distance one from the origin. Two points are *antipodal* if they lie opposite each other on the sphere—i.e., $\{\mathbf{x}, -\mathbf{x}\}$ for some $\mathbf{x}$.