SUPPLEMENTARY NOTE ON "DISTANCE SETS ON CIRCLES"

1. Introduction

In [1], the authors studied distance sets on circles, and showed the following main theorem.

Theorem 1.1. *Let X be a k-distance set with n points on S^1, where $n \geq 4$, and let*

$$M_n = \begin{cases} 3t, & \text{if } n = 4t \text{ or } 4t-1, \\ 3t-2, & \text{if } n = 4t-2 \text{ or } 4t-3. \end{cases}$$

If $k < M_n$, then X lies on a regular $2k$ or $(2k+1)$-sided polygon.

In order to prove this theorem, the authors gave several lemmas, but some of the proofs were partially left to the readers. In this note, for complementing the paper, we give the complete proofs for the lemmas. Furthermore, we give algorithms to classify by computer distance sets on $\mathbb{R}$ or S^1 with small number of points.

2. Proof of Lemma 2.5

Lemma 2.5. *Let X be a distance set with the form of type (i) A–I, (ii) A–II, or (iii) A–III with $n = 2m$ points, where $m \geq 4$. Then, $|D(\ell X r) \setminus D(X)| \leq 4$ for each type of the form of X if and only if (i) $\ell = r = 1$, (ii) $\ell, r \in \{c, 1+c\}$, or (iii) $\ell = r = 1+c$, respectively.*

Proof: For each case, ℓ and r given in the statement of this lemma satisfy $|D(\ell X r) \setminus D(X)| \leq 4$. It is enough to prove that there is no $\ell > 0$ other than those in each case of (i), (ii), and (iii) satisfying $|D(\ell X) \setminus D(X)| \leq 3$ since $|D(\ell X r) \setminus D(\ell X)| \geq 1$ for any $r > 0$ and X is bilaterally symmetric. We may assume that ℓ is expressed as $\ell = i + jc$ for $i, j \in \mathbb{Z}$ since otherwise $|D(\ell X) \setminus D(X)| \geq n$, where c is defined as in Theorem 2.4.

(i) Let $X = (\underbrace{1, 1, \ldots, 1}_{m-1}, c, \underbrace{1, 1, \ldots, 1}_{m-1})$ for $m \geq 4$. Then

$$D(X) = \{i \mid 1 \leq i \leq m-1\} \cup \{i + c \mid 0 \leq i \leq 2m-2\}.$$

Suppose that $\ell = i + jc \neq 1$. Then $\{\ell + diam(X) - 1, \ell + diam(X)\} \subset D(\ell X) \setminus D(X)$. In this case, we may assume that $j \in \{0, -1\}$ since otherwise $\{\ell + diam(X) - 3, \ell + diam(X) - 2\} \subset D(\ell X) \setminus D(X)$, which implies $|D(\ell X) \setminus D(X)| \geq 4$. If $j = -1$, then $i \geq 1$ and we have $\{\ell, \ell + m - 1 + c\} \subset D(\ell X) \setminus D(X)$. Hence, $\ell = i \geq 2$. In this case, $\{\ell + m - 1, \ell + m - 2\} \subset D(\ell X) \setminus D(X)$. Therefore we have $\ell = 1$.

(ii) Let $X = (\underbrace{1, c, 1, c, \ldots, 1, c, 1}_{2m-1})$ for $m \geq 4$. Then

$$D(X) = \{i + jc \mid 0 \leq i \leq m, 0 \leq j \leq m-1, |i - j| \leq 1, (i, j) \neq (0, 0)\}.$$

Since it has to hold that $\{\ell + diam(X) - (2+c), \ell + diam(X) - (1+c), \ell + diam(X) - 1, \ell + diam(X)\} \not\subseteq D(\ell X) \setminus D(X)$ by our assumption, we have $\ell \leq 2 + c$. If $i > j$, then $\{\ell + 1, \ell + (2+c), \ell + (3+2c), \ell + (4+3c)\} \subset D(\ell X) \setminus D(X)$. If $j - i \geq 2$, then $\{\ell, \ell + (1+c), \ell + (2+2c), \ell + (3+3c)\} \subset D(\ell X) \setminus D(X)$. Furthermore, these also imply that $i \geq -1$ and $j \geq 0$. Therefore we have $\ell \in \{c, 1+c\}$.

(iii) Let $X = (\underbrace{1+c,\dots,1+c}_{m_1},\underbrace{1,c,1,c,\dots,c,1}_{2m_2-1},\underbrace{1+c,\dots,1+c}_{m_1})$ for $m_1 \geq 1$, $m_2 \geq 2$ and $m_1 + m_2 \geq 4$. Then

$$\begin{aligned} D(X) = &\{i + jc \mid 0 \leq i \leq m_2, 0 \leq j \leq m_2 - 1, |i - j| \leq 1, (i,j) \neq (0,0)\} \\ &\cup \{i + ic \mid m_2 \leq i \leq m_1 + m_2 - 1\} \\ &\cup \{(i+1) + ic \mid m_2 \leq i \leq 2m_1 + m_2 - 1\}. \end{aligned}$$

Suppose that $\ell \neq 1 + c$. Then $\{\ell + diam(X), \ell + diam(X) - (1+c)\} \subset D(\ell X) \setminus D(X)$. If $i > j$, then $\{\ell + m_1(1+c) + 1, \ell + m_1(1+c) + 2 + c\} \subset D(\ell X) \setminus D(X)$. Since $\ell + m_1(1+c) + 2 + c < \ell + diam(X) - (1+c)$, $|D(\ell X) \setminus D(X)| \geq 4$. If $0 \leq i < j$, then $\ell + (m_1 + m_2 - 2)(1+c), \ell + (m_1 + m_2 - 1)(1+c) \notin D(X)$, and hence $|D(\ell X) \setminus D(X)| \geq 4$. If $i < j$ and $i < 0$, then $j > 1$ and $\ell, \ell + 1 + c \notin D(X)$, which implies that $|D(\ell X) \setminus D(X)| \geq 4$. Finally, if $i = j \geq 2$, then $\{\ell + (m_1 + m_2 - 1)(1+c), \ell + (m_1 + m_2 - 2)(1+c)\} \subset D(\ell X) \setminus D(X)$, and hence $|D(\ell X) \setminus D(X)| \geq 4$. Therefore we have $\ell = 1 + c$.

The proof is now complete. □

3. Proof of Lemma 2.8

Lemma 2.8. *Let X be a distance set with the form of type (i) B–I, (ii) B–II, or (ii) B–III with $n = 2m$ points, where $m \geq 4$. Then, $|D(\ell X r) \setminus D(X)| \leq 3$ for $\ell, r \in \mathbb{R}$ for each type of the form of X if and only if (i) $(\ell, r) = (1,1)$, (ii) $(\ell, r) = (1+c, 1)$ or (iii) $(\ell, r) = (1+c, 1+c)$, respectively.*

Proof: In each case, we first determine $\ell > 0$ such that $|D(\ell X) \setminus D(X)| \leq 2$ since $|D(\ell X r) \setminus D(\ell X)| \geq 1$ for any $r > 0$. We may assume that ℓ has the form $\ell = i + jc$ for $i, j \in \mathbb{Z}$ since otherwise $|D(\ell X) \setminus D(X)| \geq n$, where c is defined as in Theorem 2.4.

(i) Let $X = (\underbrace{1,1,\dots,1}_{m}, c, \underbrace{1,1,\dots,1}_{m-2})$ for $m \geq 4$. Then

$$D(X) = \{i \mid 1 \leq i \leq m\} \cup \{i + c \mid 0 \leq i \leq 2m - 2\}.$$

Suppose that $\ell \neq 1$. Then $\ell \leq 2$ and $\{\ell, \ell + m, \ell + m + c\} \subset D(X)$ by Lemma 2.7. If $j > 0$, then $\ell + m + c = (m+i) + (1+j)c \notin D(X)$. If $j < 0$, then $\ell \notin D(X)$. Hence, we have $j = 0$. If $\ell = 2$, then $\ell + m \notin D(X)$. Hence $\ell = 1$ and $D(\ell X) \setminus D(X) = \{m+1, diam(X)+1\}$.

Next, we find $r = i + jc > 0$ such that $|D(Xr) \setminus D(X)| \leq 2$. Suppose that $r \neq 1$. Then $r \leq 2$ and $\{r, r + m - 2, r + m - 2 + c\} \subset D(X)$ by Lemma 2.7. If $j < 0$, then $r \notin D(X)$. If $j > 0$, then $r + m - 2 + c \notin D(X)$. Hence, we have $j = 0$. If $r = i \geq 3$, then $r + m - 2 \notin D(X)$. Therefore, we have $r \in \{1, 2\}$. In particular,

$$D(Xr) \setminus D(X) = \begin{cases} \{diam(X) + 1\}, & \text{if } r = 1, \\ \{diam(X) + 1, diam(X) + 2\}, & \text{if } r = 2. \end{cases}$$

Since it has to hold that $|(D(\ell X) \cup D(Xr)) \setminus D(X)| \leq 2$, we have $(\ell, r) = (1,1)$.

(ii) Let $X = (1+c, \underbrace{1,c,1,c,\dots,1,c}_{2m-2})$ for $m \geq 3$. Then

$$\begin{aligned} D(X) = &\{i + jc \mid 0 \leq i \leq m, 0 \leq j \leq m-1, |i-j| \leq 1, (i,j) \neq (0,0)\} \\ &\cup \{m + mc\}. \end{aligned}$$

Suppose that $\ell \neq c$, then $\ell \leq 1 + c$ and $\{\ell, \ell + 2 + c\} \subset D(X)$ by Lemma 2.7. If $i > j$, then $\ell + 2 + c = (i+2) + (j+1)c \notin D(X)$. If $i + 1 < j$, then $\ell \notin D(X)$. These also implies that $i, j \geq 0$. Therefore $\ell \in \{c, 1+c\}$. In particular,

$$D(\ell X) \setminus D(X) = \begin{cases} \{diam(X) - 1, diam(X) + c\}, & \text{if } \ell = c, \\ \{diam(X) + 1, diam(X) + 1 + c\}, & \text{if } \ell = 1 + c. \end{cases}$$

Next, we find $r = i + jc > 0$ such that $|D(Xr) \setminus D(X)| \le 2$. Suppose that $r \ne 1 + c$. Then $r \le 2 + c$ and $\{r, r + c, r + 1 + c\} \subset D(X)$ by Lemma 2.7. If $r = 2$, then $r + 1 + c \notin D(X)$. If $i > j + 1$, then $r \notin D(X)$. If $i < j$, then $r + c \notin D(X)$. Hence, $i = j$ or $i = j + 1$, which also implies that $i, j \ge 0$. Therefore, we have $r \in \{1, 1 + c, 2 + c\}$. In particular,

$$D(Xr) \setminus D(X) = \begin{cases} \{diam(X) + 1\}, & \text{if } r = 1, \\ \{diam(X) + 1 + c, diam(X) - 1\}, & \text{if } r = 1 + c, \\ \{diam(X) + 2 + c, diam(X) + 1\}, & \text{if } r = 2 + c. \end{cases}$$

Since it has to hold that $|(D(\ell X) \cup D(Xr)) \setminus D(X)| \le 2$, we have $(\ell, r) = (1 + c, 1)$.

(iii) Let $X = (\underbrace{1 + c, \ldots, 1 + c}_{m_1+1}, \underbrace{1, c, 1, c, \ldots, c, 1}_{2m_2-1}, \underbrace{1 + c, \ldots, 1 + c}_{m_1-1})$ for $m_1 \ge 1$, $m_2 \ge 2$. Then,

$$\begin{aligned} D(X) = &\{i + jc \mid 0 \le i \le m_2, 0 \le j \le m_2 - 1, |i - j| \le 1, (i, j) \ne (0, 0)\} \\ &\cup \{i + ic \mid m_2 \le i \le m_1 + m_2\} \\ &\cup \{(i + 1) + ic \mid m_2 \le i \le 2m_1 + m_2 - 1\}. \end{aligned}$$

We consider the two cases: (A) $m_1 = 1$; (B) $m_1 \ge 2$.

(A) Suppose that $\ell \ne 1$, then $\ell \le 1 + c$ and $\ell \in D(X)$ by Lemma 2.7. It is clear that $|i - j| \le 1$ with $i, j \ge 0$ in order for $\ell \in D(X)$. Then, $\ell \in \{c, 1 + c\}$. If $\ell = c$, then $\{(1+c)(m_1+m_2)+c, diam(X)+c\} \subset D(\ell X)\setminus D(X)$. If $\ell = 1+c$, then $\{(1+c)(m_1+m_2+1), diam(X)+1+c\} \subset D(\ell X)\setminus D(X)$. If $\ell = 1$, then $\{(1+c)(m_1+1)+2, diam(X)+1\} \subset D(\ell X) \setminus D(X)$.
Next, we find $r = i + jc > 0$ such that $|D(Xr) \setminus D(X)| \le 2$. Suppose that $r \ne 1 + c$. Then $r \le 2 + 2c$ and $r \in D(X)$ by Lemma 2.7. It is clear that $|i - j| \le 1$ with $i, j \ge 0$ in order for $r \in D(X)$. Then, $r \in \{1, c, 1 + c, 2 + c, 1 + 2c, 2 + 2c\}$. If $i > j$, then $\{r + 1, r + c + 2, diam(X) + r\} \subset D(Xr) \setminus D(X)$. If $i < j$, then $\{diam(X) + r, (m_2 - 1)(1 + c) + r\} \subset D(Xr) \setminus D(X)$. If $r = 1 + c$, then $diam(X) + 1 + c \in D(Xr) \setminus D(X)$. If $r = 2 + 2c$, then $\{diam(X) + 1 + c, diam(X) + 2 + 2c\} \subset D(Xr) \setminus D(X)$. Since it has to hold that $|(D(\ell X) \cup D(Xr)) \setminus D(X)| \le 2$, we have $(\ell, r) = (1 + c, 1 + c)$.

(B) Suppose that $\ell \ne 1 + c$, then $\ell \le 2 + 2c$ and $\ell \in D(X)$ by Lemma 2.7. It is clear that $|i - j| \le 1$ with $i, j \ge 0$ in order for $\ell \in D(X)$. Then, $\ell \in \{1, c, 1 + c, 2 + c, 1 + 2c, 2 + 2c\}$. If $i > j$, then $\{\ell+(m_1+1)(1+c)+1, \ell+(m_1+2)(1+c)+1, \ell+diam(X)\} \subset D(\ell X)\setminus D(X)$. If $i < j$, then $\{\ell+(m_1+m_2)(1+c), \ell+(m_1+m_2)(1+c)+1, \ell+diam(X)\} \subset D(\ell X)\setminus D(X)$. If $i = j \ge 2$, i.e., $\ell = 2 + 2c$, then $\{\ell + (m_1 + m_2 - 1)(1 + c), \ell + diam(X) - (1 + c), \ell + diam(X)\} \subset D(\ell X) \setminus D(X)$. Hence, $\ell = 1 + c$ and $D(\ell X) \setminus D(X) = \{(m_1 + m_2 + 1)(1 + c), diam(X) + 1 + c\}$.
Next, we find $r = i + jc > 0$ such that $|D(Xr) \setminus D(X)| \le 2$. Suppose that $r \ne 1 + c$. Then $r \le 2 + 2c$ and $r \in D(X)$ by Lemma 2.7. It is clear that $|i - j| \le 1$ with $i, j \ge 0$ in order for $r \in D(X)$. Then, $r \in \{1, c, 1 + c, 2 + c, 1 + 2c, 2 + 2c\}$. If $i > j$, then $\{r + (m_1 - 1)(1 + c) + 1, r + m_1(1 + c) + 1, r + diam(X)\} \subset D(Xr) \setminus D(X)$. If $i < j$, then $\{r - (1 + c) + diam(X), r - 2(1 + c) + diam(X), r + diam(X)\} \subset D(Xr) \setminus D(X)$. Hence, we have $r \in \{1 + c, 2 + 2c\}$. In particular,

$$D(Xr) \setminus D(X) = \begin{cases} \{diam(X) + 1 + c\}, & \text{if } r = 1 + c, \\ \{diam(X) + 1 + c, diam(X) + 2 + c\}, & \text{if } r = 2 + 2c. \end{cases}$$

Since it has to hold that $|(D(\ell X) \cup D(Xr)) \setminus D(X)| \le 2$, we have $(\ell, r) = (1 + c, 1 + c)$.

The proof is now complete. □

4. Proof of Theorem 2.4

Theorem 2.4. *Let X be a k-distance set with $n \geq 3$ points on $\mathbb{R}$ containing both rational and irrational intervals. Then, the following hold:*

1. *For $m \geq 2$, $(n,k) = (2m, 3m-2)$ if and only if the form of X is equivalent to that of either type A–I, A–II, or A–III in Table 1;*
2. *For $m \geq 7$, $(n,k) = (2m, 3m-1)$ if and only if the form of X is equivalent to that of either type B–I, B–II, or B–III in Table 1;*
3. *For $m \geq 5$, $(n,k) = (2m+1, 3m)$ if and only if the form of X is equivalent to that of either type C–I, C–II, or C–III in Table 1,*

where c is any irrational positive number and all the forms have the 2-element base $\{1, c\}$. There are exceptional examples of $2m$-point $(3m-1)$-distance sets for $m \leq 6$ and $(2m+1)$-point $3m$-distance sets for $m \leq 4$ not contained in the classifications above, which are listed in Tables 2 and 3.

Proof: We verified by computer that there is no exceptional example for the cases where $n = 2m$ and $n = 2m+1$ with $m = 7$. We use these cases as starters of our induction.

1. Let X be a $(3m-2)$-distance set with $2m$ points containing both rational and irrational intervals, where $m \geq 8$. By Lemma 2.1, $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| \geq 3$. If $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| \geq 4$, then $X \setminus \{a_0, a_{n-1}\}$ contains only rational intervals by Lemma 2.3. In this case, we have $|D(X)| \geq 4m-4$ by Lemma 2.2, which is a contradiction. Hence, we can assume that $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| = 3$, i.e., $X \setminus \{a_0, a_{n-1}\}$ is a $(3m-5)$-distance set with $2m-2$ points. In particular, $X \setminus \{a_0, a_{n-1}\}$ contains both rational and irrational intervals since otherwise $k \geq 4m-4$ by Lemma 2.2 again. By induction, the form of $X \setminus \{a_0, a_{n-1}\}$ is equivalent to that of type A. Then, by Lemma 2.5, the form of X is also of type A.
2. Let X be a $(3m-1)$-distance set with $2m$ points containing both rational and irrational intervals, where $m \geq 8$. By Lemma 2.1, $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| \geq 3$. If $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| \geq 5$, then $X \setminus \{a_0, a_{n-1}\}$ contains only rational intervals by Lemma 2.3. In this case, we have $|D(X)| \geq 4m-4$ by Lemma 2.2, which is a contradiction. Hence, we can assume that $|D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| = 3$ or 4, and $X \setminus \{a_0, a_{n-1}\}$ contains both rational and irrational intervals. By induction, the form of $X \setminus \{a_0, a_{n-1}\}$ is equivalent to that of type A or B according to $D(X) \setminus D(X \setminus \{a_0, a_{n-1}\})| = 4$ or 3. Then, by Lemmas 2.5 and 2.8, the form of X is of type B.
3. Let X be a $3m$-distance set with $2m+1$ points containing both rational and irrational intervals, where $m \geq 7$. If $|D(X) \setminus D(X \setminus \{a_0\})| \geq 3$, then $X \setminus \{a_0\}$ contains only rational interval by Lemma 2.3. In this case, we have $|D(X)| \geq 4m-2$ by Lemma 2.2, which is a contradiction. Hence, we have $|D(X) \setminus D(X \setminus \{a_0\})| = 1$ or 2, i.e., $X \setminus \{a_0\}$ is a k'-distance set with $2m$ points, where $k' = 3m-2$ or $3m-1$. By the claims (1) and (2) of this theorem, the form of $X \setminus \{a_0\}$ is equivalent to that of type A or B according as $k' = 3m-2$ or $3m-1$. If $k' = 3m-2$, by Remark 2.6, the form of X is of type C. If $k' = 3m-1$, by Remark 2.9, the form of X is of type C.

We now obtain the assertion of this theorem. □

5. Distance sets with a small number of points on $\mathbb{R}$.

In this section, we treat distance sets with a small number of points on $\mathbb{R}$. Consider an n-point k-distance set X for $(n,k) = (2m, 3m-2), (2m, 3m-1), (2m+1, 3m)$, and assume that X contains both rational and irrational intervals.

It is direct to see that an n-point k-distance set X for $(n,k) = (3,3)$, $(4,4)$, $(4,5)$, $(5,6)$ is equivalent to either of the forms below:

$$
\begin{array}{ll}
(1,c), & \text{if } (n,k)=(3,3),\\
(1,c,1), & \text{if } (n,k)=(4,4),\\
(c,1,1+c),(1,1,c), & \text{if } (n,k)=(4,5),\\
(c,1,c,1),(1,1,c,1),(1,c,c,1),(1+c,1,c,1), & \text{if } (n,k)=(5,6).
\end{array}
$$

Furthermore, a 6-point 8-distance set X such that $X \setminus \{a_0, a_5\}$ contains only rational intervals is determined as $(c,1,1,1,c)$ by Lemma 2.2. All other n-point distance sets X for $n \geq 6$ should be obtained by adding two endpoints a_0 and a_{n-1} to $(n-2)$-point distance sets containing both rational and irrational intervals by Lemmas 2.1, 2.2 and 2.3. We assume that $X \setminus \{a_0, a_{n-1}\}$ has the base $\{1, c\}$ and $diam(X \setminus \{a_0, a_{n-1}\})$ has the form $s + tc$ for $s, t \in \mathbb{Z}$ with $s, t \geq 0$. Let $x := a_1 - a_0$ and $y := a_{n-1} - a_{n-2}$, and then these have the forms $x = i_1 + j_1 c$ and $y = i_2 + j_2 c$ for $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ with $-s \leq i_1, i_2 \leq s$ and $-t \leq j_1, j_2 \leq t$ since otherwise $k \geq 2n - 4$ similar to Lemma 2.2. This observation enables us to check by computer the form of X for small n inductively, where we use n-points k-distance sets with $(n,k) = (4,4), (4,5), (5,6), (6,8)$ above as starters. See Tables 5, 6, 7, 8, 9, and 10 for the outcome.

$m=3$		$m=4$		$m=5$		$m=6$		$m=7$
$(1,1,c,1,1)$	$\to$	$(1,\dots,1)$	$\to$	A–I				
	$\to$	$(c,\dots,c)$	$\to$	$(1,\dots,1)$	$\to$	$(1,\dots,1)$	$\to$	$\times$
					$\to$	$(1+c,\dots,1+c)$	$\to$	$\times$
	$\to$	$(1+c,\dots,1+c)$	$\to$	$(1,\dots,1)$	$\to$	$(2+c,\dots,2+c)$	$\to$	$\times$
	$\to$	$(2+c,\dots,2+c)$	$\to$	$\times$				
$(c,1,c,1,c)$	$\to$	$(1,\dots,1)$	$\to$	A–II, A–III, B–III				
	$\to$	$(1+c,\dots,1)$	$\to$	B–III				
	$\to$	$(1+c,\dots,1+c)$	$\to$	A–III				
	$\to$	$(c,\dots,c)$	$\to$	$\times$				
$(1+c,1,c,1,1+c)$	$\to$	$(1+c,\dots,1+c)$	$\to$	A–III				
	$\to$	$(1,\dots,1)$	$\to$	$(1+c,\dots,1+c)$	$\to$	$(2+c,\dots,2+c)$	$\to$	$\times$
	$\to$	$(2+c,\dots,2+c)$	$\to$	$\times$				
$(1+c,1,c,1,c)$	$\to$	$(1+c,\dots,1)$	$\to$	B–III				
$(-1+c,1,c,1,-1+c)$	$\to$	$(1,\dots,1)$	$\to$	$(c,\dots,c)$	$\to$	$(c,\dots,c)$	$\to$	$\times$
$(2+c,1,c,1,2+c)$	$\to$	$\times$						

Tabe 5. Distance sets obtained inductively from the starter $(1,c,1)$

$m=3$		$m=4$		$m=5$
$(1,1,1,c,1)$	$\to$	B–I		
$(c,1,1,c,1)$	$\to$	$(1,c,1,1,c,1,1)$	$\to$	$\times$
	$\to$	$(1,c,1,1,c,1,1+c)$	$\to$	$\times$
$(1+c,1,1,c,1)$	$\to$	$(1,1+c,1,1,c,1,1)$	$\to$	$\times$

Table 6. Distance sets obtained inductively from the starter $(1,1,c)$

$m=3$		$m=4$		$m=5$
$(1,c,1,1+c,1+c)$	$\to$	B–III		
$(1,c,1,1+c,1)$	$\to$	$(1+c,1,c,1,1+c,1,1+c)$	$\to$	$\times$

Table 7. Distance sets obtained inductively from the starter $(c,1,1+c)$

6. Distance sets with a small number of points on S^1.

In this section, we consider n-point k-distance sets X on S^1 satisfying $k < M_n$ for

$$n \in \{5, 7, 9, 11, 13, 15, 17, 19, 23\}.$$

$m=2$		$m=3$
$(1,c,1,c)$	$\to$	C–II, C-III
$(1,1,c,1)$	$\to$	C–I
$(1+c,1,c,1)$	$\to$	C–III

Table 8. Distance sets obtained inductively from the starters $(1,c,1,c)$, $(1,1,c,1)$ and $(1+c,1,c,1)$

$m=2$		$m=3$		$m=4$		$m=5$
$(c,1,1,c)$	$\to$	$(1,c,1,1,c,1)$	$\to$	$(1,1,c,1,1,c,1,1)$	$\to$	$\times$
			$\to$	$(1+c,1,c,1,1,c,1,1+c)$	$\to$	$\times$

Table 9. Distance sets obtained inductively from the starter $(c,1,1,c)$

$m=3$		$m=4$		$m=5$		$m=6$		$m=7$
$(c,1,1,1,c)$	$\to$	$(1,c,1,1,1,c,1)$	$\to$	$(1,1,c,1,1,1,c,1,1)$	$\to$	$(1,1,1,c,1,1,1,c,1,1,1)$	$\to$	$\times$

Table 10. Distance sets obtained inductively from the starter $(c,1,1,1,c)$

In [2], all three-distance sets on $\mathbb{R}^2$ were classified. In particular, we know that any 5-point k-distance set with $k \le 3$ lies on R_5 or R_6. Hence, we now assume that $n \neq 5$.

By Lemma 3.1, we can assume that $|R_{a_0}| = |L_{a_0}| = (n-1)/2$. Furthermore, by Lemma 3.6, we can assume that $R_{a_0} \cup \{a_0\}$ contains both rational and irrational intervals. Then, by Theorem 2.4, $R_{a_0} \cup \{a_0\}$ is of type A or B in Table 1 or either of types in Table 2.

We first assume that $R_{a_0} \cup \{a_0\}$ is not of type A, B–I, nor C–I, and eliminate these cases by computer as follows:

(1) Choose a form from Tables 1 and 2 except for types A, B–I, and C–I as that of $R_{a_0} \cup \{a_0\}$.
(2) Choose a_u, $1 \le u \le (n-5)/2$, so that each of $R_{a_u} \cup \{a_u\}$ and $L_{a_u} \cup \{a_u\}$ contains both rational and irrational intervals. Then, by Lemma 2.3, we have $|R_{a_u} \cup \{a_u\}| = |L_{a_u} \cup \{a_u\}| = (n+1)/2$.
(3) Let $x_i := al(a_i, a_{i-1})$ for $i = 1 \ldots, n$. Choose $x_{(n-1)/2+i}$, $1 \le i \le u$, and $x_{u+(n+3)/2+i}$, $0 \le i \le -u + (n-3)/2$, from $D(R_{a_0} \cup \{a_0\})$ so that
$$D(R_{a_u} \cup \{a_u\}), D(L_{a_u} \cup \{a_u\}) \subseteq D(R_{a_0} \cup \{a_0\}). \tag{6.1}$$
(4) Choose $x_{u+(n+1)/2}$ from $D(R_{a_0} \cup \{a_0\})$ so that
$$D(L_{a_0} \cup \{a_0\}) \subseteq D(R_{a_0} \cup \{a_0\}).$$
(5) Change a_u, $0 \le u \le n-1$, and check again whether Eq. (6.1) holds.

Then, we have no outcome for x_i's, and hence these cases are eliminated.

We now treat the case where $R_{a_0} \cup \{a_0\}$ is either of type A, B–I, or C–I. By the argument in the previous paragraph, we can assume that each of $R_{a_u} \cup \{a_u\}$ and $L_{a_u} \cup \{a_u\}$ has the form equivalent to that of type A, B–I, or C–I if they contain both rational and irrational intervals. Under this assumption, this case can be eliminated by a proof quite similar to that of Lemma 3.7 or 3.8.

References

[1] K. Momihara, M. Shinohara, Distance sets on circles, to appear in the American Mathematical Monthly.
[2] M. Shinohara, Classification of three-distance sets in two dimensional Euclidean space, *European J. Combin.*, **25** (2004), 1039–1058.