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The Moore Method

A Pathway to Learner-Centered Instruction

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W. Ted Mahavier
E. Lee May
G. Edgar Parker

Mathematical Association of America

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[1] This chapter contributed by Jennifer Christian Smith, Sera Yoo, and Stephanie Ryan Nichols.

In Memory of R. L. Moore

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[1] www.educationaladvancementfoundation.org

[2] www.discovery.utexas.edu/rlm

"…the secret of education lies in respecting the pupil..."[1]
— Ralph Waldo Emerson

1
Introduction[2]

Quietly but steadily, the ground is being prepared for an eventual shift in American colleges away from a teacher-oriented system featuring lectures delivered to passive audiences to a more learner-centered process in which students become more actively involved in their own education and professors adapt their teaching in accordance with more complex understandings of human learning. (Bok, 2006)

A child learns, at any moment, not by using the procedure that seems best to us, but the one that seems best to him; by fitting into his structure of ideas and relationships, his mental model of reality, not the piece we think comes next, but the one he thinks comes next. This is hard for teachers to learn, and hardest of all for the skillful and articulate, the kind often called 'gifted.' The more aware we are of the structural nature of our own ideas, the more we are tempted to try to transplant this structure whole into the minds of children. But it cannot be done. They must do this structuring and building for themselves. (Holt, 1965)

(Mahavier[3]) What is the Moore Method?

Moore-method teaching has been associated with pedagogies such as discovery-based, inquiry-based, student-centered, Socratic, and constructivist, yet is not fully encompassed by any of these. The bulk of a Moore-method course will consist of student presentations of solutions to problems provided by the instructor that they produce individually without external aids. Such a course meets the students where they are, guides them at a fair and challenging pace through the material, and requires them to construct and present the key mathematical ideas before their peers, a discerning but supportive audience. The primary goal of such a course is to address the content in such a way as to develop in the students the ability to investigate problems independently. The instructor serves as a coach, mentor, collaborator, and guide, as well as a source of positive reinforcement. R. L. Moore, for whom the method is named, developed the method at the University of Pennsylvania and used the method at the University of Texas from 1920 until his retirement in 1969.

[1] Edward Emerson, *The Complete Writings of Ralph Waldo Emerson*.

[2] This chapter has only one author, Mahavier. All writing in this book is preceded parenthetically by the contributing author's name.

[3] Please direct all correspondence to W. Ted Mahavier at wtm@mathnerds.com.

Who was R. L. Moore?

Moore received both his bachelor's and master's degrees in mathematics from the University of Texas in 1901 where he studied under G. B. Halsted. In 1903, the University of Chicago became his academic home and there Moore studied under E. H. Moore (no relation). Upon receiving his doctorate in 1905, Moore held positions at the University of Tennessee (1905–1906), Princeton University (1906–1908), Northwestern University (1908–1911), and the University of Pennsylvania (1911–1920) before returning to the University of Texas in 1920. By the date of his retirement in 1969, he had produced fifty doctoral students, sixty-seven publications, and the teaching legacy that precipitated this book, commonly referred to as the Moore Method or the Texas Method. Today, the Mathematics Genealogy Project[4] reports more than 2,200 descendants.

What's in this book?

This book provides practical guidance for those considering teaching a course using the method, as well as insight for those seeking a deeper understanding of the intricacies of the method. While our primary aim is to address the mechanics, we open with three chapters intended to set the stage. The second chapter, "Moore's Moore Method," describes the method as practiced by Moore himself to provide a context in which the remainder of the book might be interpreted. Because no deep pedagogical method can be described by a list of techniques, rules, or features, we follow this by two chapters as philosophical as they are practical: "What is the Moore Method?" and "On Culture." These define the authors' broader interpretation of the method and the culture that we attempt to establish in our courses to maximize opportunities for the mathematical development of our students. Like the second chapter, they provide context in which the practical guidance that follows may be accurately interpreted. The three chapters delineating the mechanics of the method, "Development and Selection of Materials," "In the Classroom," and "Grading," answer the questions we have been asked throughout our careers: How do we develop or select materials in preparation for teaching a Moore Method course? What actually occurs day-to-day in the classroom? How do we grade our students fairly in such a course? In response to questions that may be answered more succinctly, we include the chapter "Frequently Asked Questions." We supplement this practical guidance with "Why Use the Moore Method?" to share our own motivation for using the method and to offer evidence from the literature supporting a move toward learner-centered approaches to teaching. "Evaluation and Assessment: Effectiveness of the Method," contributed by Jennifer Christian Smith's research group at the University of Texas, references the latest mathematics education research on the method. Extensive appendices include sample resources such as class diaries, course notes, handouts, syllabi, and tests.

What isn't in this book?

It does not contain biographical or historical information. Several articles and books (Devlin, 1999), (Parker, J., 2005), (Traylor, 1972), (Zitarelli, 2004) have addressed these

[4] http://genealogy.math.ndsu.nodak.edu/index.php

topics. Aside from this treatise, there is no book on the implementation of the method, although numerous papers on this aspect (Chalice, 1995), (Cohen, 1982), (Halmos, 1974), (Halmos, 1994), (Jones, 1977), (Mahavier W.T., 1997), (Mahavier W.S., 1999), (Parker, G., 1992), (Renz, 1999), (Whyburn, 1970) do provide the reader with a glimpse into some of the topics that we address. The authoritative on-line collections of materials addressing the method are the *Legacy of R. L. Moore* website and *The Journal of Inquiry-Based Learning in Mathematics*.

How might I read this book?

This book is intended as a resource, and, as such, need not be read linearly. For those seeking a quick, let-me-see-if-I'm-interested approach, we recommend either reading a few of the paragraphs introducing the chapters or skimming the chapter titled "Frequently Asked Questions." For those readers connecting with a particular author's style, reading only that author's contributions might give the reader a relatively thorough insight into the method. Still, each author offers a unique perspective and each chapter is ordered assuming that within this chapter, the authors' contributions will be read in order. Often one author will provide succinct practical guidance and another will provide a deeper philosophical perspective. We have carefully considered the ordering of the presentations in an attempt to maximize the benefit to the reader and have introduced each chapter with a brief paragraph or two delineating the commonality of the four authors' perspectives. To aid the reader who connects with a particular author's writing, we have preceded each author's writing parenthetically by his name. Finally, for those enjoying the occasional anecdote or student comment, we use bordered text to highlight these supporting and entertaining tidbits.

Why did we write it?

Each of the coauthors has mentored many faculty on the method and each has spoken on the method at conferences. With more than twenty-three presentations to date, I have not only lectured on, but also demonstrated the method on stage. Each time I felt confident that my audience had fully absorbed and understood my presentations. Yet after such talks, I was inevitably met by attendees who would surprise me by querying, "I really like what you talked about and I really want to try it, but *how* do I do it? *What do you do?*" After each such talk I ended up mentoring at least one individual.[5] Eventually what should have been obvious dawned on me; perhaps lecturing was not the best approach to make my point. Since my coauthors and I can't mentor all of those interested in using the method, the next best approach seemed to be a book that would at least help potential users determine if this is a method of instruction appropriate to their own tendencies and, if so, provide them with practical guidance to facilitate a successful implementation. This book is the result.

The decision to coordinate the writing of this book was in direct response to demand from would-be practitioners of the method lacking resources. Stanley Yoshinobu of California State University Dominguez Hills hosted and filled to capacity workshops on the method during the summers of 2006, 2007 and 2009 where participants requested such a resource. Since 1998, the annual Legacy of R. L. Moore Conference has attracted faculty

[5] The Legacy of R. L. Moore website now includes a Mentoring Project and a Visiting Speaker's Bureau.

who use, support, or wish to learn the method. A significant number of attendees have been NExT Fellows seeking resources, mentors, and information concerning the method. Harvard University, the University of Chicago, the University of Michigan, the University of Texas, and the University of California Santa Barbara have established Moore Centers implementing the Moore Method. Annually, faculty and staff manning the Legacy of R. L. Moore booth at the Joint Mathematics Meetings field questions on the method and pass out hundreds of pages of materials. Each coauthor has mentored mathematicians teaching their first Moore Method course. All of our mentoring relationships share common threads not addressed in the current literature. Collectively, these experiences demonstrate the need for multiple perspectives on how to implement the method. This is why we chose to write as we did.

How did we write it?

We wrote independently and each author's individual writing is preceded parenthetically by his name. This introduction was penned by Mahavier. Chapter 2, Moore's Moore Method, was written by Coppin. Chapters 3 through 8 and 10 have contributions from each author. Chapter 9 was written collaboratively by Jennifer Christian Smith, Sera Yoo, and Stephanie Ryan Nichols.

The most challenging hurdle in preparing this particular book was deciding how to intertwine the writings of four coauthors in a way that demonstrates four unique approaches with significant overlap. We ultimately decided to include the union of our collective writings, as we felt that the intersection would inevitably be too small to be of value. We also felt that with multiple authors there was a greater chance of the reader connecting with one author's writings. Each chapter contains our individual essays authored with limited collaboration. Once the contributions to a chapter were collected, we shared these materials for the purpose of editing, ordering, soliciting feedback, and developing an introduction addressing the commonality of the essays. Because we wrote independently, two authors would, at times, quote the same work. To eliminate repetition, a quote appearing in one essay will be referenced in another. Striking a balance between retaining the theme that four authors using very different implementations of the method share a broad common element and omitting redundancy was a nontrivial challenge. We hope that the commonality is reinforcing. After the first draft was completed, we were familiar with each other's writing and therefore, to further eliminate repetition, we occasionally refer to writing by another author from another chapter. It was an unconventional way to write a book, but certainly compatible with the philosophy of Moore Method proponents. Each coauthor was able to first create his own articulation of an issue without being influenced by a colleague's thinking and then to display his creation before his peers for constructive criticism.

Who are we?

With a combined Moore Method experience of more than 130 years of teaching, the four coauthors have taught courses in mathematics ranging in level from elementary school through graduate school. We have taught at summer camps, community colleges, small liberal arts colleges, mid-sized regional universities, and large state universities. All have published and remain active in both mathematics and mathematics education. We have developed and implemented a combined total of more than forty-four Moore Method courses. We studied under Moore, Wall, and fourteen of their mathematical descendants. At least sixty-five of our students have gone on to earn graduate degrees beyond the degrees offered at the institutions where we instructed them. Full biographical sketches follow the appendices.

Is what we do *really* Moore Method?

Each coauthor has strong opinions on how to implement the method and even of what constitutes the method. The astute reader will note elements on which the coauthors disagree. One point on which we all agree is that each instructor must find what works for him or her. Another is that the only *defensible* definition of the Moore Method is "the method that Moore used" and our teachings are more aptly labeled as the Coppin, Mahavier, May, or Parker Method, respectively. However, for the sake of brevity and to avoid addressing the definitions of and distinctions among "student-centered," "inquiry-based," "discovery-based," "problem-based," "Socratic," "Moore Method," and "modified Moore Method," we will refer to all of our teaching practices as "Moore Method" except in rare instances where we are attempting to describe some specific teaching method that we consider to be sufficiently removed from the basic tenets of the Moore Method to be labeled otherwise. There are instances in the book where what we describe does not exactly parallel the practice of Moore. Moore did not attempt to define his method for his students and his only formal statements on his method appear in the film *Challenge in the Classroom*.[6] Upon the offer of an honorary doctorate in mathematics education from Carnegie Mellon University, Moore is said to have replied, "I never taught students a method for teaching. I taught mathematics. If they *chose* to teach in a certain way, it was of their own accord." Certainly we consider ourselves Moore Method practitioners and adhere to the basic tenets of a method that produces in our students the same life-changing attitudes as it produced in each of us.

[6] Originally filmed in 1966, *Challenge in the Classroom* is a documentary on the life and work of R .L. Moore. It was produced by the Mathematical Association of America and is currently available as *The Moore Method: A Documentary on R.L. Moore* from the association.

> "The greatest sign of success for a teacher… is to be able to say, 'The children are now working as if I did not exist.'"[1]
> — Maria Montessori

2
Moore's Moore Method[2]

> "Education is what remains after one has forgotten everything one learned in school."[3]
> — Albert Einstein

(Coppin) Herein is an attempt to give the reader a snapshot of what an R. L. Moore class was like by examining a particular course of his I took in 1964 and 1965. Contemporaneous classmates will have an account that may vary somewhat from my own but I believe those differences are not of a different species.

Some background may be in order so that the reader may have a backdrop against which to judge a class like this. I went to graduate school during the social revolution of the 1960s that included civil rights marches, demonstrations in the streets and the assassinations of John F. Kennedy, Martin Luther King and Robert F. Kennedy. My contemporaries were products of the decades containing World War II, the Korean Conflict, the Vietnam War and, of course, the Cold War. This was a time colored by a potential worldwide nuclear conflagration. We were Sputnik's children. We thought our country was behind in the space race. Many of my generation decided to go into mathematics and science because we wanted to contribute to our nation's defense and its existence. This was the milieu in which my classmates and I found ourselves doing mathematics.

At the University of Texas, the mathematics department had split into two factions, the third floor and the second floor. The third floor consisted of Moore, H. S. Wall, R. G. Lubben, H. S. Ettlinger and Ralph Lane, who had died just before I arrived at Austin. In my opinion, the primary difference between the two factions was that the third floor was of the old school. On the third floor, one had to satisfy the master and not traverse prescribed coursework, qualifying exams and dissertation committees such as is the practice today and was the practice on the second floor. The atmosphere on the third floor may have been much like that of the University of Göttingen in its heyday before World War II. Göttingen

[1] http://en.wikiquote.org/wiki/Maria_Montessori

[2] This chapter has only one author, as Coppin is the only coauthor who took a course under Moore. All writing in this book is preceded parenthetically by the contributing author's name.

[3] http://en.wikiquote.org/wiki/Education

> The Big Three of the third floor at the University of Texas, H. J. Ettlinger, R.L. Moore, and H. S. Wall, have an interesting and significant mathematical genealogy worthy of note. During the time period 1876–1900, the United States was transformed from a mathematical backwater to a major presence in the international community of mathematicians. According to Parshall and Rowe (Parshall, 1994), it was three giants of mathematics, J. J. Sylvester, Felix Klein and E. H. Moore, who were major figures in this transformation and who were, not so incidentally, mathematical ancestors to Ettlinger, Moore and Wall. Moore's teacher at Texas, G. B. Halsted, was a student of Sylvester and his dissertation supervisors at the University of Chicago were E. H. Moore and Oswald Veblen, also a doctoral student of E. H. Moore. Wall was a mathematical grandson of Klein by way of Edward Van Vleck, a student of Klein's and the doctoral supervisor of Wall. Ettlinger was a student of G. D. Birkhoff, a student of E. H. Moore, making him a mathematical grandson of E. H. Moore.
>
> Coppin

was the academic home of the likes of Karl Gauss, Johannes Dirichlet, David Hilbert, and Richard Courant. Some even considered Göttingen to be the center of the mathematical universe before World War II.

Moore was a formal man, very much adhering to the mores of the Old South. He always wore a black suit, white shirt, cuff links and fedora hat as he walked to and from work. I remember observing him from the University of Texas' tower as he was walking on the mall. As he approached two women, he tipped his hat as any gentleman would. He called his students by Mr., Miss or Mrs. followed by their surname. We certainly did not call Moore, Robert or Bobby. Heaven forbid! In his eighties, he would stand on the curb at Guadalupe Street, adjacent to the mall, perched and ready to fire out of the blocks so as to get across the drag as fast as possible. He was an intense, competitive man as well as a formal man.

At the beginning of my first year of graduate school, H. S. Wall, who would later be my supervising professor, had wanted me to enroll in Moore's Math 688/689 courses entitled *The Foundations of Mathematics*. Moore thought I had an unacceptable background and turned me down. Moore was selective as to whom he would allow to enroll in his courses. If you knew too much, too little or had learned bad mathematical habits, you might not be allowed into Math 688. However, the following fall, in 1964, for a second time, I asked to join his class. This time, he allowed me to enroll in Math 688, his year-long course. I had just recently learned how to prove theorems.

688.	Foundations of Mathematics. — A considerable degree of mathematical maturity required. Prerequisite: Consent of Instructor. Three lectures a week for two semesters. Mr. Moore. Catalogue Number: Part VII 1964–1966.

In the fall of 1964, I found Math 688 to be small (fewer than fifteen students) and somewhat stressful, at least for me. I was very shy and, in light of the fact that our grades relied on getting up in front of our peers to make presentations, I could have had a problem. However, I knew that going in and I had a great deal of drive. We were to prove theorems, solve problems or answer questions completely on our own without recourse to a book or any outside aid. Each student was on his or her own. To receive credit for a result, a student had to make a presentation at the board and, under the guidance of the teacher, allow fellow classmates to ask critical questions. If a student had already seen the proof, the solution, or the answer, he or she was disqualified for credit for that result. Although the process seems harsh by today's standards, I did not think twice about it. As a result, I matured mathematically as the semester progressed. I think others in the class did the same although I think we all matured at different rates and to different levels. Moore's 688 class was a success for me: I would say it was the best-taught class I ever took, but, of course, it meshed with my learning style. Looking back, the idea behind Moore's method is that he had arranged a collection of theorems and questions that allowed the students to work through the material completely on their own. The content of Moore's graduate courses and upper level courses was Point Set Theory[4]. Relative difficulty ranged from easy to extremely difficult during the long term and even during a week of class. A basic requirement was that each student did the work by himself or herself. As students, we knew that the teacher would not bail us out. Below are signal characteristics of Moore's course:

1. A student's grade was derived primarily (or, solely) from his or her presentations.
2. Each student's presentation was his or her own work. Students were not to get aid from anyone. They were not to go to any book for help.
3. A student could not receive credit for any result he or she had seen.
4. When someone was at the board, no one in the class was allowed to give aid or hints. A class member could ask questions but not in a way as to offer direct help to the one at the board.
5. A student was allowed to step out of class under certain conditions if he or she did not want to see someone else's proof.

On the first day of class or very soon thereafter, Moore stated the parameters concerning the format of the class. He emphasized that we were to work on our own. We could not look at books or get help from anyone. He would also dictate to us the initial axioms, pertinent definitions and the first two or three theorems to work on. The reader should note the word "dictate." Since no notes were handed out in advance, the material was given to us in small chunks, which required students to translate the material delivered orally into the written word. I actually kept two notebooks. One was for in-class, quick, cursory notes and the second one contained my detailed and thorough rendition of what actually took place that day. Here, for later reference, I include what he dictated to us the first two or three days of the course. I have taken these from my own notes; however, I have taken the liberty of correcting minor note-taking errors. Please note his use of language as compared with the current use of linguistic shorthand. I have found it profitable to study his writing of mathematics. The reader may also profit from such a study and may wish to refer back to the definitions of words as these words appear later in the essay.

> Assume the notion of point and of region to be undefined.
>
> Definitions. S is "the set of all points," the point set containing every point. Point p is a limit point of a point set M if and only if every region that contains p contains a point of M distinct from p. $\overline{M}$ is a point set such that x belongs to it if and only if x is a point of M or a limit point of M. The point set M is said to be closed if there is no limit point that does not belong to M. Point p is said to be a boundary point of the point set M if and only if every region contain-

[4] To my memory, Moore did not like to use the word "topology"; thus, out of respect for him, I do not write Point Set Topology, as is the common practice today.

ing p contains a point in M and a point not belonging to M. The collection G, each element of which is a point set, is said to cover M if and only if every element of M belongs to an element of G.

Axiom 0. Every region is a point set.

Axiom 1. There exists a sequence G_1, G_2, …. such that (1) for each n, G_n covers S (2) for each n, G_{n+1} is a subcollection of G_n (3) If R is a region and a is a point of R and b is a point of R, then there exists a positive integer m such that if g is a region of the collection G_m containing a, then $\overline{g}$ is a subset of R and if b is distinct from a, then $\overline{g}$ does not contain b.

Theorem 1. No point of a region is a boundary point of that region.

Theorem 2. If M is a point set, $\overline{M}$ is closed.

Definitions. The collection G of sets is said to be a monotonic collection if and only if for every two sets of the collection G, one is a subset of the other one. The point set M is compact if and only if there is no infinite subset of M that has no limit point. The point set M is perfectly compact if and only if it is true that if G is a monotonic collection of subsets of M then either the sets of the collection G have a common point or they have a common limit point. The set M is said to be ordered with respect to a certain meaning of precede if and only if for that word precede it is true that if x is an element of M and y is an element of M then

1. if x precedes y, y does not precede x.
2. if x is distinct from y either x precedes y or y precedes x.
3. if x, y, and z are elements of M and x precedes y and y precedes z, then x precedes z.

The set M is said to be well ordered with respect to the meaning Q of the word "precede" means that for the meaning of that word precedes M is ordered and for every nondegenerate subset K of M, some element of K precedes every other element of K.

Theorem 3. If M is a closed and perfectly compact point set and Q is a meaning of the word precedes and (1) β is a collection of regions which is well ordered with respect to the meaning Q of precede and (2) M is covered by the collection β, then there exists a region g of β such that M is covered by the collection whose elements are g and the regions of β that precede g.

Theorem 4. If M is a closed and bounded perfectly compact point set and G is a collection of regions covering M and there exists a meaning for the word "precedes" with respect to which G is well ordered, then some finite subcollection of G covers M.

Throughout the course, Moore would ask for examples in the same manner as he would assign theorems; however, in my class, he rarely asked for them to be presented; but, much to my surprise, we saw some of them on the final, the only examination in the course. One sidebar concerning his finals was that they were the most challenging examinations and, at the same time, the most satisfying ones I ever sat for.

During each class period, Moore would ask for volunteers, although it has been said by others that he would call on students in reverse order of their successes: weaker students would be called on first. Supposedly, this would give less successful students ample opportunity to be more successful. Moreover, in this way, one would know his or her standing in class. However, in my classes, Moore seemed only to ask for volunteers but he could have used the ranking technique in some way unknown to me.

When a student was asked to go to the board, generally, he or she would present a proof, and sometimes, an example or a counter example. The presentation was a combination of written work and diagrams with oral work dominating the presentation. Presenters could work from notes while at the board. However, I seldom saw a proof written in its entirety on the board. The class would ask questions or point out potential errors. The presenter was required to respond to the criticism; however, the class was never to offer any help, not even to shorten a long proof.

If the presenter ran into trouble, he or she was given the opportunity to patch the hole in the proof. The class and Moore would patiently wait on the presenter. Of course, the class might ask critical questions. If unsuccessful, the presenter would be seated, possibly with the theorem "reserved" for continuation during the next class period. Moore would then move to the next theorem, problem, or question. If the presenter was unsuccessful and Moore did not give him or her the opportunity to work on the theorem next time, he would then call on someone else to go to the board.

> I remember, in the beginning of Math 688, watching student after student get "shot down" on Theorem 3. As for myself, watching the attempts of others, time and time again, allowed me to calibrate the difficulty of the theorem. I then knew it was a theorem worthy of attack. As a result, I was inspired to give it a try. In that class, it was my first time at the board. I remember starting my proof at the end of a period and, as a result, not having time to finish it. The proof hinged on the concept of well ordering. (See the material from Math 688 previously stated for a definition of well ordering and Theorem 3.) Before the beginning of the next period, I had arrived early. Moore entered the classroom and asked me to come next door to a conference room. We sat opposite one-another at the short sides of a long table. No one else was in the room; it was a lonely place. I was nervous and thought I was in serious trouble, although I had no idea for what. He asked me, "Mr. Coppin, have you ever heard of well ordering before?" I responded, "No sir, I haven't and wished I never had." He smiled—a rare event—and dismissed me. I got up to present Theorem 3 and successfully finished my proof.
>
> Coppin

One characteristic of Moore's class is that we would never attempt to settle a result on the day it was posed. In this way, all of us had equal opportunity to tackle the result. That worked to my favor, as I never really was very good at responding "on the spot" but I was pretty good if given time. Sometimes, a theorem or question was left over indefinitely. I can remember proving the theorem that every set is well ordered some two years after it was posed.

To emphasize a point, each student's presentation was based on his or her own work. Moore believed that mathematical creation is a solitary activity. As a result, students were to receive no help from anyone. They were not even to go to any book for help; in fact, Moore would tell his classes that they could read only their notes and what was written on the board. In the film *Challenge in the Classroom,* Moore is asked if he allows students to work together. He, of course, states his distaste for such a thing. This was a most important feature of his method.

One point that seems to be rarely mentioned in any discussion of his method is Moore's use of language and naturalistic understanding of mathematical entities. I believe that his philosophy of mathematics was part and parcel of his method. Examine the material given earlier. Note his word choices, his exact use of grammar and his limpid use of language. In regard to his mathematics, he relied on the efficacy of the English language to communicate clarity and power. As a result, Moore used very few symbols. For example, and there are many, contrast his definition of closed set, "M is closed if there is no limit point that does not belong to M," with its modern counterpart, "a set M is closed iff its complement

$M^c = X \setminus M$ is open." Notice how his definition is characterized by its simplicity and its primitive nature.

I never knew how Moore evaluated us. In each course, he had only one examination, the final examination. The 688 final examination was a doozy, but I found it doable and fun. I suspect most of our grade came from presenting proofs at the board. Most likely, he was ranking and evaluating each one of us throughout each course. We had no syllabus. Of course, universities at that time did not require of their faculty the structure that they do today. Thus, I suspect that Moore's evaluation was highly subjective and most likely was based on the dynamically changing day-to-day evaluation of each student.

In the film *Challenge in the Classroom,* as a voice over, Moore is heard saying that if someone would offer to take him by the hand and lead him through a forest to show him the birds and the animals, he would decline. He would rather take the time and find his own way through the forest. This was the spirit he inculcated in his students.

"There is one effective way of learning mathematics and that is by DOING it. This will always be my point of view."[1]

— H.J. Ettlinger, circa 1979
Chairman, Department of Pure Mathematics, University of Texas

3 What is the Moore Method?

The Moore Method, as practiced by Moore himself, relied on the rigid policies delineated in Chapter 2 on page 9. The consequences of these policies, along with Moore's own unique personality, created the learning environment he desired. Since the 1950s, many individuals assumed that these rules, and these rules alone, defined the method and applied the rules with widely varying results. Moore's rules alone are probably not adequate to define the method *and* assure a successful implementation. Therefore, in describing the Moore Method as second- and third-generation practitioners, we articulate our commonalities by describing the method in terms of the learning environment and student outcomes we hope to create rather than by the rules we establish for our students. Six principles common to our four versions of Moore Method are:

1. the goal of elevating students from recipients to creators of knowledge,
2. the commitment to teaching by letting students discover the power of their minds,
3. the attitude that every student can and will do mathematics,
4. the time for students to discover, present, and debate mathematics,
5. the careful matching of problems and material to students, and
6. the material, varying widely in difficulty, to cover a significant body of knowledge.

The essays that follow elucidate different modes of execution that can occur within this common philosophy.

(Parker) When writing "Getting More from Moore" (Parker, G., 1991) I considered the question that is the title to this chapter to be so important that I devoted virtually the entire introduction of the paper to it. At that time, I took pains to point out that what Moore did in the classroom likely differed significantly from what I did, but that nevertheless the philosophy from which he apparently worked remained viable and guided my version of the Moore Method. Looking back at that exposition after seventeen years, during which I taught enough courses to have more than doubled my university-level teaching experience,

[1] From *Notes on 66 years on Forty Acres,* H. J. Ettlinger, R. L. Moore Collection of the Center for American History at the University of Texas.

I am pleased that, in the context of that paper, I would write the same thing again. Thus I will begin where that exposition ended: "The Moore Method … is not necessarily Moore's method; rather it is a *commitment to teaching by letting students discover the power their own minds have*."

What, then, characterizes the pedagogical strategies that enable us to pursue the commitment? The introduction to this chapter reminds us of Moore's classroom rules and a consensus description of the learning environment and student outcomes favored by the authors. They represent two poles. Moore's rules state how his Moore Method course would be conducted, but contain no description of how Moore would play his role. The authors' list, on the other hand, describes a perceived fertile context for accomplishing the learning goals they have identified, but offers no prescription for how the courses will be conducted. This essay is designed to clarify what strategies I identify as being Moore Method strategies.

At some level, all instructional strategies are student-centered. Even in a course that is given by a lecture-delivery system, the students have to prepare for and take the tests. But in the lecture model, the students' roles are reactive; that is, they are asked to internalize patterns of thought explained to them by an expert and then to make those thought patterns a part of their own particular intellectual repertoires. In the Moore Method, students are put in a proactive, rather than reactive, learning mode. Problems are presented to the students with the expectation that the class, through the *individual* efforts of its students, will solve sufficiently many of them by the end of the semester to constitute adequate coverage of the curriculum for the course. Class time is spent with the students *collectively* analyzing the work of whoever is presenting.

The professor, acting as a member of the class, is privy to the student presentations, but assumes the role, along with the rest of the class, of understanding whether or not the presentation constitutes a valid argument. The professor, as the teacher, listens in order to gather information concerning how the presenter is thinking about the problem and, from the questions raised by the presenter's classmates, what other members of the class may be thinking.

Generally, a Moore Method course takes the following form:

1. A curriculum for the course is preplanned; that is, the professor prepares a problem set that, if the right problems in it are solved, meets the curricular goals of the course.
2. From day to day, students address the problems that have been given to them and, individually, convince the rest of the class of the correctness of an argument or are convinced by the class that the argument presented does not settle the problem.
3. Each day, the professor listens to what the students say in their pursuit of arguments.
4. Consequent to what is proven, additional preplanned problems are distributed; or consequent to what the professor observes, problems designed to take advantage of ideas that surface during the student presentations are improvised; or general encouragement for the efforts expended is given. Any or all of these three may occur.

The dynamics of the Moore Method classroom are built from repetition of this cycle: Problems generate ideas. Ideas are put before the class for validation. Members of the class validate or reject the arguments for the ideas. The professor assesses the ideas presented and decides what problems come next.

The professor/student roles within this dynamic are effectively reversed from those of the lecture model. Students are actively involved in trying to show why their arguments for the problems are correct, a proactive mode, while the professor is trying to make sense of what he or she has heard said, a reactive mode. This is not to say that the Moore Method professor does not put his or her stamp on the course; the choice of what problems are addressed will certainly reflect the prejudices of the professor. However, the ideas through which the solutions are pursued are consequent to the students' thinking.

The preceding idea is perhaps sufficient to distinguish the Moore Method from most brands of lecture-based pedagogy. The insistence on individual work outside of class distinguishes it from the usual cooperative-learning pedagogies. Separating the Moore Method from other active-learning pedagogies, particularly those that are problem based, may require a sharper razor. Indeed, I would not characterize the teaching I do in many of my introductory classes as being Moore Method courses although they are clearly not lecture-based. They are, however, problem-based and committed to active learning on the part of the students. They are not examples of Moore Method classes, however, because of the level of interaction I am willing to take as the "class leader" and the nature of the problems I am willing to state. In a Moore Method class, when a problem does not get solved, I am only willing to highlight ideas that have been presented (this may be in discussion or through an additional problem germane to the problem at hand that I know such an idea is adequate to solve) or to reiterate the issue on which a failed argument crashed. In a guided-discovery setting, I am willing to ask questions and/or state problems that have solutions that are "clear" consequences of concepts that have been identified as being true or to suggest connections among concepts that have already been established.

To illustrate the distinction, consider the following "identical" problems, one from first-semester calculus and the other from the beginning of Analysis.

> Definition: If x is a number, then $Q(x) = x^2$.
>
> For calculus: *S*how that if p is a number, then the limit of Q near p is p^2.
>
> For analysis: Show that Q is continuous on $\Re$.

In the calculus context, this is one of the problems on the basic functions from which the algebra of functions builds the classical functions, and it is there to begin to teach the students how the logic of quantification consolidates things learned from doing specific examples.

Thus, before being asked to do this problem, the students will have been asked to solve problems such as the ones below.

1. Suppose the error estimate to be guaranteed on the output of Q is .1. Define an open interval containing 3 so that if x is an element of it, then the distance from x^2 to 3^2 is less than .1.
2. Suppose the error estimate to be guaranteed is the positive number E. Define an open interval containing 3 so that if x is an element of it, then the distance from $Q(x)$ to 3^2 is less than E.

3. Suppose that p is a positive number and that the error estimate to be guaranteed is .1. Define an open interval containing p so that if x is an element of it, then the distance from $Q(x)$ to p^2 is less than .1.

The students would also be working concurrently on similar questions regarding root functions and the reciprocal function. Watching the students work on these problems will indicate to me their level of algebraic proficiency relative to using inequalities and indicate which parts of high school algebra are in need of re-emphasis. The problems themselves give me a chance to emphasize to the students that the mathematics that we make, as articulated in the definitions or proven in the theorems for limits, is intended to have utility for applications that involve approximation.

In the analysis context, this problem serves a very different purpose. Here it is present so that the students can verify that a function that they said was continuous in calculus is continuous because it meets the standards set forth in the definition of "continuous." If they nail a proof right off, it confirms their intuition from calculus. If a proof comes hard, then their attempts to frame their intuition within the rigor of the course raises discussion points that, with appropriate follow-up questions, can validate both the ideas they bring in with them and the choice of definition.

What separates my teaching in these two contexts is, for the calculus problem, if they cannot get it, I will lead the class in a discussion that "points the way." For instance, if no one has been able to do Problem 1 above, I might have the class work collectively on "for the estimate .25, find an open interval containing 1 that guarantees the estimate" and ask leading questions to try to get someone to say something that suggests either "With what does Q pair a number close to 1?" in hopes of triggering a response that will make the binomial theorem an attractive option for an estimate or "Is either $1^2 + -.25$ or $1^2 + .25$ in the range of Q?" in hopes of triggering a response that will suggest an algebraic equation that might help. Also, I can attribute the idea to whoever makes the suggestion and try to foster a "see, you can do this" attitude. What keeps this from being the Moore Method (it might well be called Socratic since the questions are intentionally leading with a very restricted set of expected responses) is that I am trying to draw out of them a particular attack on the problem, rather than letting them conceive one on their own. The approach is "Moore-like" in that, even when the modified problem is solved, there still remains the issue of modifying what has been done to handle other estimates at other localizing conditions, or to generalize to a nonnumerical (E instead of .25 or p instead of 1) estimate and/or localizing condition.

For the analysis problem, a question from the students indicating that they have made no progress on the problem gets a very different response. The class will likely already have proven:

> Suppose that m is a number and b is a number and $L = \{(x,y) : x$ is a number and $y = (m * x) + b\}$. Then L is continuous on the numbers.

Here, I might simply ask if there is anything from the argument for the "theorem on lines" that might suggest an attack, or if they have tried to prove that Q is continuous at 3. I might review with them the guidelines for working on problems as I suggest that thinking about what has been done or making examples might give them a productive idea. This is the Moore Method; I am encouraging them to keep working, but not directing them to a specific attack that is not already in evidence because of their own devices.

Notice that, in both examples, the focus of the pedagogy is to elicit student responses that are being driven by questions that are posed consequent to a careful exposition of a definition, rather than as follow-up to a presentation on how to do it. This is a fundamental characteristic of the Moore Method as I practice it, that the mathematics, if carefully presented, can speak for itself, so long as the students are given the grammar (logic) through which to hear it. The active characteristic of the Moore Method as I practice it is that the students will find, using their own faculties, answers to the questions.

There are, of course, many techniques and subtleties that each Moore-style teacher brings to the table as he or she works within the framework of the plan, listen, revise plans model. That is why this book is ten chapters long instead of ten paragraphs long and why there are four authors instead of one. As you use basic Moore Method principles, you will put your own special stamp on the pedagogy and create a "your name goes here"-method that is not only unique to your personality and own special outlook on mathematics, but appropriates the vitality and learning possibilities of the Moore Method.

(May) The Moore Method, as I practice it, is a style of teaching that places the student, the subject, and their interaction at the center of the learning experience. At the same time, it transforms my role as teacher from that of dispensing knowledge to one of facilitating learning. It repositions me, physically, from the front and center of the classroom to some place in the middle or back of it, as it subtly, yet significantly, increases my involvement in the activities taking place.

> "BEST way to have class. Really gets students involved."
> Anonymous Evaluation from the sophomore-level course, "Discrete Mathematics."

The daily life of my courses reflects this repositioning. Class meetings are devoted almost exclusively to the presentation by students of their solutions to problems or their proofs to theorems, and to the discussion, primarily by the students, of the results presented. My day-to-day tasks are many. I decide who should go to the board to solve a particular problem or prove a given theorem. I moderate the discussion that accompanies the solution or proof. As I do this, I wait patiently for the members of the class to catch any flaw that appears in a presenter's argument. When one does arise but seems to have been overlooked by all, I decide whether to point it out, or, without saying anything, give the students a few days to notice it themselves. When a flaw is discovered, it is my job to weave it, as the valuable step that it is, into the learning process. It is also my responsibility to offer hints on problems that seem to have the entire class stumped. Finally, I maintain a diary of the proceedings of every day's class. Included in the diary are comments on the progress of each student. (I ask never to have any two of my four courses per semester scheduled back-to-back, so that I can use the time immediately after each class to update its diary.) I might not be lecturing much, but I am busy.

Lecturing is not entirely excluded, however. Occasionally, especially in courses such as freshman calculus or statistics, I do lecture in the traditional sense. Such a performance is, however, rare. It occurs almost always on the first day of class. After that, it happens only when the class, especially a freshman- or sophomore-level one, comes upon a topic, such as integration in calculus or the Central Limit Theorem of statistics, that the syllabus of the course requires be covered more quickly than the Moore Method might allow. On most days, however, I can practice my preferred version of instruction, *teaching with my mouth shut*.[2]

The Moore Method is a continuation of the learning-by-immersion process that was fundamental to us as children. Something persuaded us that we ought to walk, so we tried. We fell down. We got up and tried again. We fell again. We continued in this way—encouraged, comforted, and applauded by our parents—until we succeeded more often than we failed. Amazingly soon, we were able not only to walk but also to run, skip, jump, hop, and perhaps even dance. Similar events occurred with regard to talking, feeding ourselves, writing and drawing, and interacting with other beings. The amount of learning that takes place in most people's lives before the age of five is staggering, and it happens by the Moore Method. As I employ it, then, the Moore Method is the assumption and execution of the role of good "parent" toward my student "children" as they learn to walk and talk mathematics. Don't get me wrong here. I do not condescend to my students, as the use of the terms *parent* and *children* might suggest. On the contrary, the Moore Method forces me to be anything but condescending, based as it is on the premise that students are able and willing to do mathematics at a high level. I treat my students as adults—in particular, as junior colleagues. I view the relationship between student and teacher as one of junior and senior professionals working together to achieve the common goal of advancing knowledge.

This description is, by design, brief. It almost certainly is insufficient as a guide to your practicing the Moore Method in your own classroom. It is probably inadequate even for you to understand the method well enough to make a decision as to whether you want to try it. Like the first problem in a set of Moore's notes, however, it is, I hope, enough to intrigue you and whet your appetite for more knowledge about the subject. That knowledge—some specifics of the ways in which my coauthors and I practice our versions of the Moore Method, and some reasons for our having chosen to use it—follows in the rest of the book.

(Coppin) For years, I have been asked, "What is the Moore Method?" I have been using the method or some modified version of it for my entire teaching career. My quick answer is that it is a teaching method that puts the responsibility for learning squarely on the shoulders of the student. Each student must prove theorems, solve problems or answer questions completely on his or her own without recourse to a book or any outside aid. To receive credit for a result, a student must present it at the board and, under the guidance of the teacher, allow fellow classmates to ask critical questions. If a student has already seen the proof, the solution, or the answer; he or she does not qualify for credit. Although the process seems harsh, it is not. As a result, each student matures mathematically as the semester progresses. Some undergo a veritable metamorphosis. How might we achieve this in our students?

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[2] I have taken the liberty of using Finkel's phrasing. See Finkel, Donald, *Teaching With Your Mouth Shut*, Boynton/Cook Publishers, Heinemann, Portsmouth, New Hampshire, 2000.

Consider the Monarch butterfly emerging from its chrysalis, in its pupal (immature) state. As time nears for the butterfly to emerge, the chrysalis darkens. The butterfly begins to emerge and is struggling. It is important that the butterfly break through the hard casing of the chrysalis on its own without help. If someone were to be sympathetic to the butterfly's plight and cut open the chrysalis in order to assist the beautiful Monarch to emerge, the butterfly *would* die. The Monarch must fight through the resistance that the hard covering of its pupal home provides in order that the blood vessels in its wings develop. If they do not develop, the butterfly cannot fly and, as a result, cannot feed itself. The beautiful Monarch dies!

What are the Problems?

First, let us state the problem we are attempting to resolve. Many courses in mathematics yield one or more, if not all, of the following problems, which might be alleviated with an alternative method of instruction:

1. Students do not have "bedrock" understanding of the mathematics; that is, students leave courses with only a superficial understanding of the mathematical way of thinking.
2. Students seem not to have enough excitement about mathematics and the confidence to do significant mathematics. Many teachers work examples in class similar to those they will assign for homework. This approach does not challenge nor does it excite the students.
3. Students lack the ability or motivation to grapple with difficult mathematical problems and lack the will to succeed. Students who are not challenged are less likely to grow dramatically and become self-motivated.
4. There is no feeling for proper usage of language, e.g., surprisingly many bright students do not seem to see a distinction between a universal quantifier and an existential quantifier. Lessons in symbolic logic may not provide a solution here, since the student needs to go beyond learning to perform algebra on statements in order to internalize the deeper concepts behind the statements.

To my mind, the best way to remedy all the problems just mentioned is to involve the students in the process of doing genuinely significant mathematics, almost doing research. The Moore Method and the modified Moore Method of teaching do offer viable, exciting, alternative ways to engage many students in mathematics.

The Method

The idea behind the pure Moore Method is that there must be a collection of problems, questions and theorems representing a single genre of mathematics such as geometry, graph theory, or modern algebra. A survey of topics is not what is meant here. There should be a strong, dominating, unifying theme. The students engage problems that range in difficulty from easy to very difficult and are to be done totally by the students. The content is normally composed of axioms, definitions, theorems, problems, conjectures and invitations to raise questions or problems. Generally speaking, the material should be arranged in ascending order of difficulty, with much thought applied to motivation. The students do the work by themselves: they know the teacher will not bail them out! You have heard of "tough love." As parents, we fail if we over-function. As teachers, we fail if we over-function. If we really care for our students, we will challenge them. Here are characteristics of courses I teach by the Moore Method:

1. I generally present no proofs, but I will respond to incorrect statements or to discussions arising from incorrect statements, providing examples or logically sound statements to clarify objections or promote understanding.
2. The students present all material. Each student's grade will be derived solely or predominantly from the student's original presentations. I attempt to make my Moore Method classroom a universe of mathematics unto its own. The students are mathematicians: they "publish" their proofs, solutions and answers as presentations at the board.
3. Students receive no help whatsoever from anyone or from any source such as books, articles, etc.
4. A student must disqualify himself if he has previously seen a proof, solution, or answer to the theorem, problem, or question.
5. Students cannot give help in class. They are to develop skills of critical analysis, and must carry the burden of refereeing the arguments presented.
6. A student may leave the classroom while a result is being presented if:
   a. the student feels that he "has it" and wants to present it later;
   b. the student feels he is very close to a final resolution and wants to have more time; or
   c. the result is one the student feels he must get on his own.
7. However, no student should abuse this privilege, and as a rule, he should not step out of class if he has at least two unresolved results from which he has previously excused himself.
8. Statements are very carefully worded, precise but limpid.
9. The method stresses the individual, not teamwork or collaboration.

To summarize, my students create their own mathematics and are evaluated only by their original creations, just as research mathematicians.

"To know is nothing at all; to imagine is everything."[3] Students find tremendous pride in being able to prove a theorem, solve a problem, or answer a question completely on their own. This is a good thing! It is a large part of being human. This is a most powerful emotion to tap into; it will drive the learning of authentic mathematics. I believe this emotional engine is diminished or crippled when we allow students to collaborate and/or read a proof in a book. To that end, the first six characteristics above allow an opportunity for students to do research that is original to them. Sometimes, they just flat out do original work!

Not Angel

When asked how he carved beautiful angels from blocks of marble, the classical Italian sculptor Michelangelo replied that it was easy. There was already an angel within each

[3] This quote of the French novelist, Anatole France (1844-1924), is referenced by Moore in the film, *Challenge in the Classroom*.

block of marble and it was his task to merely chip away everything not angel. I will follow Michelangelo's example. The block of marble will be the extant, collective teaching methods. Hopefully, when we reach the end of this piece, we will see that what is left, although not angel, is Moore Method.

In my years of teaching, I have learned that when more than one method of teaching is used in a course, the method most used is the one that dominates all others used in the class. For example, when one allows students to collaborate sixty percent of the time on their lessons or evaluates students sixty percent based on collaborative components, then whatever else is done, regardless of how effective it is, supports only the collaborative portion of the course. Thus, if forty percent of a course is taught by the Moore Method and the remainder is collaborative learning, then that course remains essentially collaborative in nature. Using this principle, with a single blow of the mallet on chisel, I eliminate most learning theories and teaching methods from the Moore Method rubric. To reiterate, if less than half of the students' grades are derived from a portion of the course that follows the above rules, then, in my opinion, that course is not an authentic Moore Method course. This would eliminate courses that predominately rely on the lecture method or rely on groups. Other teaching methodologies, such as the Montessori, HOTS, and the SEED[4] program, although effective, are removed. These are not Moore Method courses. The Moore Method is not known to most of the practitioners of those approaches. This is indeed unfortunate because conferences or workshops where the many different approaches are discussed in a constructive way would help the Moore Method to become more mature and more generally accepted.

Allow one more blow with the mallet and chisel. I must address what I will call Neo-Moore Method courses.[5] The teachers of these courses have no qualms about divulging answers when no student has successfully resolved the problem of the day. Some will even allow class collaboration when developing a proof, solving a problem or answering a question. This is done routinely. Therefore, as you see, at the very least, the spirit of the Moore Method is violated. *In my opinion, these are not Moore Method courses*.

Modified Moore Method (M^3)

In a Modified Moore Method course, or M^3 course, I de-emphasize somewhat the objective of maintaining the maximum "learning gap" that I believe characterizes the pure Moore Method. This can be done in one of the following ways:

1. Divide the class into groups and treat the groups as you would have treated individuals. R.F. Jolly has suggested this in his book, *Synthetic Geometry*, (Jolly, 1969). I had a colleague who divided his class into groups. Students were allowed to collaborate within their group but not outside. When the group proved a theorem, solved a problem or answered a question, then the group made certain each person in the group was able to successfully present the proof, solution or answer. Then, the teacher called on someone in the group at random to make a presentation. I believe groups actually can work in this case. It solves one of the problems with groups, namely, that one student will do all the work but all members of the group get the same grade. One leads, the others follow; one learns, the others know little.

4 www.ProjectSeed.org and www.hots.org

5 Mahavier and Parker refer to this type of teaching as "problem-based."

2. Break theorems or problems down into smaller, more palatable pieces. Over the years, I have seen the following "problem." If I design a sequence of theorems, problems or questions that are broken down into pieces that are too small, students may not grow; therefore, the teacher needs to be careful. After all, students develop best when challenged.
3. Talk with students on an individual basis or as a class about their use of language, minor mathematical points, fear, and even hysteria but always protecting the key of the theorem, problem or question. The students need to feel that you are not "hanging them out to dry."
4. Lecture and/or discuss some of the material but leave some designated results for the students to do on their own without aid of any sort. This has worked quite well for me. Still, I reserve well over eighty-five percent of the material for the solitary work of students.

The Dark Side

It is said that Moore fostered intense competition, even cut-throat competition. Moreover, many who practice the Moore Method seem cultish, almost arrogant. Moore had his foibles to which others can attest. Many have attempted the Moore Method and failed. Many have alienated colleagues. I reject these darker elements. Even Moore's successes were gained, I believe, in spite of this darker side. He was not perfect, far from it! I heard complaints while a student. Moore might not even survive today's classroom. I believe it is much tougher to succeed as a teacher in the 21st century than in the days when I started teaching, which was a period when Moore's career was reaching its end.

I advise anyone who wishes to experiment with the Moore Method not to emulate or incorporate the dark side of the method. Use your pedagogical discernment, your understanding of human nature and your understanding of contemporary culture to use the positive side of the Moore Method. Then, you will have greatly increased your chances and your students' chances of success.

Conclusion

At this point, I have managed to chip away from the pedagogical monolith most that is not Moore Method. However, as you may have anticipated, what is left is not a finished work. You, as a prospective teacher of the Moore Method or M^3 course, are like a painter standing before a canvas who must allow his or her artistic talents to flow through the brushes and the paints. You must allow your pedagogical talents to flow through the exchanges with students, the creation of materials and the management of the warp and weft of all the many intangible issues and forces that play throughout a class. The invention of the rest of the method or the job of finishing the final piece of our sculpture is, in the true spirit of the Moore Method, left to you. You must find your own way. Let's not let our students down!

(Mahavier) To state a list of rules or features that define the Moore Method would be as incomplete as to define a martial art by the types of kicks that students of that art master, or the rules of the dojang[6] in which they train. Training in the martial arts transforms a student

[6] Dojang is Korean for any facility where students train in the martial arts. When used for martial arts, students

with minimal control over mind and body to one with a mind capable of fully controlling a body powerful enough to break boards. Certainly this transformation has little to do with the rules of the dojang or the types of kicks mastered by the students, and everything to do with the fact that it is, like the Moore Method, truly an art. If pressed for rules defining the Moore Method, I would reiterate those enumerated on page 9 of Chapter 2 and argue that any course taught using these rules is in some sense a Moore Method course. Yet this answer is dangerously incomplete precisely because it does not address the art. Over the years I have altered the rules in order to meet the needs of widely varying courses and students, while still remaining true to the art. Hence, while rules may be used to define the *method*, for a successful implementation of the *art*, we must ferret out the essence of the art alongside those ingredients that complement the rules and are invariant between distinct iterations.

The art of the Moore Method is that it rekindles the independence and inquiry that are innate in all children at birth, but are often whittled away and lost during childhood as social and intellectual challenges take a toll on the child's natural-born ego and curiosity. Once independence and inquiry are re-established, no problem seems intractable and the ego of the student thrives. Success in life begets success, and whether success occurs in the dojang or the classroom is of little consequence. The dojang is an environment in which students develop success at controlling their mind and body. The Moore-method classroom is an environment in which students develop success at exploring and creating their own mathematical knowledge. Just as mental and physical strength transcend the martial arts, the ability to create knowledge transcends the field of mathematics and permeates all aspects of the students' lives, as we see illustrated in the broad and varied successes[7] of products of the method.

Assuring that the opportunity for such success exists in the classroom boils down to three essential ingredients: time for discovery, appropriate materials, and teacher attitude.[8] In my mind, these more fully define the method than can a list of rules or features.

Time

John Holt wrote (Holt, 1965), "When kids are in a situation where they are not under pressure to come up with the right answer, far less do it quickly, they can do amazing things." In the martial arts, there are no deadlines and students train as much or as little as they choose. Upon mastering a certain skill set addressing etiquette as well as mental and physical strength, students are told that they are ready to test for the next rank. Success on the test is defined by students demonstrating their personal best, since the instructor, by asking the students to test, has already acknowledged the necessary progress. While the constraints of academia do not allow the complete elimination of deadlines, allowing time for discovery in mathematics creates a more level playing field for all students, benefiting both the weak and the strong. It allows for significant individual successes and a deeper understanding of fewer problems rather than repetition and superficial understanding of many problems.

in the facility are expected, much like in the Moore Method classroom, to conduct themselves with the highest standards of respect and etiquette.

[7] See boxed text at conclusion of this essay for examples.

[8] Throughout *Teaching with Your Mouth Shut*, (Finkel, 2000) Finkel develops the theme of appropriate materials, time for the students to create, and appropriate attitudes by the instructor.

Because materials for Moore Method classes are typically self-contained, every student is working within the same mathematical notation and framework. The student who has seen less mathematics is not necessarily at a disadvantage, since all the problems can be worked using the tools at hand and there is no timeframe attached to any one problem. Because not every student is expected to solve every problem and because there is significant variation in the level of difficulty of the problems, it is common for some problems to remain unresolved for quite some time—perhaps even the entire course. An apparently weak student who continues working on a problem and solves it may surprise you and the class. This provides an opportunity to praise the student and point out that expanding on one great idea in mathematics is what constitutes a dissertation in the field. In many instances, I have seen this extra time allow students who "never really got math" or "never liked math" to create their own correct mathematics, master the art, and go on to major in mathematics.

Adequate time benefits the better prepared and more talented students as well. Such students are quickly identified, given the quality of their presentations. If two students have a solution, then students who have presented less are called on first to present. Thus in order to have the opportunity to present, a strong student who has presented several times, must solve the more challenging problems. These challenging problems may not be resolved in the short time a strong student is used to spending on a problem and the process steels the student to working longer and harder. Often I have been able to push talented students by extending the problems they got hooked on with phrases like, "Well, that's very nice—do you suppose that result holds if we make this change in the statement?" Such individual successes for both weak and talented students depend on having the time to create. The result is confidence and rich knowledge that breeds success in future classes.

Materials

The second key component is to have in place materials (axioms, definitions, problems, and theorems) that foster discovery yet are challenging enough to create the transition from, "I just worked another problem" toward "I just created a great idea that solved this problem." In the dojang, junior ranks visually observe future challenges during their training. Despite rules governing at what rank a student may be taught a given technique, motivated students will often train out of class, striving to individually master advanced techniques before they are formally instructed in them. Most of us never forget our early successes, no matter how insignificant they may seem in retrospect. In an early topology class, I created a definition for a "center point" in order to solve a certain problem. While my definition turned out to be equivalent to the definition of an "interior point," I had discovered it independently. Such early successes tend to generate confidence and ensure the success of more significant research in the future while creating a rich conceptual understanding of the subject. Materials can be written in a fashion whereby students discover key definitions and concepts rather than following the lead of the instructor. While providing guidance is naturally appealing and provides *self*-satisfaction as we lead our students toward "discovery," the danger is the possibility that in so doing we are embedding in the students the belief, either conscious or subconscious, that without an instructor's guidance they cannot learn independently. From the fact that roughly two-thirds of those with doctorates in mathematics publish three or fewer papers (Grossman, 2005), we may infer that the majority of doctorates graduate either incapable of original work beyond that demonstrated by their dissertation or not desirous of pursuing research in their chosen fields. Hence, the development of materials that produce students who enjoy creating mathematics should contribute significantly to our field.

Polya's quote from *How To Solve It* (Polya, 1973) addresses the importance of carefully developed materials.

> *Thus, a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.*

In this work, Polya describes a process of instruction that is Socratic at its heart, where students are given problems and the instructor serves as a coach, questioning and encouraging the students. In many ways, this parallels the Moore Method. A key difference is that in the Moore Method we do not provide a thread of just-in-time questioning to the students in order to lead them to solutions during the questioning. Rather, we ask questions only when the student has worked significantly on a problem and either demonstrated an attempted solution or queried us regarding their progress. Because of the importance of the material, the chapter "Development and Selection of Materials" focuses fully on this subject.

Attitude

Central to successful Moore Method teaching is the attitude that all our students are capable of doing mathematics, that all of them will succeed, and that we, as teachers, care deeply about our students' personal and mathematical development.

While the martial arts instructor would never assume that a particular student would accomplish a given task by a certain date, the instructor would assume that every student would eventually master and even excel at numerous techniques. Considering the feats that martial artists accomplish, including aerial kicks, weapons training, board breaking, and gymnastics, to assume that every student will eventually succeed might be considered naïvely optimistic. All these tasks are demanding, and predicting when a student will be mentally and physically able to tackle such feats might be considered foolish. I claim the same is true of mathematical maturity and the mastery of mathematical concepts. To assume that every analysis student will independently prove the Heine-Borel Theorem and the Bolzano-Wierstraus Theorem from a few axioms and definitions might also be considered naïve. Still, what we do in the dojang and in the Moore Method classroom is to provide the environment wherein a student has the necessary path, the necessary tools, and the necessary nourishment to succeed. And we promote and presuppose success, even if we do not put a timeline on it.

We encourage the attitude that most difficult tasks can be boiled down to key examples that, if we allow our students to discover and study closely, are likely to lead to a deep and applicable understanding of the subject. Not every student of martial arts accomplishes every physical feat and not every student in the Moore Method analysis course proves the Heine-Borel theorem. But, all students have enough individual successes to develop a rich

During the academic year of 2006–2007, a student took a Moore Method analysis course under my direction. Upon the conclusion of the course, he was unable to resolve a particular problem that would have earned him an *A* instead of the *B* that he received in the course. Throughout the next year, he reported to me that he was working on the problem. I offered encouragement, but no help, and I'll confess that I had little faith that he would resolve the problem. When, during the spring of 2008, he showed me a truly elegant solution, I invited him to visit class as a "guest lecturer." He presented his solution to the problem, which was also unresolved in this class. His comment to me in the hall the day after the presentation was, "That problem really made my semester." His presentation was flawless and far superior to his presentations during the course the previous year.

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understanding of the materials and to build their confidence that, should they choose to continue, they will succeed in the subject at hand.

The second notion, that we care about our students' personal as well as mathematical development, has long been known to be important for students' successes. I would be unlikely to write it more succinctly than this excerpt (Davis et al, 1990) regarding the teaching of Jaime Escalante.

> *He conscientiously builds relations of care and trust with each student. He shows steady concern for the integral development of his students — how they are doing in English, how their home lives are going, what jobs and sports they participate in. This attitude and the effort that accompanies it are part of teaching mathematics.*

Because university students are typically more mature than high school students and because I believe in maintaining a professional distance from my students, the concern that I show for my students' development is far more formal than that described by Davis and more focused on their professional than their personal lives. I escort my students to conferences, introduce them to colleagues, and attempt to teach not only the essential mathematics, but also the culture of research, conferences, and the profession. In my classes, I assert my assumption that what we are doing is important and that by striving to excel, we will become better individuals and better mathematicians. By becoming better individuals and mathematicians, we will help to ensure the forward motion of society.

Conclusion

Just as the dojang creates an environment that encourages fundamental changes in the students' mental ability to control their bodies, the Moore Method classroom encourages fundamental changes in the students' mathematical ability and underlying approach to learning and therefore to life. The teacher's attitude, the materials, and adequate time for creativity are at least as important as the rules governing the structure of the class. Together, these provide the students the power to develop significant mathematics beyond their own estimates of their abilities, just as students of the martial arts master physical feats exceeding their initial estimates of their ability.

The motivation to learn in this environment is strengthened by the group mentality that we are inquiring as a whole, even if we are working as individuals to solve problems, to master and understand the subject at hand. Even the teacher may learn something new –

some alternate proof, nuance, or additional insight into the subject. The method strives to change the student from a recipient of knowledge to a creator of knowledge. Not every student achieves this fundamental transition, but every student is made aware of this life-changing goal. Certainly, when I teach a calculus course with forty-eight students, it is unrealistic to expect to achieve this goal with every student. Still, I make clear to the students in the class that it is within their power to make this fundamental transition from one who absorbs material to one who creates material and that the difference between the two is the difference between leading and following. I stress that, regardless of career choice, those who create have the most freedom in their chosen careers and the most rewarding ones. Statements like the following reinforce this theme: "When your managers need a research problem solved, they cannot tell you when it is due. If they could solve the problem, they would have already done so." I stress the value of making this transition and even if only the seed is planted, perhaps it will someday grow into a mature plant. For many of my students, the plant begins to grow immediately as they cease to look up answers and strive to find the answers within themselves. This early transformation can easily awaken the creative spirit, transforming the engineer into the mathematics major, and the mathematics major into the mathematician. Ultimately, if I cannot make every student into a mathematician, I can at least impart the philosophy of the mathematician to all my students and that should benefit them regardless of majors.[9]

> At each of the past eleven Legacy of R. L. Moore Conferences, entrepreneurs, mathematicians, businessmen, and even one theologian credited their success to studying under the Moore Method. Harry Lucas Jr., founder of the Educational Advancement Foundation which hosts the conference each year, credits Moore himself with his success in the oil industry. Entrepreneur James Lemke (Lemke, 1996) credits the method with his success. One of his companies patented recording equipment that at one time recorded fifty percent of all broadcast hours on television. WesternGeco is the world's largest geophysical company and Dr. Craig J. Beasley, WesternGeco's Chief Geoscientist, was educated using the method. Retired from AOL/Time Warner at forty-nine, Allen Stenger credits his early training via the method with his success in the computer industry. These are a few representative, if perhaps less publicized, successes. More renowned successes (Devlin, 1999) include three members of the National Academy of Sciences, at least one member of the National Academy of Engineering, three presidents of the American Mathematical Society, three vice-presidents of the American Mathematical Society, and five presidents of the Mathematical Association of America. Further successes are documented in John Parker's biography of Moore (Parker, J., 2005), and include seminal work in space travel, medicine, and warfare. Of course, each of the authors has his own stories of success—students who are graduate students, heads of mathematics departments, researchers, or industry employees who contact us, sharing their latest successes.
>
> Mahavier

[9] Moore stated this quite forcefully in the quote, perhaps apocryphal, "Not everyone can be a mathematician, but everyone can *want* to be a mathematician."

4

“We have to believe—before students can believe—that hard work pays off, that effort matters, that success depends not on your genes but on your sweat. We must convince students that they can do it when they have teachers who insist on quality work and give them extra remedial help when needed.”[1]

Gene Bottoms, Director, High Schools that Work

On Culture

We address the culture in the classroom before addressing the mechanics of the method, for we believe that the mechanics have the best chance for success within a classroom of appropriate culture. As our method of instruction will likely be radically different from that to which our students are accustomed and will likely place more responsibility for active learning on them, part of the established culture is to provide some justification to the students as to how this approach can enhance their intellectual growth. By the same reasoning, we need to provide an environment that has the capacity to maximize the success of each student. The culture should emphasize student responsibility for learning and creating mathematics, while reducing the need for the exercise of authority by the instructor. We strive for a low stress environment and a relaxed atmosphere within the classroom because we are asking students to produce, present, and defend the mathematics of the course. Demonstrating and defending such a personal product can be stressful even in a friendly environment. Who hasn’t had butterflies before his or her own research talks at conferences? Assuring a culture in the classroom that is supportive helps offset the stress on students that accompanies their being responsible for defending their mathematics. We desire an environment where conjectures arise naturally within discussions, and where questions during presentations are not posed in order to challenge the presenter, but rather to address the correctness of the mathematics, thereby increasing the collective understanding of the class. By creating a culture in which students succeed in creating mathematics and feel an ownership of the mathematics after they defend it, we seek to accelerate the transition in our students from being passive learners to being active agents in the creation and use of mathematics.

(Coppin) Mark Hopkins was a professor of moral and intellectual philosophy at Williams College from 1836 to 1872. He taught at a time when the idea behind an education was that knowledge was so deeply learned it was just as much a part of the students’ constitution as walking or talking. One would learn a way of thinking; process was of paramount importance. It was liberal learning, a very rare thing today. Hopkins taught in the

[1] G. Bottoms, “Effort, not ability,” www.sreb.org/main/Publications/Other/Effort,_not_ability-Op_ed.pdf.

Socratic tradition, which is at the essence of the Moore Method style of teaching. Hopkins' former student and twentieth president of the United States, James A. Garfield, as education lore has it, said, "The ideal college is Mark Hopkins on one end of a log and students on the other." This is an archetype or picture of what the Moore Method can be for you as the teacher and your students. Keep this image before you. This *is* the essential idea behind the culture of learning we want in our class—this is what we all seek to capture for our students and as teachers. It is this excitement that we all want to share with other teachers and their students. This is why we teach and why we seek to share what we have learned.

Why is concern for class culture so important? Culture is the language, the portal, by which we see reality. If we attempt to have social interaction with a group of people, the more we understand their culture, the more successful we will be. This is no less true of our students, especially when we implement a radically different teaching method than what they are accustomed to. Moreover, mathematics, as R. L. Wilder (Wilder, 1968) has pointed out, is a subculture unto itself. Thus, as we attempt to acculturate our students to the mathematical way of thinking, we are also changing their learning culture. The more aware we are of this fact, the more successful we will be in implementing the Moore Method.

First, allow a view of the average student of mathematics as he or she understands how to learn mathematics. Generally, in spite of many reform initiatives, mathematics students still expect the teacher to be the authority in the classroom, all knowledge emanates from the teacher, and students receive or assimilate that knowledge. Students expect guided drill with a step-by-step approach to learning mathematics or an approach that presents a mathematics that is essentially algorithmic. This is a teacher-centered class. Many classes taught as we have described here are very successful. Students *do* learn and *do* enjoy the experience. Frankly, those who teach students by the Moore Method also teach other classes by a teacher-centered method when the situation dictates it. I personally believe that teachers should not adhere to just one approach to teaching because of some ideological stance in teaching. We all have the idealistic hope of actually playing a significant part in giving the students in our charge a transformational experience that is also exciting. We teach most effectively when we seek learning; that is why I tend to use a learner-centered method when I can even if I am teaching a class that is not a Moore Method class. Remember Mark Hopkins.

> "When I visited the University of Dallas during sleeping bag weekend, I had the pleasure of sitting in on the linear point set theory class. No textbook. No lecture per se. Instead students received a list of axioms and theorem statements; the content of the class then consisted in students going up individually to the board and proving the theorems, with critiques from the teacher and others. I was impressed. Although the LPS class was rather unique among math classes you would take, nevertheless, one always found a more-than-average emphasis on the proof and theory, i.e., the *mathematics*, at all levels of courses. No part of my education at the University of Dallas was more important in my formation as a theoretical physicist than was that of the math department."[2]
>
> Jonathan Engle (2001)

Maryellen Weimer (Weimer, 2002) has coined the term "the learner-centered class." She chooses the words "learner-centered" instead of "student-centered" because, as she

[2] http://www.udallas.edu/math/testimonials.cfm

writes, "Being learner-centered focuses attention squarely on learning: what the student is learning, how the student is learning, the conditions under which the student is learning, whether the student is retaining and applying the learning, and how current learning positions the student for future learning." In Moore Method classes, most knowledge originates with students and is received or assimilated by the teacher and fellow students. Moore Method classes are learner-centered. It is exactly this issue we are concerned with in this chapter.

To aid you in your journey to model Mark Hopkins by way of the Moore Method, I discuss the student culture, the teacher culture and finish with the collective culture in the class.

Student Culture

In speaking of the student culture, we note some of its properties.

- **Imagination.** To be successful, students must want to know; however, they must want to delve into the deep ideas of a subject. They must want to imagine. This is the knowledge that remains invariant from antiquity to modern times. Information changes but imagination remains. A proof done solitarily yields deep knowledge. Teachers must supply the materials and the class setting so that imagination flourishes.
- **Courage.** To have fear is inevitable. However, students should control fear and *not* let fear control them. Fear will abound. I empathize with the students at all times. My goal is to allay fears. I use personal stories and experiences of previous students. For example, I tell them that I was as shy as some of they are, and how, as a student, I would not indicate that I had a proof just so that I would not be called to the board. But, *surprisingly*, I found that I could be successful albeit after some prior failures. I was amazed at how much I learned from going to the board.
- **Encouragement.** Encourage the students and communicate to them that they should encourage each other. They should learn to work through failures. I attempt to communicate that it is okay to fail – that is how you learned to talk. Infants had to coo and babble before they could speak. However, they needed to be understood by individuals who were not their parents. The parents had learned to understand the difference between "goo-goo" and "goo-ga." As teachers we must not over-function or else we will cripple our students. But, those of us who are parents did encourage our children while they were learning to talk. Likewise, our students must strive to be the masters of the material, not enslaved to whatever information and processes we teachers give them.
- **Relaxed atmosphere.** Students should relax and they should realize they can get help from you, the teacher, outside the class as well as during class. You are attempting to develop an environment in which the students will need to be creative. In my opinion, real creativity occurs when the mind is relaxed and at peace.

Teacher Culture

Here are some ideas teachers should think about.

- Sell. Because the students have been immersed in a different learning culture for at least twelve years, you may need to sell, sell, sell what you are doing on a daily basis. You need to answer the questions: "What?" and "How?" and "Why?" Most teachers are romantics. They want to affect a transformation. I would always keep in mind the picture of Mark Hopkins sitting on one end of a log with my students on the other end, except I *am* Mark Hopkins.

- Care. You must care, genuinely care, about each person in the class. Be unbiased whether it be SAT scores or attire. Do not steal learning opportunities. As a result, you will be surprised at what students can do. They will astound you at times and will frustrate you at other times. You will have an urge to just show the students at the board how the proofs are to be done. You will want to show them slicker proofs. You will want to show off your knowledge. But do care about their learning.
- Be Aware. At all times, especially when I have someone at the board, I maintain awareness of the class atmosphere. How well are students paying attention? Is someone distracted or has someone initiated a conversation that is off topic or is distracting others? The person at the board selects his or her own style of presentation and may use any notes he or she wishes. Sometimes, students may want to do an oral presentation with some diagrams. That is okay but I must be aware that the class will want to fully understand the proof. As a result, I will give the students a choice of writing up the proof or I will write the proof on the board with guidance primarily coming from the student but I will determine some way to engage many other students as well. I must be aware of legitimate modes of learning, imagination and communication styles and react constructively.
- Have Courage. Some years ago, a fellow mathematics teacher was observing one of my classes. I had a student at the board who was presenting a very unique proof. Other proof ideas were being voiced over that proof via student questions. I had to deal with multiple ideas very different from my own. My colleague remarked, "I don't see how you have the courage to teach that way." However, I feel compelled to teach in this interactive manner—I could no more teach the way she taught than I could stop breathing. I will say this. Whatever style of teaching one chooses, excellent teaching is a heroic act.
- Be Relaxed. As teachers, we must strive to be comfortable. Put ego aside. You want the class to relax; therefore, you must relax. Enjoy yourself.
- Finally, I must reiterate that I try to be aware of feelings: fear, points of confusion, and different perspectives. I try to be sensitive to every little nuance of emotion or cognitive dissonance. Again, sell, sell, sell.

Collective Culture

There are several properties of the collective culture, some of which I have alluded to earlier.

- Camaraderie. I want my students to develop a sense of camaraderie. I strive for a collegial atmosphere. I want them to support one another. From time to time, when a student has finished a proof, applause breaks out. It is hard to explain how and why but it does happen. I believe it occurs because of camaraderie. If a critical mass of collegial students exists then, sometimes, good human response occurs. Again, it is not something that is to be contrived or manufactured. It must occur of its own accord.

- Constructive criticism. I encourage students to develop a critical attitude. I tell them if you learn to criticize another's work, you will learn how to find the flaw in your own work. Even if a student already has a proof that is being presented by someone else, I still want that student to pay full attention to the presentation. In this manner, attempting to follow another's work makes you stronger. Secondly, and more importantly, following someone else's work demonstrates support. According to the Moore Method, one cannot supply a step in a proof to a fellow student at the board. However, one can ask questions. "How do you support step *x*?" "Please explain where that step came from." "I don't understand how *x* follows from *y*. Please explain." What do you do when students don't know where to start or don't seem to have questions? I usually have the student at the board "walk" us through the proof. This will tend to focus the class on each line, line by line. If that fails, I will point to the first line of the proof and ask the class if they accept that one line. Is it true? Is it supported? Do all the terms make sense? Collective criticism is a very important feature of my classes. They must not collaborate on proofs but they can collaborate with very civil and polite criticisms.

I have attempted to communicate the culture that I strive for in my own Moore Method classes. I have been purposefully oblique for the very reason that its nature is very elusive. To use the language of literary criticism, we are attempting to describe the "muse" of the class culture. It is there; it is real. We see it in our peripheral vision and what I remember is all I have been able to tell you. However, I think you know enough that you will recognize it when you see it. Keep working to that end. I am sure that Mark Hopkins had two qualities: love of his subject and love of his students. If you do as Mark Hopkins and love mathematics and love your students and use the Moore Method with the correct spirit, you will greatly increase your probability of success.

(May) In his book, *The Courage to Teach* (Palmer, 1998), Parker Palmer contrasts traditional teaching with the kind that he advocates. The former, according to him, is characterized by knowledge passing to the students from someplace "on high," external to the classroom. It does this by first entering into the mind of the teacher, who masters and digests it and then attempts to inject it into the minds of the students, who, it is hoped, will absorb it for their own use. In Palmer's teaching paradigm, it is the subject under study—not the teacher, not even the students—that is the central element of the course, not something external to it. Around the subject sit the students, one of whom is the teacher. All study the subject, and all offer insight. The teacher makes an observation only when to do so seems preferable to attempting to draw it out of the class. As I read Palmer's exposition of his paradigm, it occurred to me that it was simply a description of my version of the Moore Method. It also was painting a picture of the culture that I try to create in my classroom.

That culture is the culture of mathematics itself. Above everything else, it is one in which the students' main goal is to become better mathematicians as they learn more mathematics. Their learning takes precedence over their earning a particular grade. The most important abilities for a mathematician are formulating conjectures; testing them; and, by means of proof or counterexample, settling them. Better than any other method of teaching, the Moore Method presents students with myriad opportunities to develop all of these skills.

> "(The Moore method) afforded me more freedom to explore topics of interest than in any other class."
>
> Note from a student about to graduate.

Whereas placing learning above earning can be difficult, putting into the course a spirit of competition is almost impossible to avoid. Competition seems to be a fundamental element of life, academic and otherwise. Almost every student, from the moment when he or she finds out how the course will be conducted, wants to become a better mathematician —better than he or she has been, and, often, better than at least some other members of the class. In addition, almost everyone wants to contribute to the success of the class. The way to accomplish each of these goals is to present proofs and counterexamples; and this is best achieved by inventing proofs and counterexamples soon enough to be prepared to go to the board when called upon by the teacher. This develops a spirit of healthy competition, as most students adopt an attitude of, "I don't want to pulverize everyone else in this class, but I surely would love to be the first one to put up a correct proof of the next theorem." The main factor with which I have to deal is to minimize discouragement and a loss of the competitive spirit. This occurs in the students who are not achieving enough to please themselves in the face of the seemingly greater success of other students.

To support the notion of learning over earning, and to keep the competition healthy and productive, I inject four elements into my classroom culture. They are respect, responsibility, democracy, and relaxation.

Respect

I have great respect for college students. Especially today, they have accomplished much just to have completed twelve years of schooling and gained admission to an institution of higher learning. Life is significantly more challenging for young people today than it was for me when I was growing up. College freshmen of the twenty-first century have faced the problems of broken homes, drugs, societal violence, terrorism, sexually-transmitted diseases, and a general decline of the positive influences of family and religion in American life, in addition to the half-century danger of nuclear war. They also have lived through the over-burdening of the schools with many of the responsibilities that used to be discharged by parents and religious institutions. Finally, they have dealt with our society's rising expectations for their work inside the classroom – more advanced-placement courses, for example – and outside – the requirement of community service for graduation, to mention one. Somehow, through all of this they have managed to become kind, respectful, and winsome human beings. I take my hat off to them, and I tell them so often.

Responsibility

I also have confidence in them. I trust that they can and will assume full responsibility for learning the concepts of the course of study on which they and I have embarked. I let them know this by treating them as the adults who they, at least legally, are. Whether they are freshmen in a general-education course or junior and senior majors in an upper-level one, I try to relate to them as a senior mathematician to his junior colleagues. The Moore Method provides me with an excellent vehicle for doing this.

Democracy

Democracy is built upon respect and confidence. I begin on Day One to manifest the idea of a democratic learning partnership. As I present my policies for the course, which cover topics such as grading, attendance, and academic integrity, I solicit discussion from the students. I encourage them to suggest changes to any policy that they think could be improved. Each suggestion is discussed and, I tell them, will be voted on by them at the next meeting of the class. I exercise my dictatorial prerogative with regard to only a very small number of issues. For example, a vote by the class to transform the course from a Moore Method one to one with a lecture format would be vetoed by me. (Such a proposal has never been made, however—at least not during the first or second day of class.) With regard to grading, I allow them, again with some restrictions, to choose their own weights for the various components of their grade: presentations in class, tests, and a project. (Details of this aspect of my courses appear in the chapter on grading.) Also on Day One, I hand out the evaluation forms for the course. I encourage the students to turn in a filled-out form—or, if they don't mind being less anonymous, to visit or email me—at any time during the semester when they think that doing so is warranted, whether for the purpose of noting some praiseworthy occurrence, correcting some problem with the running of the class, or simply "letting off steam." Right from the start, I emphasize that it is they, not I, who will determine what learning and other benefits they will derive from the course.

Relaxation

Most of the students whom I have known during my career have had no need for more stress in their lives. Rather, they welcomed almost any aid in reducing the amount of their tension. I share their desire. I agree with Michael Henle, who wrote, "It is best to read mathematics at a leisurely pace" (Henle, 1997). As a consequence of this, and of my having placed a great deal of trust in them and responsibility on them, I see my role as that of creating as relaxed a classroom environment as possible. One way to do this is to begin each meeting with a discussion of some topic of current interest: yesterday's baseball, basketball, or football game; the release of a much-hyped movie; the latest episode of a popular television show; gossip about some celebrity; or an important piece of political news. I try to cause the theme of light-heartedness to follow us as we move from chit-chat to the rest of the day's agenda. I use terms such as *play* rather than *work*, and *exercise* rather than *problem*. In addition, I praise effusively a correct solution, and, perhaps more importantly, point out the value of learning from mistakes. I remind everyone, especially in a theorem-proving course, of how difficult it is to do mathematics. Finally, on a continuing basis, I try to drive home the point that, above all else, if all of us constantly improve our ability to learn, there is no limit to what we can accomplish.

A hallmark of the Moore Method, as it was practiced by my professors, is to allow, and even encourage, students to leave class. Suppose, for example, that I have called upon

Jason to prove Theorem n, but he has not yet been able to produce a correct argument. Nicole, on the other hand, has a proof ready to present, so I ask her to go to the board. Jason believes that he is "within epsilon" of having his own proof; he simply needs a little more time. I encourage Jason to leave class and repair to a room not far away. Once in that room, he can, on his own, attempt to complete his proof while Nicole presents her argument to the rest of the class. If Jason is successful, he may show his proof to me in my office at a later time and receive just as much credit as Nicole. If he is not, then he may ask Nicole, one of the students who understood her proof, or me to show him a correct argument. Best of all, for Jason anyway, if he comes up with a correct proof and Nicole has presented a flawed argument that she is unable to correct and ultimately "yields the floor," Jason might end up presenting his proof of Theorem n to the whole class after all. The practice of having students leave the classroom accomplishes at least two purposes. It reinforces the principle that they, and not anyone else, are responsible for mastering the concepts and methods of the subject. At the same time, it relieves the stress that having to solve Problem X by time Y can impose.

As one out-of-the-ordinary way to decrease the level of stress in the classroom, I devote a large portion of at least one meeting of the class to a "working meal." This is simply a combining of food with a discussion of the work of the day. In a morning class, the students bring with them their own sustenance—from the dining hall, some local fast-food restaurant, or home. For a class that meets around noon, something such as pizza is ordered to be delivered at the beginning of the period. The class spends approximately the first third of the period eating and socializing. When everyone has finished, we return to the regular format: "The floor is now open for offers of and requests for solutions to exercises" in a freshman-sophomore course, and, "Does anyone have a proof to Theorem m?" in an upper-level one. In either venue, the working meal seems to relax the environment.

These, then, are the ingredients that I stir into my Moore Method classroom. My belief, reinforced by the evaluations of the students themselves, is that the culture that results is one of lightheartedness and productivity.

> "Great way to teach. I enjoy this method b/c it is relaxed yet intense. You are not forced to turn things in. This method makes me work harder."
>
> Anonymous Evaluation from senior-level Geometric Structures.

(Parker) A Moore Method class, like any other class, develops a character of its own as it evolves over a semester. In my experience, there is radical diversity in the form the culture of a particular class may take. I have had classes where the students adopted an "all-for-one, one-for-all" attitude early on and shared ideas in ways that I would rather they had not, and they achieved. I have had classes where the students competed strenuously as individuals and they achieved. I have had classes where the students accepted the challenge of making mathematics in good spirits from Day One. I have had classes where my main early contribution was to find ways to enable the students to convince themselves that they could actually make mathematics themselves. I have had classes where students were open and fun-loving in their criticisms and accepting of criticism. I have had classes where I never succeeded in fostering the critical attitude, yet the students still made good mathematics. I no longer have any fixed expectations of what the culture of a particular class will be. On the other hand, I have become particularly diligent in trying to be aware of the culture as it develops and to be, at worst, benign, and at best, a catalyst, in its development.

Fostering the development of a classroom culture can be accomplished in several ways. At the beginning of a course, I emphasize the significant difference between validating someone else's mathematics and making an argument oneself without being shown how or why.[3] I reassure the students that they can pass the course just by reproducing their classmates' work, but I make it clear that my expectation is that each student will be able to make mathematics and contribute in this way as a member of the class. I then do my best to describe the thrill of having and validating one's own idea. I talk about effective use of time and the importance of considering thinking about mathematics as studying and try to make it clear that success does not always come quickly or necessarily at the time of perceived maximum effort. It is terribly important that the tone of the first day of class overpower any feeling of dread with a sense of how exciting not only the results, but the pursuit of the results, is going to be.

> "Before I took this course, I hated mathematics. I still do. But now I think I understand why some people love it."
>
> Anonymous Evaluation from Mathematics 103: The Nature of Mathematics (1991)

Regardless of how good a job one does with introductory propaganda, the culture we hope for will not bloom unless the students possess tools adequate for dealing with the problems. It is also necessary that some of them experience early success. Essentially these two issues reduce to answering the questions: "How will the students get the logic they need?" and "What are appropriate starter problems?"

As my Moore Method teaching has evolved, I have come to address the logic issue by setting a standard for language and dealing with points of logic as they arise. The more naïve the class, the more extensive work I do in making sure that concrete issues associated with use of the logic[4] arise so that they can be addressed explicitly. Nevertheless, the objective is the same for all classes: whatever the mathematics to be made is, the students need to have from their previous experience, from direct exposure to the course notes, or from their classroom experiences in the course, logic adequate to the task of validating any ideas that they can articulate within the vocabulary of the mathematics in which they have been asked to work.

We will deal with the issue of creating effective problem sets elsewhere, but in addressing the question of creating early success, I think it is impossible to make beginning problems too easy. Even a problem that is a direct logical consequence of a single definition is not out of the question. If multiple students get such a problem, you know that the point of logic has registered and the success snowball has started to roll. If the class does not get the problem quickly, you know that their "logic kit" needs to be enhanced and that an attempted remedy is in order.

Assuming that one succeeds in supplying adequate tools and some students experience early success, we come to the problem of dealing with the subcultures within the class. *Students develop mathematically at different rates.* The point in time during a semester

[3] See Appendix IV.A, "Policies" and "An introduction to doing mathematics."

[4] See Appendix IV.B, "Notes on logic and sets."

at which a student begins to experience success is inconsequential so long as the student eventually makes mathematics. The students who pick the cherries at the beginning of the course are not necessarily the best students (although they may be); a Moore Method instructor must constantly remain aware that a student may be making the transition from passive to active learning even deep into the semester and attempt to find cues that enable encouragement. One of the critical experiences early in my teaching career came when a student dropped my course at about the halfway point. She was discouraged, I presume (she never came to talk to me), because she had not yet made a proof. On the other hand, I was delighted at her progress. Each thing she did showed greater insight than what she had done previously and she had not duplicated logic flaws. I had no doubt that she would be a top student by the end of the course. She was clearly learning, but because I did not find ways to encourage her and get her to see how well she was progressing, she apparently measured herself against the most successful student in the class to that point as the standard and quit. Ever since then, I have made a point of finding something to praise about every presentation (finding ways to do this without being patronizing is not necessarily easy) and I make a special effort, when a problem is solved, to point out during the recap where ideas from previous attempts live in the successful argument. I try to appear really eager when the turns of "struggling" students come up and act crestfallen if they do not present. However, I *never* go into accusatory mode and I *always* try to act as if I believe they are working by making conversation about the problems that are available.

It is in deference to the difference in development rates that I do not commit to a finished problem set at the beginning of a course. I hand out the notes a few problems at a time so that, when I observe an opportunity to build on an idea that did not succeed, I can tailor a problem or problems to clarify what its impact might be. This also enables me to incorporate their vernacular, when appropriate, in definitions or for techniques or theorems. For example, in an algebraic structures class, existence of a one-generator subgroup within a group became a theorem known as "Fran's Run" in honor of the student who first articulated a process by which it could be built (by the way, she did not make it work!) because that was what the class called it. In a similar vein, the existence of a countable dense set in the place-value model became, in one class, "The Bead-Chain Theorem" because that was the way the class began describing a construction made early on that was used as the key idea in the proof. Opportunities to reinforce the notion that the mathematics being made belongs to the class usually present themselves routinely and often. Use them. It is fairly common for a student's argument to contain a portion that can be extracted and stand on its own as a theorem. Point such things out and add such theorems to the list of completed theorems; it is a theorem of the class's making and increases the class's sense of ownership of the course.[5] Even something as simple as naming a theorem after the person who proved it can be a morale builder.

Despite their usefulness as morale builders, the techniques described above are just social amenities meant to enhance the sense of student ownership of the work. The primary substance for nurturing an appropriate culture lies within the notes. My goal is to have, at any point in time during the course after the first two weeks, problems available that the students whom I perceive to be the most slowly developing should, according to what

[5] Such problems can also be stashed for possible use in future offerings of the course.

I have observed, be able to do if they can do anything, and problems available that will stretch the students whom I perceive to be the most rapidly developing. The culture will develop in a positive way so long as students solve problems and are acknowledged for doing so. Our job as a culture facilitator is to create contexts for success that make students, individually, reach a little farther than they did in their previous successes.

In summary, a culture will develop in every classroom. I want to optimize the chances that it will develop around the making of the mathematics. Critical to this is giving the students something to work with and creating some form of early success in the course. Subsequent to this, paying careful attention to what each student does can signal ways to abridge the problem sequence on the fly to fit a class and the individual students in it. Social devices can enhance morale.

(Mahavier) Addressing the goal of a liberal arts education, A. Whitney Griswald (Holt, 1965) wrote, “The purpose of a liberal arts education is to expand to the limit the individual’s capacity, and desire for self-education, for seeking and finding meaning, truth, and enjoyment in everything he does.” My goal for my students, and the culture of the Moore Method, is to provide students with a firm base of mathematical content while developing in them the ability, the confidence, and the desire to study new topics independently. If I must choose between emphasizing the knowledge base or emphasizing the desire and ability to study independently, I will err toward the latter since I believe, as Griswald’s quote indicates he believed, that the student who lacks a bit of knowledge but has the desire and capacity for self-education will progress, while the student with a broad knowledge base but no desire or capacity for self-education will stagnate.[6] I want my students to leave the academic setting with the belief that there is no topic to which it is beyond their abilities to contribute. Essentially, I want them to embody the mindset of the research mathematician, and I believe that this mindset is a result of the culture we create in the classroom and that it will benefit them regardless of their choices of majors and careers. The anecdotal evidence seems to bear this out, with hundreds of Moore’s descendants having pursued successful careers in academia, government, and industry. While the chapter “Why Use the Moore Method?” delineates numerous successes, the following quote (Renz, 1999) specifically addresses the culture in the classroom.

> *Martin Ettlinger, who took an MA with Moore before going on to graduate study at Harvard and a distinguished career as a plant products chemist, recently described the atmosphere in Moore’s classes as extraordinary. Every student’s ideas were listened to carefully and critically. No sniping or discourtesy was tolerated, but every idea was tested before being accepted. Ettlinger said the only other place he encountered this atmosphere was as a Junior Fellow of Harvard’s Society of Fellows.*

This research mindset in a student is born from success on a challenging problem or set of problems and infuses in the student a strong sense of inquiry to question the assumed principles on a subject, to explore new avenues, and to clearly communicate new discoveries and observations before a courteous, critical audience. Hence, the *culture* in the class-

[6] This does not imply that I believe that Moore Method courses cover less material than alternatives. See Chapter 10—Frequently Asked Questions—for a discussion of this topic.

room that I wish to develop is one that fosters the development of these abilities.

My perspective on culture is naturally biased by my own educational experience. Before entering graduate school from industry, I spent three months traveling the states by motorcycle, camping, visiting friends and family, reading and thinking. During that tour as I would contemplate graduate school, and in particular teaching, I would imagine Socrates, Plato or Pythagoras with their pupils sitting cross-legged in a circle under a tree, each engaged either by asking questions or working on problems. While earning my master's degree, I found a different model – professors lecturing to students taking notes at a furious pace, the same model I experienced in many of my undergraduate classes.[7] I often wondered, if most of these students are absorbing the materials either from the text or from their notes, then what is the role of the professor? Wouldn't a good text, DVD, or web-based course with the ability to review the lecture multiple times be as effective? I later discovered that this question was addressed in the Northern Polytechnic Experiment (McLeish, 1968) where one group of students attended a lecture while a second group reviewed the text of the lecture. After testing both groups, McLeish concluded that "… It is also clear that reading the text is more effective than listening to the lecture, if, as in this case, equal time is available for the "readers" as for the "auditors," and that minimal context for relevance is provided." On the same subject, Martin Anderson (Anderson, 1992) wrote, "The classroom lecture is a one-way flow to the student and has limited value, perhaps not much more than watching a good videotape. The two-way flow between professor and student is where the real teaching is done. This interaction takes place when there is classroom discussion, a dialogue between the class and the teacher where students learn to test their own powers of thinking, presenting ideas and defending them." The picture of the interactive model that I naïvely imagined prior to graduate school has remained with me and is the model after which I build my classes. In short, this is the culture I want to promote; one that has students actively engaged in, working on, presenting, and questioning a significant body of mathematical material.

> "The teaching method (the instructor treating us as if we could prove theorems before we could)."
>
> Anonymous evaluation response to "I believe the most effective part of his course was…" from Real Analysis II.

The culture I strive for accepts the students at whatever level they are, assesses this level carefully by providing problems at a wide level of difficulty early on, and keeps them working on significant, nontrivial problems with a well-defined content and theme. As material is presented, questions from students are not considered as issues to be addressed by the instructor, but as avenues to be explored by the class in the hopes of discovering new ideas and generating new questions. The culture treats the students not as they are, but as they can be. The culture embodies the ideal that Johann Wolfgang von Goethe advocated when he wrote, "When we treat man as he is, we make him worse than he is; when we treat him as if he already were what he potentially could be, we make him what he should

[7] With hundreds of students in biology lecture sections at Auburn requiring attendance, I would occasionally pay students to sit in my assigned seat, K–13, rather than attend class, but graduate school was too small for this tactic.

be." The culture allows time for exploration, and has every student involved and working on some problem with the goal of presenting it to the class at some future date. The presentations are communications of the underlying heart of the problem along with carefully written or verbalized arguments so that other students attain an understanding both of the correctness and the intuition of the problem. The materials are adequately challenging so that the students are proud of their successes and presentations. The culture supports extensive student-teacher interaction both inside and outside the classroom. The culture creates students who are independent in their approaches, wanting to solve the problems "their way," and the instructor should vociferously protect and guide the students' independence of thought by avoiding the presentation of "better" or "quicker" or "alternative" solutions either by the instructor or other students. The culture promotes the notion of students' ownership of the material, as well as ownership of the responsibility for their own educations. The culture may create a naturally arising healthy competition due to multiple students all wanting to master the same material, but should also promote a relaxed and enjoyable environment in which to question and to learn.

Annie and John Selden (Selden & Selden, 1993) describe quite succinctly the culture I strive to avoid. In discussing the four schools of thought that influence mathematics education research today, they write that while there are both conflicts and consistencies among these four schools,

> *…they might even agree that for various reasons, not necessarily under teachers' control, mathematics is now often learned in small, isolated bits, which tend to be computational or procedural, devoid of conceptual understanding, and largely useless in applications requiring much originality.*

Halmos (EAF, 1999) refers to the Moore Method culture as "(students) doing research at their level" and Peter Renz (Renz, 1999) sums this culture up quite clearly when he writes, "Motivate what is to be done. Let the students discover how to do it. Have the students present their results in good order before a critical but friendly audience."

One of the aspects of my classes that I find most refreshing is a direct consequence and an illustrative example of the culture in the classroom. This aspect is the ability of the class to proceed in my absence.[8] For fourteen years while teaching classes from freshman to graduate level, I have consistently traveled to meetings while my classes covered new material with good results. Each student is required to email me a report of the class day(s) I miss. At a minimum, the report must include which problems were presented, who presented them, and who raised questions. Responses will range from a numbered list of problems alongside the presenter's name to lengthy reports describing the techniques of proofs and who raised what questions. From these emails, I quickly piece together the progress of the day. These messages also let me know if there is an issue that a student feels needs addressing, but which was not addressed in class. This happens occasionally due to time constraints or the personalities of the students. Upon returning to a class where this has occurred, I'll say that I am particularly interested in the proof technique used on a certain problem and ask to see it again. More often than not, the class agrees that a cer-

[8] The type of whole-class independence required is described nicely in D.L. Finkel's *Teaching with your Mouth Shut* in his chapter entitled, "Refusing to 'Teach'."

tain set of problems has been properly presented and we move on to new material upon my return. Often students already have determined which problems will go on the board during my first day back. This feature is due to the established culture in the classroom of student responsibility and student ownership of the course. The culture makes them very comfortable with presenting their own work and with validating the presented results of their peers by questioning. They are proud of their work and willing to display it not just for the sake of the instructor, but for the sake of their own education and the education of the other students. When students solve a problem, it is not a slightly modified example from a text, rather it is their own creation and they are proud to display it, and anxious to see if it meets with class approval.

Evaluations often contain statements supporting this cultural difference.

> "I liked how the class was very interactive and we were highly influential in our overall grade and learning experience."
>
> Anonymous evaluation from a Calculus II course taught Moore style.

This student is not an anomaly. Statements like this appear in every set of evaluations, and are representative of the attitude that *they* are responsible for their own education and for their own grade. Developing this culture in the classroom is as easy as treating students as though they were already the mature research mathematicians we expect them to become.

How do you know if the culture in a Moore Method class has been achieved? That's easy. Do freshmen decide to minor or major in mathematics? Do majors decide to present at regional conferences? Are the students occasionally producing original mathematics? Are the students in the classroom before and after class talking with you about the math-

> The first course I taught using the pure Moore Method from notes developed completely on my own was trigonometry. The course had an interesting conclusion. On the last day of class, there were several problems outstanding, as is normally the case. I wanted to share with my students something about the Moore Method as the issue of the methodology's history had not been raised during the semester. Furthermore, on the previous day there had been little sign of progress on the outstanding material and I had that "we're about done" feeling from the class that one gets when students are tired and facing exams at the end of a long semester. So I brought a copy of the video *Master of the Game* (EAF, 1999) to class and prepared to present it. As I negotiated with the video equipment before class, one student came into the classroom and after a brief exchange regarding the presence of the video player, she said to me in a tone that was forward, yet respectful, "You told us that what we had to present was always what was most important to you and that anytime we had problems to present you would see those first." I responded that I had indeed said that (well, at least it sounded like something I probably had said) and asked if she had anything to present. She said she had problem 142 and I pushed the video player to the side and the class participated in a very nice presentation of this key problem. Not wanting to be caught off guard twice (really, it had never happened before ;>), I asked if anyone had problem 143. She did and she presented it. She continued until the class period ended and I don't recall if we had to use some of our examination time for her to complete the problems, but I do recall that she fully and correctly presented the last four problems of the problem set. We never did see the film, and I think the class was better off for it. Certainly the student was.
>
> Mahavier

ematics of the course? Are they stopping you in the hall to ask questions? Are they in your office asking questions? Are your students absorbing and regurgitating material or are they intimately involved in the pursuit and understanding of the underlying principles of the material? Do your students ask you questions that you do not know the answer to? Have the students taken an active stake in the material and the class? If the answer to most of these questions is yes, then the desired culture has been achieved.

“That student is taught the best who is told the least.”[1]

R. L. Moore

5 Development and Selection of Materials

This chapter addresses the generation of materials to be used to support Moore Method teaching. Development and usage of materials by the prospective user is treated from various perspectives, including

- guidelines for writing your own problem sequences,
- adapting textbook treatments,
- adapting pre-existing Moore Method course notes,
- modifying materials to fit the realities of a particular class,
- addressing learning-theory issues, and
- obtaining support when using another’s notes.

These perspectives illustrate principles that may guide the construction of problem sets, resolve the question of authoring notes or adapting pre-existing notes, and provide sources for course notes that have been written for use in Moore Method classes other than your own.

The essays will likely offer the greatest benefit if read in the order presented, especially for the reader entertaining thoughts about using the Moore Method for the first time. May’s essay is both highly practical and succinct in its outlook. Mahavier deals, in some detail, with making the decision whether to author notes or use a pre-existing problem sequence. He then segues into guidelines for penning materials. Coppin focuses exclusively on authoring one’s own notes, addressing issues from learning theory and illustrating how course notes can address these theories directly. Parker writes from the viewpoint of generating course notes, then adapting them consequent to classroom dynamics. When taken in order, the essays progress from a general palette to experience-based specifics and the union of what precedes each essay provides a richer context from which to consider what follows.

(May) There are several ways to gather materials for a course that you wish to teach by the Moore Method. Some of them follow.

[1] *Challenge in the Classroom*, documentary of R.L. Moore, Mathematical Association of America, 1966

1. Take a course from a practitioner of the method.
2. Establish a mentoring relationship with a practitioner of the method.
3. Obtain notes from the Educational Advancement Foundation's Legacy of R. L. Moore Project.
4. Obtain notes from the *Journal of Inquiry-Based Learning in Mathematics*.
5. Assemble problems from existing textbooks.
6. Develop notes from scratch.
7. Attend a workshop on inquiry-based learning.

Some comments about the items might be helpful.

1. Take a course from a practitioner of the method. This is the best way not only to develop a set of materials for your course, but also to immerse yourself in the method. I am fortunate to have spent my graduate life in the classrooms of some of the best practitioners of the Moore Method. I absorbed the method simply by being around people who did it as effectively, and seemingly as effortlessly, as they breathed. This does not mean, however, that someone who has not had such an experience cannot become an effective Moore Method teacher. On the contrary, the method is so appealing and effective that a single exposure to it is sufficient to produce a solid new Moore Method practitioner. Into the bargain, you will end your experience as a Moore Method student with a complete set of notes for teaching your own course. If the course that you take is taught by a first-generation Moore student, so much the better. While finding such a situation is, of course, becoming increasingly difficult to do, any of the coauthors will happily assist you. You will probably have better luck in locating an n^{th}-generation Moore practitioner with n at least two, or a practitioner such as a "graduate" from the workshops described under item 7 below.

2. Establish a mentoring relationship with a practitioner of the method. Any Moore disciple who is willing to mentor you in the teaching of a course in your chosen subject would be appropriate for such a relationship. For additional aid, contact one of the authors of this book or go to the Legacy of R. L. Moore web-site and click on "Mentoring."

3. Obtain notes from the Educational Advancement Foundation's Legacy of R. L. Moore Project. EAF publishes sets of notes in many of the courses that you might want to teach. You will arrive at them by going to the Legacy of R. L. Moore web-site and clicking on the link, "Literature, Videos, and CDs." The notes you will find there, like those obtainable from the *Journal of Inquiry-Based Learning in Mathematics*, are, for the most part, ready to use; that is, you could begin immediately working through them yourself in preparation for leading a class through them. [2]

4. Obtain notes from the *Journal of Inquiry-Based Learning in Mathematics*. JIBLM has more than 1,800 pages of materials available free for download and use. It contains the only collection of refereed Moore Method materials available on the web.

5. Assemble problems from existing textbooks. When I was preparing to teach real analysis, number-theory, and abstract algebra for the first time, EAF did not exist, there was no formal network of Moore Method practitioners, and the internet had yet to be put

[2] Editor's Note: EAF and JIBLM now have a Memorandum of Understanding in place whereby notes submitted to EAF in the future are referred to JIBLM first. For details, contact the Managing Editor for JIBLM.

into place. So I did what seemed natural: I "lifted" materials from existing textbooks. With regard to number-theory, I asked a senior member of my department for a recommendation of a text. I didn't say anything about wanting to use it simply as a source of theorems for a Moore Method course; I simply asked for any suggestion he had about a good book out of which to teach. He suggested one by J. E. Shockley. I looked at it and almost immediately agreed with him that it would be an excellent choice – for piracy. I proceeded to sit down and work my way through it. Often as I did so, I would partition one of Shockley's theorems into two or three, as the needs of my future students seemed to dictate. I increased the number of my theorems still further by placing, as theorems in my notes, lemmas that I had used to prove some of Shockley's theorems. When the semester to actually teach number-theory arrived, I had assembled a set of Moore-style notes sufficient for at least the first month of class.[3] I was ready to start, and confident that I could stay far enough ahead of my students to complete the semester. The results were pleasing. One of the greatest benefits to me in preparing my own notes is the opportunity that the activity gives me for taking a hard look at what is important about the subject of the notes. For example, as I was preparing my first Moore Method real-analysis course, I stumbled upon an idea that, I believe, improves the course in either a Moore Method or a lecture format. Because the course was a terminal one-semester one, I wanted my students to end their experience as close as possible to the finish-line established for it by my department's (lecture-format) syllabus. One way to accomplish that, I concluded, was to omit the topic of limits *per se*. I reasoned that my students would have plenty of engagement with the ideas surrounding limits – epsilon-delta proofs, primarily – as they worked their way through continuity and differentiability. In addition, although the concept of limit can be a fascinating study in its own right, its primary function in real analysis is to enable the student to deal with continuity, differentiation, integration, and the convergence of sequences and series. (The historical record shows that the modern, rigorous definition of *limit* succeeded, rather than preceded, its application to these concepts.) Consequently, limit, as a topic in its own right, was expendable. Not only have I never regretted that omission; I have continued it ever since.

6. Develop notes from scratch. When no single text exists for the subject that I want to teach, I sometimes take it on as a topic of research. I assemble as many of the relevant books and articles on the topic as I can, study the topic for myself, and produce, from scratch, my own sequence of theorems. That happened recently with affine transformations. Ever since encountering them in graduate school, I had been fascinated by these almost-but-not-quite-linear functions. Finally, as a sabbatical leave approached, I chose, as my sabbatical project, the studying of and the construction of a course in affine transformations. As I conducted my research, I kept my eye on my intended audience, graduate and advanced undergraduate students, and accordingly exercised care about stating and sequencing my definitions, lemmas, and theorems. By following this procedure, by the end of the sabbatical I had produced a set of notes that was ready to be used in a Moore-style course. I have taught the course once, as a special topic for advanced undergraduates.

[3] Most Moore Method teachers advise against handing out the entire term's set of notes at the beginning. Rather, one should hand out only enough to last the students for a week or so. Then, as the students prove their way through this first portion of the notes, pay attention to their skill at handling the level of difficulty built into the problems and, if necessary, adjust accordingly the level of difficulty of notes subsequently to be handed out.

"As an educational experience, this class was a great preparation for graduate school. The experience at the board sharing and teaching theorems to others helped me to feel more comfortable at the board. Also, having just finished an advanced linear algebra course, I felt this reinforced the material in a new light."

Anonymous evaluation from MATH 490,
Discovering Affine Transformations.

7. Attend a workshop on inquiry-based learning. During each of the 2006 and 2007 summers, Dr. Stan Yoshinobu of California State University, Dominguez Hills, implemented an inquiry-based-learning workshop dedicated to the modified Moore Method. These conferences, funded by the Educational Advancement Foundation, used presenters well-versed in the method, including Parker. They were so successful, in the eyes of the participants as well as the organizers, that, as of this writing, plans are being made to continue the workshops into the foreseeable future. Attending one of them would be isomorphic to, and easier than, following Suggestion 1 above. In addition, it would yield at least the beginnings of a set of self-produced Moore Method notes, and would almost certainly lead to a mentoring relationship of the sort described in Suggestion 2.

Conclusion

Teaching a Moore Method course allows me to optimize my two favorite professional activities: studying and teaching. Assembling materials for and teaching a Moore Method course never fails to lead to fascinating questions. Attempting to answer those questions always yields new insights – new not only to my students and me but also, in some cases, to the larger mathematical community. Even when the insights are original only locally, however, teaching in the spirit of Moore constantly renews my enthusiasm and appreciation for mathematics and for students, among whom I hope always to be counted. The Moore Method provides me with a wonderful opportunity to realize that hope.

(Mahavier) Your choices are simple: write your own materials, seek out materials written by others, or use a text. Provided you are teaching in an area in which you are particularly well versed, I believe the best approach is to write your own materials. Having said this, I have both taken and implemented courses where each of the three approaches was implemented with success. Hence, while there are no rules about where the materials come from that define a successful course, I strongly believe that a well-written set of notes can really make the class. A set that starts off where the students are, carries them at a fair and challenging pace through the material, and requires them to construct the key mathematical ideas of the subject will make for a true Moore Method course. Few books can allow this because most present the key ideas. Authors presuppose that the level of every student is the same; thus content level varies little from one chapter to the next. In the Moore Method, to increase the probability of success of less-prepared or less-mature students, early material is typically less demanding than later material.[4] I am not advocating a world without mathematical texts. I have benefited from many of them! But until our students are masters of creating mathematics, texts that display pristine proofs will hinder their development.

[4] This calls into question the common supposition that the Moore Method works best with the better students, which I dispute. See Chapter 10—Frequently Asked Questions—for more on this topic.

Reading someone else's proof of a theorem in a book won't train you to prove theorems any more than reading a book on how to rock climb will teach you to scale a rock face. Because there are advantages and disadvantages to each approach and there may be cases where the use of a text is appropriate or mandated, I'll address all three choices. Let's tackle creating your own materials first.

Write your own materials

Writing your own materials assures you are working in an area that you are overly qualified to teach. If you can write your materials without looking to the literature, then you are in a particularly good area to start the method, because you will be able to spend much of your preparation and class time addressing the mechanics, and the pedagogical and psychological nuances of the class. When using others' materials, a certain amount of time is spent thinking about whether a given problem or theorem is true as stated or working through the material and trying to determine how someone else's exposition of this material dovetails with your own conceptual framework. Furthermore, just as writing a research paper solidifies your understanding of some of the finer points of the subject, writing your notes will cause you to think about the inter-dependence between problems that you might not see merely working through or teaching out of another's notes. A nice perk of writing your own notes might be a publication. In addition to papers, the following are a few of the many books that have resulted from Moore Method courses: (Burger, 2000), (Clark, 2008), (Hale, 2003) (Hodge, 2005), (Moise, 1963), (Moise, 1972), (Moore, 1962), (Neuberger, 2000), (Penney, 1972), (Schumacher, 2000), (Stopple, 2003), (Wall, 1948), (Wall, 1963), (Whyburn, 1942), (Youse, 1965), (Zettl, 2005).

Here are some guidelines for developing your own notes.

- Approximately 150 significant conceptual statements (axioms, definitions, lemmas, problems and theorems) seem optimal for a three-credit-hour, one-semester course. For example, my Trigonometry notes have 163 items and my Analysis notes have 140 even though I did not consider any such number as a goal during development. My Calculus III course has 163 problems for a four-credit-hour, one-semester course. These are all problems and I introduce definitions and topics via mini-lectures in this course, so that it is closer to problem-based than to Moore Method. In this class, numerous students place problems on the board at the same time and I introduce new concepts via examples. Teaching in this manner is a concession to my preferred mode of instruction due to the size of the classes and the fact that the course services the College of Engineering and has a rather firm and full syllabus.
- The beginning should include a brief introduction to the topic. This is simply a paragraph or two that tells the students where they are going mathematically. They will be working among the trees; you know the whole forest. Draw them a map of where they are going and reiterate the key results of the course throughout the semester. The freely downloadable "Trigonometry" (Mahavier, 2007) includes an example of this.
- Syllabi, encouragement, and any other notes should be separate from the mathematical thread to emphasize that *mathematics* is what is important. Several handouts that I share with students are included in Appendix II. These address grading, guidelines for how students might approach such a class, and guidelines for student conduct in class.

- The sequence should start at an elementary level. The first problems in the sequence should be so elementary as to engage the class successfully on the first day. Student success in the first two days is critical to a successful class and the first week can be frustratingly slow as you learn exactly where your students really are! Be sure and start them where they are, not where you think they should be. As an example, consider the first and last page of "Analysis" by Mahavier, W. T. < Mahavier W. S.[5] (Mahavier, 2009) included in Appendix II. The first page has students showing that zero is a limit point of the open interval, (0,1). It's hard to imagine a simpler analysis problem. The last page has nothing but a multitude of analysis theorems—the core of the course. Starting at a low level allows time to build the students' proof-writing skills and confidence in their ability to *do* mathematics. Once that skill is established, one need not break larger theorems into smaller pieces, as the students will do this automatically, creating their own lemmas as needed.
- The sequence should include at least one early problem that is quite challenging to assure there is something for every student. Some students really are where they need to be, or perhaps even far beyond. Should you have a particularly gifted student who appears bored, you can say, "Have you tried Problem *X*?" Now you have enabled both the weaker students to have a fighting chance and given the more talented or better-prepared student something to sink his or her teeth into.
- Even if you have all the materials written out before the first day, don't pass them all out. Pass problems out a page at a time or, better still, write them on the board. This allows you to add material based on student interest, to add problems that students raise in class (Put their names next to them!), and to assure you do not overwhelm the students early on. If you give them 150 problems on the first day, tell them that these fifteen pages are all they have to cover, and tell them that they have to do all of them, some will be concerned about the lack of expository material in the notes and others may become discouraged if they struggle with the first few and can't even read the last page.
- The first few pages of mathematics should be perfect. Logical and typographical errors can damage a class culture. Until you have established a good rapport with the class, such mistakes can shake the confidence of a class in the instructor. The student who trusts you, spends four hours on a problem, and then finds that there is a mistake in the problem may tune out for the rest of the semester. The flip side of that coin is that the student who catches your mistake and provides a counter-example to a statement cannot be praised too highly. Because you are quite possibly asking them to do more and harder work than they might do in another section on the same subject, their enjoyment, trust, and confidence are the first things you must develop. High-quality notes help establish that trust. If you let the students know you are developing the materials so you can teach a better class, then it is my experience that the students will fight side-by-side with you to get the notes in the best possible shape for the next class. I have gone so far as to give points to students for finding mistakes in the notes.

[5] The ordering (<) implies that the original notes came from my father and I have modified them significantly. They are definitely not a coauthorship, as my father may not agree with all of my modifications and would not use my version where he teaches, just as I needed to modify his to meet my students' needs.

The culture that "we, the students, are helping improve the notes for future classes" can further motivate the students.

- The language should be used carefully, consistently and sparingly. This is not unique to the Moore Method. Askey (Askey, 1997) writes, "Do not lie to your students… . Words are important and their meanings should not be changed without very good reasons." Knisley (Knisley, 1997) writes, "Good theorems are the stuff of graduate courses. Good definitions are the stuff of introductory calculus." Krantz's article (Krantz, 1999) speaks to the careful use of language in a lecture-style calculus course. All elementary mathematics courses have the opportunity to develop mathematical ability, yet we must speak and write in mathematically precise language and, more importantly, train our students to do the same if we expect them to embrace the beauty of our subject. Most departments have "bridge" or "transition" courses between elementary and advanced courses to address just this issue. Such courses are intended to train students to read, understand, produce, and write mathematics. Given that elementary courses such as calculus often constitute between nine and twelve semester hours of the mathematics taken by students majoring in mathematics, the sciences, or engineering, I ask, would training the use of correct language in early courses facilitate the transition to higher mathematics and remove the need for transition courses?
- Don't confuse a "problem-based sequence" with a "Moore Method sequence." This is a delicate distinction. A problem-based sequence is a collection of problems that leads a student through a topic. A Moore Method collection leaves large enough gaps that students must actually discover some of the fundamental ideas and techniques individually. When writing, the distance between problems is important and is dependent upon the level of the class.[6] Gaps are smaller at the beginning of the notes and larger at the end. What might be a problem-based set of notes at the junior level might be a Moore Method set of notes at the freshman level. Consider the analogy of crossing a stream. If you want to teach hikers with heavily laden packs to cross a fast-moving stream successfully, you might place the rocks so close together that crossing the stream was nothing more than an exercise in walking. Everyone succeeds; no one learns. If you place the rocks too far apart, each hiker falls into the stream and loses confidence, most choosing not to return. The rocks must be placed just far enough apart that each hiker is challenged but successful. And occasionally you might need to add a rock for one hiker, but remove a rock for another. This analogy supports the student-teacher interaction that mathematics educators have identified as so important to successful educational practices.
- Don't restrict students to your way of thinking. Breaking a powerful theorem into "five easy lemmas" preceding the theorem is tempting, but forces the students to follow *your* path and denies them the opportunity to develop useful lemmas and insights while "taking the problem apart." If you believe a problem is simply too hard for the level of the class, create supporting problems and place them earlier in the notes but not in such a way as to tie them to the problem at hand. That way, the

[6] These gaps are closely related to the Zones of Proximal Development that Coppin refers to and defines in Axiom 2 of his essay in this chapter and in the chapter "Why Use the Moore Method?"

> After a student independently discovered an algorithm equivalent to Cantor's Diagonalization argument to show that the real numbers are not countable, students were chatting after class. The presenter apologized for the lack of coherence of the presentation and another student exclaimed, "Are you kidding?! I couldn't wait to see what was going to happen next; it was like watching a movie!"
>
> From the second semester of Analysis, Spring 2008

student may use the previous results as lemmas or may pursue his or her own path. Alternately, wait until a student attempts the problem. If he or she struggles, then facilitate the extraction of a lemma based on the language and techniques of the student's attempted proof which, if solved, will yield the desired result. Do so in a way that credits the student with the idea. You might say, for example, "Let me see if I understand this. Are you saying that if you could show P implies Q then you would be able to finish the problem? I'd very much like to see that." You might say this even if the conjecture were false or would not resolve the problem, as long as it was what the student was striving to state. The goal is to allow the student to follow his or her path because unsuccessful paths may yield as much knowledge and understanding as a path straight to the result. Your path may not be the best path for students and allowing them the freedom to follow any path to the result is important. There are no "right" paths in original research.

- Intertwine multiple threads. Most texts are compartmentalized in an effort to break the material into bite-sized pieces for the students (Schoenfeld, 1988). One disadvantage to this is that students mentally compartmentalize the materials and fail to recognize connections among the topics. When students are working through the material independently, we wish to maximize their potential for success. Keeping multiple threads intertwined in a sequence assures that if a student is not having success on one particular thread, he or she may well latch on to, and succeed, in another thread. For example, if you are writing a standard trigonometry sequence, then both the graphing of the trigonometric functions and the solving of identities can be treated simultaneously. This not only optimizes the chance of success for the individual students, it also keeps the course interesting by showing the interactions among the different threads. Suppose an identity we have stated in this thread is that $\sin(x + \pi/2) = \cos(x)$ for any number x. The student working on the "graphing" thread may demonstrate this graphically, creating the discussion, "Is this picture really a proof?" The student who has remained focused on the unit circle may well return to similar triangles to show this. The latter approach provides a natural springboard to the summation identities and if I saw this proof technique, I would immediately state that identity in the hope that the same strategy would be generalized. When the identities are sprinkled throughout, I have been astounded by the number of different ways students have found to solve them. And it does not matter in calculus how the student came to an understanding of the validity of the identity, it only matters that the student has a deep conceptual understanding as opposed to a superficial memorization of symbols. As a second example, in analysis the ties between limit points and convergent sequences can be made more concrete by running these threads in parallel rather than contiguously.

- Omit examples. When introducing a definition in the notes, either precede or follow it with a problem statement that generates whatever example it is that you would naturally show in class. This allows the students to construct the examples just as one might present them in a lecture.

> "I like the way that the students present the material; students explain things better than teachers."
>
> Anonymous Comment from Calculus Student Evaluation

- End with a challenge. The last pages should be nothing but challenging problems and theorems. By the end of the course, the students are usually pushing to get through the material. Make sure there is too much to complete and that there is plenty to leave them with.

> A colleague (Timothy H. McNicholl) met one of Moore's former students in the lobby of an automotive dealership a few years back. The student was in his late sixties and had not followed a mathematical career. He asked my colleague if he had a copy of Moore's book on Point Set Topology so that he could obtain a solution to one of the problems from the class he took with Moore. Following the exchange, the two continued to correspond about the problem for several months. What could be a better compliment to a teacher than to have a student remember problem statements forty years after leaving the classroom?
>
> Mahavier

- Include a summary. My experience as a student of the method led me to include summaries to the notes. In the chapter entitled "Why Use the Moore Method?," Parker describes how he felt unprepared for the GRE after studying in classes using the method. When I left my undergraduate analysis class, I did not know that I knew the Heine-Borel theorem, which I had proved in a Moore Method class. Nor did I realize that I had proved that the real numbers were uncountable, although I had proved: "If p_1, p_2, p_3 ... is a sequence of distinct points in the interval [0,1] then there is a point in [0,1] that is not in the range of the sequence." In each of Parker's and my cases, we graduated knowing more than we realized! To this day, I do not share the language that other mathematicians share and I find it frustrating at times. Would I trade that for the training that has led to what I consider to be a successful and very enjoyable career? Of course not. But I strive to give my students both the training *and* the common language by concluding my materials with a summary of what the students have accomplished. This includes the names associated with significant theorems, the way in which the course fits into the larger mathematical structure, and even a list of the consequences of the results they have proved. It is my hope that this will help them see a bit more of the forest, given that most of their time has been spent studying the trees. In Trigonometry, I have the students write this as an essay before providing my own essay summarizing the course. We then use part of the examination period to discuss the various essays and draw a larger picture through the class discussion.
- Keep perfecting your notes. I never stop refining mine. When I develop a set of notes, it takes several iterations to create a set that I am willing to disseminate or mentor other faculty on. The first time through, I hand-write the notes and pass them out page by

page. The second time through, I reorder, add and delete problems as I typeset them before handing them out page-by-page. The third rendition always contains the most material and the class progresses most smoothly through this set. That is not to say that the first two courses are not successful. In fact, the first time through is often very exciting for both teacher and student as so many interesting conversations are started *because* the notes are not perfect. It is much more like doing research because no matter how meticulous we are in preparing our notes, problems are occasionally not correct as stated. Still remember bullet 7 and attempt to have the first few pages perfect! Be careful with labeling. I label statements as lemmas, problems, questions, or theorems, being sure to let my students know not to guess the difficulty by the label. Some theorems are easier than preceding problems and lemmas, since the heart of a theorem may reside in a previously presented problem or lemma. An advantage of this labeling is that when reviewing the material, theorems are the key results of the course that I would like the students to recall. Problems are typically examples building understanding and lemmas often contain key techniques that I would like the students to develop.

Use someone else's notes

Using another's notes has its own set of advantages and disadvantages. If one knows a particular set of notes has been used in a comparable course at a comparable institution, then not having to develop materials allows full concentration on the classroom and the method. I have mentored at least ten faculty members who have used a set of my notes verbatim. These individuals have claimed, perhaps out of kindness, that having a seasoned practitioner to call upon has been valuable. Not only did they ask questions about the notes and the reason for certain problems in the notes, but they reported on and questioned the day-by-day progress of the class. "What should I have done here?" "The students aren't working today. What do I do now?" "They solved Problem *X* today. Are we moving at a reasonable pace?" Other questions concerned the mathematics, grading, testing, classroom practices, individual student issues, and more. Some practitioners of the method would say that the development of the material is an integral part of the method, and I am inclined to agree. Still, for the uninitiated, it is best to learn to walk before trying to run and that is, in fact, the path that I followed in moving toward teaching using the pure Moore Method.

Within two years of my first university teaching, I was sending students to the board for at least twenty-five percent of class time without any thought of whether this was inquiry-based, discovery-based, or problem-based. At the time, I didn't even know such words, but the aspect of involving the students actively came naturally to me. Still, I was not sure that a pure Moore Method approach would be as natural and it was somewhat risky, given my record of strong student evaluations. To minimize the challenge, I taught my first Moore-style course out of notes that came from my father. With the notes in hand, I was able to spend most of my time developing my own style in the classroom because I trusted the material. Of course, at the end of the semester, I had given the students problems that were not in the notes, moved material around and modified the notes considerably. Having taught the course many times now, the notes are more or less as I want them for my own needs, but I still modify them every semester. The email mentoring by my father was important because I would often ask what he was driving at in a certain problem, and without his guidance, I might have omitted or de-emphasized a proof technique that became useful later in the thread. My students' and my own benefit from the mentoring, convinced me of the importance of mentoring for the uninitiated. My use of relatively polished notes the first time persuaded me that, at least for some people, using another's notes might be a good first start. Other mentoring experiences I am familiar with by Gordon Johnson, Ben Fitzpatrick, W. S. Mahavier, Charles Coppin, and Tom Ingram have further supported the positive effect of the three key items from this thread: mentoring by a seasoned faculty member, using time-proven notes, and the ability to modify and adjust those notes to the specific needs of the teacher, class, and institution. For first-hand experiences with mentoring, see Roe (Roe and Ingram, 2002) and Stallman (Stallman and Mahavier, 2001).

Because modifying others' intellectual property for use presents its own set of legal and ethical issues, I propose a system (mentioned briefly in an earlier footnote) similar to the open-source software movement. Notes should be freely distributed in LaTeX or TeX format and an ordering should be defined indicating where the notes originated. The only requirement of a person using and modifying a set of notes is that they title the notes with all the previous authors' names listed. The analysis notes that I adapted from my father's notes should be titled, "Real Analysis" by W. T. Mahavier < W. S. Mahavier. The ordering implies that while the original notes came from my father, I have modified them, perhaps significantly. They are definitely not a coauthorship, as my father may not agree with all of my modifications and would not use my notes where he teaches, just as I needed to modify his to meet my students' needs. In mentoring numerous people on the use of my "Mahavier < Mahavier" notes, I have discovered that a common disadvantage is that we each have preconceived notions on the development of analysis based on our own education. When using another's notes, you might be required to prove certain theorems from scratch when your training bases these results on having more powerful tools at your disposal than you currently have.

I have encountered a few drawbacks to using another's notes. Inevitably, they are not exactly what I want and the time spent working through another's notes could have been spent developing my own "perfect" set. There have been sets of notes that, after using them for a semester, I abandoned altogether, never having felt that I could modify them adequately to meet my needs. In short, you might begin using another's notes but ultimately, if you employ the method, you will end up developing your own or modifying those that you started with.

Use a text

Truth in advertising; I have never taught a Moore Method course using a text. I have taken one Moore Method course in which a text was used, and I have taught many modified Moore Method courses using texts. At the University of North Texas, Professor Paul Lewis taught the graduate analysis sequence using Royden's *Real Analysis* (Royden, 1988) and supplementing the text with his own notes on vector-valued measures. As students, we were given a list of theorems from the chapters and problems from the exercise sets that we were to prepare and present. This list, along with a problem sequence on vector-valued measures, constituted the materials that we were allowed access to. Even when presenting proofs of theorems from the body of the chapter, we were to be fully prepared to address any questions from Dr. Lewis or the class. Aside from having a text, the rules and structure paralleled that described in Chapters 2 through 4.

In the chapter entitled "Grading" I delineate a method which I call the modified Moore Method that I often use in conjunction with a book. In short, this method is a compromise between lecturing and the Moore Method that allows a combination of lecture and student presentations. It is a nice way to test the waters of having students present, while keeping both instructor and student in the safety net of the established norms in the classroom. I don't think the method is as valuable as more pure forms of the Moore Method. This too is discussed in that chapter. Still, the described method does enable one to use a mandated book and syllabus in a way that supports a more student-centered approach. Certainly, I have done this many times and it was this approach that originally moved me toward writing my own materials and teaching a more pure version of the method. For me, it is a significant improvement over lecturing (where I started), requires minimal modification of the grading and class structure, and sits well with more regimented departments. I defer further discussion of this to the chapter on grading.

Conclusion

Throughout my teaching career, I have used all three methods described above. In some classes, I use books but they play a diminished role as I write many of my own problems and de-emphasize the text. In other classes, I abandon the text completely, writing my own notes or adopting the notes of others.

(Coppin) My goal here is to present the principles that I have discovered in decades of writing materials for Moore Method courses. In the spirit of the Moore Method, I will give you a few axioms and some examples. You will teach yourself and rightly claim for yourself those principles that you discover. The essence of what I have learned is embodied in these three quotes.

> *There are two sorts of eloquence; the one indeed scarce deserves the name of it, which consists chiefly in laboured and polished periods, an over-curious and artificial arrangement of figures, tinseled over with a gaudy embellishment of words, ... The other sort of eloquence is quite the reverse to this, and which may be said to be true characteristic of the holy Scriptures, where the eloquence does not arise from a laboured and far-fetched elocution, but from a surprising mixture of simplicity and majesty, ...*
>
> Laurence Stone

> *Less is more.*
>
> Robert Browning

> *Everything should be made as simple as possible, but not simpler.*
>
> Albert Einstein

Paul Halmos said that when we write, we must write good English. I wholeheartedly agree! J. L. Kelley, a second generation student of Moore, said, "It is not enough to read the words—you've got to hear the music." Whether we are writing prose or music, similar principles apply. When we write, the words must say something. Our words must have coherence, continuity, and rhythm. Our writing will have a score and lyrics and they *must* resonate with each other. No less is required of those who write mathematics for a Moore Method class.

I submit the following axioms for consideration. Many teaching methods are a form of pedagogical micromanaging; however, the Moore Method is more holistic and bottom up teaching. These axioms are: study the masters, progress from simple to complex, calibrate the Zone of Proximal Development,[7] and write well. Remember the adage "keep it simple" when writing or selecting your materials.

Axiom 0. Study the masters

Years ago, I asked a colleague in English literature how I could learn to write well. Surprisingly, he said I should read good literature. I had expected that he would give me a list of rules. Of course, he was right. Who were the masters I "read?" I learned under Moore himself in his Math 688/689 *Foundations of Point Set Theory* at the University of Texas. Although his classes were different from Moore's, H.S. Wall's classes gave me a valuable experience base as well. In addition, early in my teaching career, I was given a set of notes in Linear Point Set Theory, which, I have always believed, came from Moore Method teachers. I taught from them countless times, making improvements over the years. *Creative Mathematics*, by H.S. Wall (Wall, 1963), was a great model as well. I have studied the movie concerning Moore's teaching method, *Challenge in the Classroom.* I learn some new twist to his method each time I watch it. I highly recommend it to you. Most importantly, it helps to be a reflective person.

What should you do? Find a good set of notes that work well and teach from them many times. Obtain a copy of the Linear Point Set Theory notes, the older the better. I have found *Synthetic Geometry* by R. F. Jolly (Jolly, 1969) extremely revealing. I highly recommend the book, *Foundations of Point Set Theory*, by R. L. Moore (Moore, 1962) as a source worthy of study for general principles on writing Moore Method materials. I am sure there are other sources worthy of your attention. Study why they work. Tinker with them. Some modifications will work like a charm and others will work as well as a Yugo; however, do not discard the latter until you have analyzed them and reflected on why they did not work. Learn from your experiences.

Study, teach and understand!

Axiom 1. Progress from the simple to the complex

The course you create should enable your students to climb to higher and higher levels of sophistication. Use your own notion of what sophistication means in this context. My own includes the ability to apply many different theorems, to deal with a concept with many different cognitive components, and to imagine. Of course, you want the topics you present to build one upon another. The topics you choose to study should generally strengthen the students so that they may have the intellectual power to move from one level to the next level where they can attempt more complex theorems or problems that require more

[7] The Zone of Proximal Development (ZPD) is a theory of social constructivism and was defined by Lev Vygotsky (Vygotsky,2006) as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers."

imagination. The good students will be the ones who will take the class to a new level. There, the weaker students may successfully attempt problems or theorems that use ideas modeled after concepts or processes learned from the presentations of the stronger students. I use words like "good," "weaker," and "stronger" advisedly. Who can really say which students are actually good, weak, or strong? Stories abound in Moore Method lore of students who appeared weak who later were discovered as strong. The reverse also occurs. Periodically, place in your materials a span of problems or theorems that will allow all students to consolidate what they have learned. This will help the weaker students as well. I demonstrate these principles by the following span of axioms and theorems from the subject of synthetic geometry.

For these theorems, the students should know "set," "is a member of," or "belong(s) to." In the spirit of the Moore Method, the reader should create proofs of the six theorems following this paragraph. In this way, you will see how a theorem builds upon a previous theorem, how students at different levels of ability prove theorems, and where weaker students progress on relatively level ground. I suggest that you prove each of the following theorems before reading my notes. Do each proof in at least two different ways. One approach could be that of a more sophisticated student and another method by a student who might be just getting by.

The words "point" and "line" are undefined. Lines will be sets of points.

Axiom 1. If L is a line, then L contains at least two points.

Axiom 2. If L is a line, then there is a point not on L.

Axiom 3. There exists at least one line.

Theorem 1. There exist at least three different points.

Axiom 4′. If A and B are two points, then there is at least one line that contains A and B.

Theorem 2. If P is a point, then P belongs to a line.

Theorem 3. If P is a point, then P belongs to two different lines.

Now, replace Axiom 4′ by the stronger statement below.

Axiom 4. If A and B are two points, then A and B belong to one and only one line.

Theorem 4. If P is a point, then there are two different points Q and R such that no line contains P, Q, and R.

Theorem 5. If P is a point, then there exists at least one line that does not contain P.

Theorem 6. If L and H are two different lines, then there is at most one point belonging to both L and H.

The following are some notes to focus your understanding of each of the aforementioned features of the sequence of theorems. Assuming you did not peek, you can compare your proofs with my observations.

My Notes:

1. The student has to apply all three axioms, admittedly making Theorem 1 a little harder starting point than desired. The proof that the three different points are *indeed* different can be bypassed. This is the teacher's call.

2. Theorems 2 and 3 have proofs that range between the two extremes. At one end is a proof that does not apply any of the preceding theorems but just applies axioms. At the other extreme is a proof that makes economical use of preceding theorems. At the lowest level, students can base or model their proofs of Theorems 2 and 3 on a proof of Theorem 1. However, they must apply a new axiom, namely Axiom 4', which is not used in a proof of Theorem 1. Thus, we have used something old and something new. Nevertheless, the proof of Theorem 3 is very much like that of Theorem 2.
3. Much as in Note 2 above, Theorems 4 and 5 have proofs that are on two extremes. However, weaker students can model their proofs on preceding proofs with a new wrinkle; that is, they must apply a new axiom in the form of Axiom 4. This reinforces the statement in Axiom 4 that for any two points there is a unique line that contains both points.
4. Theorem 6 is easy but is important for the fact that it again highlights the important distinction between Axioms 4 and 4'.

Axiom 2. Calibrate the Zone of Proximal Development (ZPD)

Think of ZPD as an intellectual gap that the student must jump cognitively if he or she will evolve to the next level of intellectual sophistication. Vygotsky's definition is "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance, or in collaboration with more capable peers." Vygotsky believed that a child would model a parent or another adult but would reach a point where the child would gain the ability to do things without help. Thus, he called the gap between what could be done with assistance and what can be done without assistance the "zone of proximal development." You, as author, have a measure of control over the level of sophistication students will need, how fast you will want them to move through the material and how deeply you will want them to learn. In particular, you may ask yourself the following questions:

- How much time do we spend here on this material? Is it worth it?
- If we move quickly through this material, will my students know enough to handle such-and-such later on?
- Are we proceeding at the right depth for later material?

The following examples from a study of the numbers demonstrate how one can calibrate the ZPD.

Example 1. Theorem. $\bar{M}$ is a closed set. This theorem can be made easier by preceding it with one or both of the following:

- If p is a limit point of the set of limit points of M, is p a limit point of M?
- Prove that M' is closed.

Note: The meaning of each of $\bar{M}$ (the closure of the set M); limit point; M' (the set of all limit points of M); and "closed" is given to the students.

Example 2. Theorem: If $p_n \to A$ and $q_n \to B$, then $p_n + q_n \to A + B$. This can be made easier by preceding it with the following two theorems:

- If $p_n \to 0$ and $q_n \to 0$, then $p_n + q_n \to 0$.
- $a_n \to L$ if and only if $a_n - L \to 0$.

Axiom 3. Write well

Here is what I have learned:

1. Use words carefully. Carefully crafting and honing just the correct sentences can help immensely, and the sloppy use of the written word can confuse. I believe students can detect when rules of good writing are broken just as a musically naïve individual can tell when a musician hits the wrong note. Some of my own pet peeves follow:
   a. When we see the phrase, "let x and y be numbers" in a mathematics text, we actually mean that x and y may be different or equal. However, proper grammar usage would dictate that the phrase "x and y are numbers" means that x and y are different numbers and are not equal. Why not state "each of x and y is a number" instead of "x and y are numbers."
   b. Choose technical terms carefully. How confusing is it when we use the words "a function is continuous when it is continuous at each point in its domain?" This sounds like double talk to the ear of a new, immature student in a calculus course. Instead consider something like "a function is *globally* continuous when it is *locally* continuous at each point in its domain" or "a function is continuous when it has Property S at each point in its domain." (Wall, 1963) Another example is the use of the word, "supremum," versus the words, "least upper bound." When I want to minimize the connotation of "least upper bound," I may use the more obscure word "supremum" if I want to really stress the deep meaning of the words and syntax of the definition. Conversely, if I don't want the students to belabor some of the finer details of the definition but I just want them to apply the definition using primarily the connotation of "least upper bound," then I will use the words, "least upper bound."
   c. Be careful using $\forall$ and $\exists$ such as:
      $\exists L \forall \varepsilon > 0, \exists \delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.
      Instead, it might be better to state:
      There is a number L such that for each positive number ε, there is a positive number δ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$.
   d. Avoid using stilted language whenever possible. The writing should have a natural flow and not be overly formal. For example, "If p is a limit point of M, p belongs to M" can be replaced with, "Every limit point of M belongs to M." The latter wording has a more natural flow.
2. Adjust for the target audience. Just as you want a proper tool for a project around your home, you want a proper set of materials for your course. Keep in mind the mathematical background, the emotional makeup and the culture of your students.
   a. Background. Consider students who have had no proof experience. You will have to bootstrap from a level of very little, or even no, sophistication. Even if

students appear to be academically mature but are mathematically naïve, you may have to lower the level. If students are academically weak or naïve, you may have to keep the language and vocabulary at a low level. For example, you might not want to make a statement such as, "No finite point set has a limit point." Instead, at the risk of using stilted language, state, "If *M* is a finite point set, then *M* has no limit point." The latter may sound wrong to a literary mind but sometimes these situations are unavoidable. Moreover, if the students have trouble with "variables," you might avoid the issue by saying "If a point set is finite, then it has no limit point." This is preferable to, "If *M* is finite, then *M* has no limit point." Some bright students might actually have problems with material that requires divergent thinking instead of convergent thinking. They might be able to handle straightforward statements such as, "If *p*, then *q*." They may stumble when faced with "Give a meaning of 'precedes' so that… ."

b. Emotional Makeup. I have found that the emotional state of students must be taken into account. For young students, make the material playful and concrete. Give students self-motivating problems by staying within their environment or comfort zone. I have found that they enjoy creating models of axiom systems. Now, suppose the students have very little confidence. It may be that their confidence has been shaken by previous bad experiences or they just are not wired to be confident people. Drill problems that start at a very elementary level and gradually become more difficult may be in order. Stay with familiar material but enrich with more challenging situations. For example, what would they do if someone asked them to compute the derivative of $f(x) = \sqrt{x-1}\sqrt{1-x}$? If they blindly follow the standard rules without thinking, they will give the wrong response. Or, ask them to find the fallacy in the following derivation:[8]

Assume that		$a = b$
Then, evidently		$ab = a^2$
Subtracting b^2		$ab - b^2 = a^2 - b^2$
Factoring,		$b(a - b) = (a + b)(a - b)$
Dividing by $a - b$,		$b = a + b$
But		$a = b$,
Therefore		$b = 2b$
Or		$1 = 2$

Examples such as the preceding may develop openness for critical thinking.

c. Culture. How you write your materials for your students has much to do with their work ethic, self-motivation, familiarity with open-ended problems, knowledge of only guided drill, shyness, rudeness and arrogance. There may be other considerations, all of which need to be addressed in different ways. Empathy for each student and each personality will guide you.

8 *Elements of the Differential and Integral Calculus* by Granville, Smith and Longley.

3. Say something. Decide what content you want to focus on. Do you want your students to develop the ability to do proofs? If so, you must have some content in mind that excites you, is meaningful and is profound. Of course, it must be coherent. For example, many transition courses tend to survey several disassociated facts from areas such as sets, logic, functions, algebra, analysis and so on. Instead, take up a single topic such as Linear Point Set Theory or Synthetic Geometry so that a coherent theme exists throughout the course that is clear to the students. In calculus, you could focus on the Fundamental Theorem of Calculus as the main character of the story you want to tell.
4. Be natural. Attempt to keep the language and the mathematics natural or close to the students' way of thinking. For example, if your students are visual learners, use visual language. Before taking up the standard textbook definition, state the following definition of convergence of an infinite sequence:

   > An infinite sequence of numbers is said to converge if and only if there is a number L such that each region[9] containing L contains a tail[10] of the sequence.

As I stated at the outset, write well. Reading great literature or even reading well-written contemporary novels will help. The following are references I have found helpful:

- *The Craft of Scientific Writing*, 3rd Edition, Springer, 1996;
- *Eats, Shoots and Leaves* by Lynne Truss, Gotham Books, 2003;
- *How to Write Mathematics* by Steenrod, Halmos, Schiffer, Dieudonné, American Mathematical Society, 1973; and
- *Style toward Clarity and Grace* by Joseph M. Williams, The University of Chicago Press, 1990.

Conclusion

The bottom line in writing good materials is to be true to yourself. Do not listen too hard to too many experts. There was a point in my career when I wanted to improve my teaching; therefore, I started listening to colleagues whom I respected as teachers. I began to apply what they told me. My teaching deteriorated! It was not that what they told me was wrong. Their advice did not work with my personality. Writing materials for Moore Method courses is an extension of our teaching. Just as you cannot be a perfect teacher, your materials will not be perfect either. We are never satisfied with our notes. Just get started and do not look back. You will be fine. What counts is what you do in class on a day-to-day basis.

(Parker)

"…but first you break the material up into this sequence of problems. Every mathematical discipline that I know of could be reduced to, say, 1 to 200 problems."

John W. Neuberger, Legacy of R. L. Moore Conference, 2004

"…There is this huge amount of stuff that you, depending on the area, might need to know. Pick it up."

Udayan Darji, Legacy of R. L. Moore Conference, 2004

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9 By region, I mean what most would call an open interval.

10 By a tail of the sequence, $a_1, a_2, a_3, \ldots$, I mean, $a_k, a_{k+1}, a_{k+2}, \ldots$ for some positive integer k.

In order to prepare or choose materials for a Moore Method course, we must initially determine what the expectation for the mathematics to be covered in the course might reasonably be. To my mind, each of us is our own best resource. Provided we already know the mathematics in the course we are to teach, we have the benefit of knowing what it took for us to understand the mathematics. Thus answering the question of how to prove a theorem provides a departure point for setting up problem sequences that may be expected[11] to culminate with arguments for the theorems deemed fundamental to the course. Furthermore, the frequency with which ideas pertinent to the course have recurred within one's own work may help in making value judgments relative to "this theorem must be included" versus "it would be nice if this theorem were included."

Even if one has firm ideas for what a course should be, it may be wise to consult other sources. At least two are likely to be of particular importance, departmental expectations and the current "state of the course" at other campuses. In my department, the minimum expectation is that coverage be compatible with the catalogue description. In addition, there is a topic syllabus, periodically updated by a committee with the express purpose of making course topics have a chance of matching current teaching realities, on file for each course. Textbooks may be useful in judging what the course may look like on other campuses. A text designed for the course in question is likely to have the feature that it was produced with the intention of selling it on a broad market. Thus, even though it may contain much more mathematics than a class of students is likely to be able to create (or perhaps even ingest), it is likely that the course you want your students to make has its main ideas stored somewhere between the covers. The down side of using textbooks for Moore Method courses is that the exposition typically shows students preferred ways of thinking about theorems, thus effectively channeling the students into a particular mindset and virtually eliminating the possibility of a novel approach and severely compromising the possibility of students re-discovering classical approaches by their own devices. Hence, a textbook is not likely to be a useful tool for the students during the course, but it can be valuable as a teacher's resource or as a source book for the students at the completion of the course.

If one is assigned a course that does not fall within one's mathematical comfort zone, the importance of departmental expectations and finding appropriate textbooks to guide one's thinking is magnified since we must teach ourselves the mathematics deeply enough to make judgments about what the most important ideas are. When put in this position, we must broaden our learning from the usual mathematical consideration of why a result is correct to include how a student can reasonably be expected to connect with these ideas. Indeed, as Darji's quote preceding this essay indicates, educating ourselves to teach can be the impetus for expanding our own knowledge.

Granted that we can put the mathematics in place in our minds, which according to Neuberger's quote preceding this essay is a reasonable expectation, how then do we go about constructing a problem sequence from which to teach? I have at least two goals for the notes for a course. One is that the problems constitute a body of knowledge that creates a background from which a student could read intelligently on the subject. As discussed above, we have sources we can use to help inform the choice of the theorems that our

[11] Expectations often do not match reality! Your students are almost certain to surprise you with how they solve the problems you present to them, so be prepared to accommodate their ideas.

problem set addresses. The other is that the notes should be flexible enough that problems can be added at any time during the semester to provide problems that anyone in the course might reasonably be expected to do.

In respect of the first goal, I begin by identifying the theorems that I consider fundamental to the subject at the level of the course. These theorems form the focal point for the rest of the problems in the course; that is, in building the problem sequence, the idea is to provide accessible problems that, if they get solved in a way you know they can be, can create access to the main theorems. This raises two additional issues. How do we identify what "accessible" means in a particular class and how do we identify reasonable paths to the main theorems?

Regardless of how well prepared we would like our students to be, we have to teach them as we find them. Logic adjudicates arguments, so we must ensure that the students come prepared with, or are exposed to, the logic necessary for doing mathematics. When fluency with logic is an expectation for a course, I always begin with a problem or two where I try to find out how much reality matches expectation. Any of these problems is typically a direct consequence of a definition.[12] If the majority of students settle them, only the time associated with a class period or two is lost and I gain, in return, some assurance that the issues to be addressed will be mathematical, rather than logic-based. If the students do not settle them, solid evidence of the nature of those parts in their various logical backgrounds that need to be remediated emerges, and can be addressed with appropriate discussion and/or problems. If the recognition of the structure of the logic or the pattern of argument is a curricular expectation, such as one sometimes finds in an introduction to proof course, I address the logic more purposefully. I will articulate for the students, verbally reiterating what is written in the notes, the part of the formal language for the course that will be used to express the logic, discuss sentence structure, and design an assignment in which the students identify, within the articulation of the axioms, definitions, or problems, instances of the use of the various structures. Thus, the introductory materials, whose primary purpose is to establish the foundation for the mathematics of the course, can also be used as a problem source for teaching, or validating the students' command and use of logic.[13]

When we have in place a plan for "teaching" the logic necessary to make arguments and that lack of logic does not block accessibility to theorems, we can address the issue of mathematical accessibility. When I was a graduate student, a joke circulated among the graduate students in mathematics about a Moore Method course being taught in the English department: "Give the students a dictionary and tell them to write *War and Peace*." This was just dark humor generated by suggesting that the Moore Method might well be practiced in a preposterous way; yet it seemed real enough in the moment whenever one came up against a particularly testy problem. Nevertheless, it also suggests the important message that, for goal theorems of considerable impact, there should likely be more preparation than just the

[12] For example: " Suppose that $Q = \{(x,y) : x$ is a number and $y = x^2\}$. Show that Q is a function from the numbers into the numbers." To validate that Q is a subset of $R \times R$ the student need only validate, by referring to the definition of Q, a universal instantiation of the definition of subset. To validate that the numbers is the domain of Q, the student will need to instantiate a universal, then instantiate an existential on the basis of the definition of Q. Validation of single-valueness often comes from an indirect argument. At a more basic level, "Suppose that X is a set with more than one element. Show that $X \times X$ is not a function from X into X."

[13] See Chapter 6—In the Classroom—for further discussion of how enhancing the quality of the class's use of logic can be addressed.

axioms and definitions. From the preceding discussions, we have armed *ourselves* with a way to prove the theorems in question and we have provided for the student the necessary axiomatic foundation, definitions, and logic preparation. But, in choosing our problems to get the students started on the mathematics, we need to ensure that ideas will emerge that will enable our students to address the main theorems. At the same time, we must not force the students to think as we do. When teaching a course for the first time, in the initial material development we have little to go on except our own experience with the mathematics. Thus, it is particularly important that we pay careful attention to students' arguments so that we do not merely judge correctness, but also look for ideas or techniques that come from the students that might be nurtured with problems not necessarily conceived as part of the original problem sequence. In subsequent offerings of the course, the experiences from previous offerings of the course will help inform problem selection.

By way of an illustration, consider the first semester of real analysis, which I build around understanding "continuous function." The first must-get theorems are the composition theorem, the sum theorem, and the product theorem for continuous functions with compatible domains. Instead of beginning with these problems, I begin with problems that ask the students to prove whether or not certain sets of ordered pairs are functions, and whether or not some particular functions are continuous. Thus if the students choose to work on the problems in order, when they encounter the three algebra theorems, the issues of what being a function means and what being continuous means have already been worked on for particular functions and there are, in addition, functions in hand with which to make examples. On the other hand, for the two "goal theorems" for the course which are stated deeper into the semester, the intermediate value and extreme value theorems for continuous functions with domain an interval, I state them without paths. If no one makes a run at either of them, after an appropriate waiting period[14] I will state, with the intention of giving some direction on a possible proof of the theorem, some problems about intervals that culminate in the nested-interval theorem. If a student (or students) makes a run at one of the goal theorems, but gets stalled, I use what they try to help me decide on whether a cut argument, a nested-interval argument, a Bolzano-Weierstrass argument, or a Heine-Borel argument is most likely to be compatible with the way they are thinking, and at the appropriate time, state problems that nudge them toward one of the theorems pertinent to their ideas. If a class gets one of them without my having to state problems that I know are lemmas, I sit back and enjoy the show. Basic derivative theorems are usually sufficient for keeping those students who are not yet experiencing success in the main thrust of the course active and, thus, address the goal of keeping problems available at all levels.

Experience in the classroom has shown me that problems stated concerning concrete models are useful in giving beginning mathematics students access to ideas that can be used to drive arguments for the main theorems of a subject. Thus, I like to keep, whenever possible, at least two strands going; one concerning the mathematical structure under consideration and the other addressing particular instances of that structure. Practically every sub-discipline of mathematics offers opportunities for doing this. In analysis, as indicated above, this might mean problems about functions and their various properties and problems about particular functions that display those properties. In linear algebra there are

[14] When to provide additional problems that you think will get them moving again is a tough call. I try to err on the side of waiting too long.

bases and bases for particular vector spaces. In algebraic structures, there are rings and particular rings such as $+$ and $*$ on the set of functions with domain [0,1] and range in $\Re$. In topology, there are topological spaces and particular topological spaces such as the topology induced by the Euclidean metric on $\Re^4$. In geometry, there are problems consequent to the axioms and the questions associated with validating that particular definitions for point, line, and plane satisfy the mathematics of the axioms. These are just several examples of many different choices that can be made to create strands that are general in outlook and strands that illustrate important general ideas but have the ideas embedded in particular examples. In addition, once seminal examples are mastered, they make excellent contexts for testing conjectures.

Standards for accessibility change as the course progresses. As noted before, at the beginning of the course, some problems need to be presented to assess the level at which the students are working and, if necessary, develop their command of logic. The "assessment" problems should be basic enough so that you expect everyone to get them. If a large percentage of your students do not turn these in, the nature of how the class does deal with them should indicate how to amend the course materials to address the particular needs of that class. If problems with logic persist, problems that address the observed deficits need to continue throughout the notes until the remediation is complete or the course ends. As the class progresses, the needs of the students dictate the number of such problems and the level at which they are presented. However, keep in mind that, as more ideas become the property of the class, it is reasonable for "directly accessible" to mean "a direct consequence of what has just been proven." The needs of the individual students dictate how many carrots are left available to enable a slower-developing student's first success.

Before I met coauthor Mahavier, I assumed that anybody who taught using the Moore Method developed her or his own materials. Not only had I never thought about using somebody else's notes, I had never even imagined that somebody other than myself might want to use those I had written. I write them for my own use. For those courses that I have taught multiple times, use and revision have given me both a feel for what may happen and an experience base from which to consider what might come next depending on what happens as a class addresses a particular problem or set of problems. I consider this experience base an important appendix to the course notes. Theorems that are added to the notes during the course to catalogue what the students have done within their proofs and the problems that are added consequent to "failed" arguments not only provide focus for and/or encouragement for the students during the course, they also provide a record that may be used as a resource the next time the course is offered. When Mahavier asked me if I would be interested in publishing the notes for my course on the number continuum,[15] my first reaction was, "Why?" But Mahavier was persuasive. The experience of having an outside arbiter ask questions such as, "Why does this problem precede that one?" and "Why would you make the definition this way?" made me answer questions that I almost certainly would not have asked myself. Even when I might have, I wouldn't have taken the time to articulate an answer suitable to explain to anyone but myself. I have no idea whether or not the resulting product is better, but the first time I used my notes in a form consequent to Mahavier's interaction, the class made the most rapid beginning burst of any class in that course I have

[15] I have two versions, one working through the place-value model and the other through the fraction model, both available at The *Journal of Inquiry-Based Learning in Mathematics*.

ever taught. Although I am still in favor of making the first offering of a course solo, I am now really enthusiastic about any collegial interaction I can get. I also have become an avid reader of others' course notes when I can get my hands on them.

In summary, a set of course notes should begin with problems that allow you to gauge some minimal level at which your students may be working. At any time during the course after the initial two weeks, there should be problems that challenge the students that have experienced the most success and problems that are accessible to the students who have experienced the least success. The course notes should be seen as a template on which to build rather than a fixed document, and adjustments made in an ongoing manner to take advantage of ideas generated by the class. You are your best first resource, but other sources are available to help you ripen your own ideas.

6
In the Classroom

> "After stating the axioms and giving motivating examples to illustrate their meaning he would then state definitions and theorems. He simply read them from his book as the students copied them down. He would then instruct the class to find proofs of their own and also to construct examples to show that the hypotheses of the theorems could not be weakened, omitted, or partially omitted. ... When a student stated that he could prove Theorem x, he was asked to go to the blackboard and present the proof. Then the other students, especially those who hadn't been able to discover a proof, would make sure that the proof presented was correct and convincing. Moore sternly prevented heckling. This was seldom necessary because the whole atmosphere was one of a serious community effort to understand the argument."[1]
>
> — F. Burton Jones

This chapter presents snapshots of the day-to-day operations in Moore Method classrooms. Several themes run common to the essays. Each author clearly tailors his instruction to individual students and uses some form of diary to track progress of the class and of individual students. Each stresses the importance of the first day of the course, of "selling" the method to the students, and of active participation by the students. Because of this participation, the unpredictability of what might happen in each class is reflected as well. Each author spends considerable time preparing for class by evaluating what has been accomplished and what might be accomplished if the right seeds are planted during the next class period. Even within a particular course with previously prepared problem sets, all observe that content is variable and determined by both the rate of progress and the focus of the path of mathematics that the class follows. Additionally, the questions of what to do when students have nothing to present and how to get students to understand the enhanced benefits of doing a course this way are answered.

The essays proceed from the general to the specific. Mahavier sketches representative classes from the first day, a mid-semester day, and a day at the end of the course. He includes lists of techniques that work and common pitfalls, derived from his experiences as a mentor to other teachers. Parker's essay also addresses typical days in the classroom, using active prose to bring the classroom to life. While all four authors maintain some form of diary entry for each class period, May and Coppin present detailed samples to demonstrate the type of mathematics and student accomplishments that may occur in the classroom.

(Mahavier) A typical first day. Aside from the mathematical material, my words and actions on this day depend very little on the level of the course, from remedial to graduate level. They do depend on whether the majority of the class has been exposed to the method. Assuming the majority has *not* been exposed, my goal on the first day of class is to set the students' minds at ease, while getting rolling on the material. The previously discussed notion of developing material that is axiomatic and elementary in the beginning, but advancing in difficulty as the course progresses, is important here.

[1] "The Moore Method," *American Mathematical Monthly* 84: pp. 273–277 (Apr. 1977)

For the first day, I want to have problems ready that the students will make immediate progress on. This will help cement, early in the course, eight features of the method: the importance of students presenting at the board; developing their communication skills; recognizing when statements are correct; defending correct statements; recognizing when something is incorrect; graciously admitting to a mistake; and, of course, the importance of *doing* mathematics itself and recognizing their own ability to *do* the mathematics. My experience is that once you have stated that students will go to the board, no amount of lecturing will set the students' minds at ease with respect to this expectation. Experience has taught me that it is best *not* to mention this aspect at the beginning of the period and just to start discussing a problem. For this reason, I go so far as to not pass out the syllabus on the first day. As we discuss the first problem, the moment a student has some good idea I hand him or her the marker to present the solution. Opening lines such as "Welcome to Calculus III. Are we all in the right place? Today I want to talk to you about functions with domains in $\Re^2$." Note the immediate emphasis on the mathematics, the most important of all subjects! "By $\Re^2$, I mean… Now suppose $f(x,y) = \ldots$ Can anyone here sketch a graph of this function without a calculator? What's your name? Ambrosia? Ambrosia, would you please do this at the board for us?" Or, in Analysis, "Welcome to Analysis. Today I want to talk to you about limit points. We'll assume a few properties of the real numbers with which you are already familiar, for example, the notion that between any two numbers there is another number. Let's call that Axiom 1." Note that the implication that I will be lecturing is a facade, but they are relaxed because I begin by lecturing. "We say that p is a limit point of the set M if …. Consider the set $S = (0,1)$. Is 0 a limit point of this set?" There will be some discussion, often with nudging by me, and at some point when someone believes he or she can argue one way or the other, I'll ask him or her to write on the board what he or she has said. It may take some gentle coaxing at first, but I have never seen a case where either that student or another would not go to the board and attempt to write out the idea. Not only do these first attempts not need to be successful, it is valuable if they are not, for students see on the first day that it is not a stressful environment if one makes a mistake. In fact, I emphasize that many, if not most, successes are the result of many failures; how many filaments did Edison try before perfecting the light bulb? What is important is that they feel unthreatened at the board and that when they sit down, I successfully get another student to the board.

At the end of the class I give some information on the structure of the class, the grading, and the integral nature of the board work to the course. Having seen at least two students at the board in a low stress environment and with the expectation that on the next day we will complete these and move on to new problems, the students leave knowing what to work on and knowing that the goal is more presentations on the next day. I am hopeful that this first day sets a precedent for some friendly competition.[2] At least one challenge has been put forth on the first day and I expect solutions on the second day. At the end of the second or third class period, I pass out some detailed information. Appendix II contains this information. This material is not all my own. It began as my own, but later incorporated writing by Parker and Robert M. Kauffman.

For the sake of brevity, I will restrict the rest of this essay to conducting a class in analysis with between ten and twenty students.

[2] See Chapter 10—Frequently Asked Questions, "Does the Moore Method foster competition among students?"

The first few weeks

At the beginning of the semester, unless students have been exposed to the method before my class, it is painfully slow going, as I find they can do very little independently, even with a bit of insight that I provide through examples and discussions that arise naturally as students struggle at the board. There is great danger in giving too much during class, as opposed to privately, because the bright (quick?) students get the subtle hints immediately and this gives them an unfair advantage. Thus, I do not like to give much guidance in class at the beginning. There is no lecturing in the beginning except to get them started on the first day, to reinforce information on how the class is conducted, and to offer lots of encouragement. This encouragement often takes the form of indicating that I know they will get there because I have done this before and know what they can do. I tell them that what I am asking them may be challenging, but I am here to help and I know they can do it because I have seen previous students succeed. (If this is your first attempt, be honest; say that you've seen, heard, or read of the success of students taught using this method and wanted to try it to benefit your own students.) At some point, I invite every student to visit my office for just a few minutes so that I can get a sense of his or her goals. Teacher? Graduate school? Industry? Undecided? After a brief interview, I transition to questions about the mathematics they are working on. I attempt to level the playing field by giving more help to ostensibly weaker students in my office. In class, I further level the playing field by letting such students have the first shot at proving theorems. So, I do absolutely no lecturing early on. I simply let them move slowly, and write up one or more of the proofs that go on the board for weekly written assignments. Within a few weeks, I have all students begin writing up original problems that have not been presented at the board for their weekly written assignments.

A typical mid-semester day

After the class is successfully rolling along, any two days are more or less isomorphic, although there are highs and lows. Some days I arrive early, casually chatting with the students. Other days a student is already at the board writing a proof or problem. In this case, in deference, I go straight to a seat in the classroom—I do not have the floor, that student does. Often students may ask about a particular problem they are working on and I will chat about it or jump up and head for their desks to help. Sometimes, no one asks a question and I'll raise one. It might be mathematical, based on something that is occurring in class. "Do you suppose it is possible to have a function defined on a closed interval that is discontinuous but still integrable?" My initial question is usually easy and someone answers it. Then I add to it. "Well, that's a nice example. Do you suppose it could have more than one point of discontinuity? Countably many?" And so on until I leave them with a question to think about. In these few minutes before class, I'm actually interspersing a more Socratic pedagogy of rapid questions that complements, but is distinct from, the Moore Method. I'm always leading them somewhere, to some interesting thread of the subject that may appear later in the notes or in future courses.

Other times, I will ask if any of them know what an REU is and we will discuss this topic. Or I'll discuss life as a faculty member and point out the many benefits, the low stress, and the enjoyment I derive from my chosen career. If no one is presenting when the time to start class arrives, I'll ask again if there are any questions and address any queries about the notes or definitions. Then, I'll ask what people have to show. Normally, I already know something of what students have because I ask them what they are working on any time I see them on campus or around town. If a student doesn't have one in mind, I'll ask if he or she has tried a specific numbered problem that I believe he or she is prepared for. If a student has identified a specific problem, then I'll ask if he or she has that problem. If not, I'll mention in each class period, "I know that Mr. Clark is working on Problem *X*. Do you have that result yet, Mr. Clark?" This puts pressure on the others to choose and work on a problem and pressure on Mr. Clark because of my clear expectation and confidence that he will solve it. This maximizes the number of problems that students have ready since some will assume that Mr. Clark is close to *X* so they move on to *Y*. Or perhaps they wish to beat Mr. Clark to Problem *X* so they work a bit harder. Assuming students have solutions, they will defend these one at a time. The student who has presented the least will get first shot at a problem. On some days, as many as three or four problems might go on the board. On other days, we could well spend the entire day discussing one problem if students have lots of questions and the presenter needs to make amends on the fly. Side discussions on related topics are not uncommon. In the middle of a presentation, Estefan might ask the *almost* well-posed question, "Does this mean that if a sequence converges to a point, then it's a limit point of the sequence?" It now becomes a class challenge to work toward a well-posed conjecture, and I'll ask leading questions such as, "I don't understand – what do you mean by a limit point of a sequence? What is our *definition* of a sequence? Can a sequence have a limit point?" If this does not work, I'll try, "Is a sequence a set or a function? Do we have a definition for a limit point of a function?"[3] Eventually, we'll either settle the issue or generate a well-posed (albeit false) conjecture such as, "if a sequence converges to a point, then that point is a limit point of the *range* of the sequence." This becomes Estefan's conjecture and, if it is not resolved, it will be asked about at the beginning of each class until it is.

> "I am sure you are excited about going back to Texas, and I know you will do well. I am happy for you but sad for the students at Nicholls. You were that rare teacher for me that motivated me far beyond my expectations. Not too many teachers can get me to stay up to four in the morning trying to do one more problem.<grin> Or make me wake up from a sound sleep and yell 'I got it!' <chuckle> Thank you, you did make a difference while you were here. Best of luck."
>
> Email from Dale Toups, a full-time welder at John Deere (2001)

I will occasionally lecture on a day when students have nothing to present. The decision to do so is based on *why* the students have nothing to present. Is the material too hard? Do they need a jumpstart in the form of a leading problem? Are they being lazy? Do they have midterms in other classes that are interfering with their studies? Is there a mistake in my notes? (Never! ;>) See the FAQ, "Do Moore Method instructors lecture?" in Chapter 10 for details.

Questions at the beginning of a course are often about the theorem statements; you may find that they have great difficulty reading the mathematics. Thus, I elaborate on the

[3] Notice how careful I am being with the mathematical language here. The student is not distinguishing between the sequence (a function) and the range of the sequence (a set).

meaning of the puzzling statement, draw pictures, and show what "could" happen. Or I may draw examples where weakening the hypothesis results in a statement that is not true. This does not include hints in class. It is strictly to assure that they understand the meaning of the theorem and learn that a healthy first point of attack on a theorem might be to learn about the theorem, not necessarily to try to prove it. Jennifer Christian Smith wrote (Smith, 2005) that the difference between Moore Method students and non Moore Method students in subsequent classes was that the Moore Method students tackled each problem by trying to understand the statement through examples in an effort to gain an intuitive conceptualization of why it might be true or even if it had to be true. Non Moore Method students sought out what technique of proof should be used to tackle a problem. I believe that the former attack is the one from which a researcher in any field would benefit.

If time permits at the end of a class period, I will talk about upcoming definitions or the bigger context of the theorems just proved. This is not a lecture on the concepts or techniques for proving the theorems. For example, one theorem reads that "if $p_1, p_2, p_3, \ldots$ is a sequence of distinct points in the interval [0,1] then there is a point in [0,1] that is not in the range of the sequence." Students often do not realize that this implies that the set of real numbers is uncountable. They are in the trees, so I spend some time talking about the forest. In analysis, I spend some time relating the theorems to calculus. Explaining that the phrase "the limit as x approaches a of $f(x)$ equals L" may not have been defined precisely in calculus, but it means that "if S is an open interval containing L then there is an open interval T containing a so that if t is in T and $t \neq x$ then $f(t)$ is in S." I put a bit of this in historical context also. Lastly, on a day when students have nothing to present, I may give an actual lecture, albeit quite interactive, where I discuss countability or the measure of a set or a function that is continuous on the irrationals but discontinuous on the rationals. This is only done after a clear work ethic on behalf of the class has been established. I would never lecture early in the term as students must believe that they are fully responsible for the classes. Of the time I spend at the board, perhaps one-fourth goes to tying analysis back to calculus, one-fourth to foreshadowing analysis in more general spaces, one-fourth to motivating examples or clarifying definitions and theorems, and one-fourth to topics not included in the notes or not appearing until the next semester. I refer to these as Hollywood does, "previews of mathematics coming soon to a classroom near you!"

The end of the course

There are two possible scenarios depending on whether I feel there is a body of material that "must" be covered. For example, if we are closing on the end of a one-semester analysis course without a second semester, then we need to make it to the fundamental theorem of calculus. Suppose we are now three quarters of the way through the semester, and I realize we are not going to make it. Some classes make it on their own; some don't. To expedite the coverage of material without sacrificing the structure of the class, I pass out a list of theorems that we are *going* to cover to cut a path toward the end goal. I eliminate some noncrucial material. From this point forward if no one has something to present on a given day, then I will ask which ones they are working on, pick one they are not working on, and prove it. Sometimes, I will prove it in multiple ways: once on one side of the board with the precision that I demand from them, a second time as it might appear in an analysis text, and a third proof as it might appear in a calculus course. And I point out the difference in the rigor that I have demanded of them and the rigor that a text might demonstrate. Now, I would *never* do this for more than two or three theorems. They might get lazy. When I do it, I make a big deal of the opportunity to present, excitedly saying, “You mean, *I* get to lecture today, really??? *Nobody* has anything? This is a favorite theorem of mine!” By cutting a path, we always get to the end of the material, the students still prove the vast majority, I prove a few, and we assume a few that we need to complete the goal theorem.

On the other hand, if this is a two-semester course, then I will simply let the class roll and we’ll continue the sequence during the next semester. We might even use the two-and-a-half hour examination period to move us further along the path to make up lost time.

The remainder of this essay is devoted to practical tips: “Classroom Techniques that Work” and “Classroom Techniques to Avoid.”

Classroom Techniques That Work

If these techniques seem elementary, it is because they are intended to assure beginning the course on a proper note. By the time the students are actively engaged, proving theorems, doing mathematics on their own, arguing, debating, and presenting among themselves, none of these techniques is needed. In a class where a sizeable portion of the students have already had a Moore Method class, none of these techniques is needed. At that point they are treating one another with respect, recognizing their own mistakes, calling one another by name, and acting as collegial adults. They are officially mathematicians and don’t need the “kid gloves” that I propose we start them off with. Having said that, the techniques are never harmful and I have met individuals who would do well to read and adhere to some of the rules regarding respect and constructive criticism.

Learn their names

At the Washington D. C. 2000 Joint Meetings during his invited address as the recipient of the 2000 MAA National Teaching Award, Arthur Benjamin, of Harvey Mudd College literally shouted, “Learn Their Names!” He then went on to explain his elaborate scheme for reviewing (via the Harvey Mudd picture book) all his students faces and names prior to the first class meeting. Nothing is more important than this simple piece of advice. Whether you call them Mrs. Lewis, Mr. Rodriquez, and Ms. Kohl or Amy, Jose, and Kenya is irrelevant. Just learn and use their names. While I have never learned their names prior to class, I promise them that I will know their names quickly and strive to learn new names every day until I have the whole class in mind.

Lecture sparsely

Contrary to many myths about the method, even Moore lectured.[4] How would I begin the first day of class if I never spoke? The question is, when and how much should one lecture? Personally, I do not like to lecture to introduce a topic, the exception being the first day. My preference is to address the consequences of a topic after students have made presentations

[4] Moore’s doctoral student John Worrell (referred to as “Mr. W.” in the MAA film, *Challenge in the Classroom*) related to me that Moore himself did not employ the Moore Method in every class he taught. See Chapter 10—Frequently Asked Questions, “Do Moore Method instructors lecture?”.

on the topic, or to tie two topics together after problems on both have been presented. An example from analysis would be the following. Suppose students have proved that a limit point of a set is the point of convergence of some sequence of points from the set and have proved that given a convergent sequence consisting of distinct points the point of convergence is a limit point of the range of the sequence. Summarizing the significance of these two results by addressing precisely when a point of convergence of a sequence is a limit point is an appropriate mini-lecture. Solidifying the concept with examples can provide a broader perspective than the students received by proving the individual theorems.[5] I will occasionally lecture on a day when students have nothing to present, as discussed earlier in this essay.

Other miscellaneous lectures include discussing what theorems we have not proved because they are not listed in the notes, but that we could easily prove now. For example, we prove that the sum of two continuous functions is continuous, but omit that a constant times a continuous function is still continuous. A good test question is "Rewrite the phrase, 'a constant times a continuous function is still continuous' with the same rigor as that with which we write the problems in the notes." A nice miscellaneous lecture is that since multiplication is a continuous function, once you prove that composition of continuous functions is continuous, you can get for free that each of the product, quotient, and scalar multiples of continuous functions is continuous.

Be positive

Everything in class is positive, even the criticism. If two students are talking in freshman calculus, while two others (Ashley and Carl) are silent, I may say to the class, "I appreciate how thoughtful Ashley and Carl are in class. They are always careful not to disturb the other students by talking." If a proof is on the board and there is a glaring hole that has not been discovered, I follow one of the martial arts training rules. *Always* criticize by saying something positive alongside something requiring improvement. For the flawed proof, "Jim, can you explain why the 5th line states that 1 = 0?" While Jim is thinking about that, I add, "Anne, did you note what a very nice start Jim has made on this problem – I especially liked the fact that he stated the hypothesis using his own unique notation." There is a second benefit to this exchange, since Anne *never* gets her hypothesis out right on the first try, and now she has an example to which I may draw her attention, while drawing attention away from Jim so he can decide if he can fix the problem or if he needs to be seated.

Be gentle when it is necessary for a student to leave the board

When a student is at the board and realizes he or she has a mistake that can't be patched on the fly, the student needs to be seated. Often, this is an awkward moment in the class as the student is either still thinking about the problem or is embarrassed by the mistake and quite likely both. Personally, I rarely ask a student to sit down. More often, I say something like, "now, let's remember the rules, if we are stuck, we can either fix it on the fly, reserve the problem and readdress it at the next class, or allow someone else to take a stab at it today." A student then should be given significant time to think about both the problem and the decision. Diverting attention away from the student while he or she thinks is a good tactic;

[5] Such an approach is supported by the mathematics education theories. See the discussion on social constructivism in Chapter 10—Frequently Asked Questions—suggesting that isolated learning is reinforced by communication with peers and instructors.

asking a related question of another student in the class can serve as a valuable diversion. Reminding the student of the options and allowing the student to choose his or her own course of action is significantly more respectful than asking the student to be seated.

Cover significant mathematical ground

During mentoring, faculty often ask about covering material, with statements such as, "I wanted to be at problem x by day y." For a course under development, the first three times I teach the new sequence, the class covers more material with each iteration. The third iteration often contains as much as thirty percent more material than the first iteration. As the notes are developed there are consistently problems that take significant time, but offer little benefit later in the course. These can be removed or replaced. Still, I *never* address where to be on a given day. The class can move only at the speed at which it is capable of moving unless you are willing to allow only bright talented students to present the vast majority of the material so as to minimize mistakes. If you are going to allow students to go to the board and make mistakes, then it will take time. At the beginning, it can be painfully slow. Students are learning how to write and communicate mathematics and at the same time are learning new material. Often only one or two problems will go on the board in the earlier days. By mid-semester, depending on the difficulty of the problems, it is not uncommon to have three or more problems on the board each day.

Leave the classroom

There will be times when a student presents a flawed proof to the class. My general rule, often stated to the class, is that any proofs on the board with mistakes will reappear on the mid-term. Usually, this keeps the class active and on task. Still, if a mistake is on the board, unnoticed, and if I feel the class will be harmed by allowing this mistake to pass, then I may give up my poker face and address the problem. Typically, I will simply say that I am concerned about a point on the board, but since my throat is dry, I will go get some coffee while the class seeks a resolution. Inevitably, in three-to-five minutes, I return to a class engrossed in discussion about grammatical, notational, and logical errors.

Let students use notes at the board

Many mentorees have asked me about letting students take notes to the board. I allow my students to take their notes to the board. Students divide into categories—some write verbatim what they have in their notes, practically shaking with nerves. Others take their notes with them as a crutch and look at them occasionally. I find that this group makes a lot of typos—which is fine, because it encourages more discussion, questions, and corrections on the fly. Some just go the board and write a proof without aides. Some copy and pass out notes so they can explain at the board without writing every word. I do tell my students that when they offer to go to the board, they should have a perfectly written proof, because in the past students have wasted class time by assuming they had a proof, going to the board, and then a few minutes later finding a flaw that they can't fix at the board. When I was a student, I not only took notes to the board, but I wrote defenses for many of my statements on the back of the paper. My confidence was low in early Moore Method classes and only later did I get to where I could "wing" a proof. Thus, disallowing notes might have stopped me from going to the board.

Discourage note taking

Because the goal of the classroom is the presentation, defense, and discussion of mathematics, too much note taking can interfere with students' abilities to engage in the mathematics at hand. On the other hand, students rightfully fear going home without notes because the key elements of a proof fully understood in class may be easily lost and difficult to reproduce. During the past several years I have implemented an idea borrowed from Cornelius Stallman and Bernd Rossa. I take pictures of the proofs on the board and upload them to the web after class, a process that takes less than ten minutes. This way, my students take notes only as an option. This method has worked well to facilitate the productive use of class time. It works far better than the techniques I have employed in the past including offering to reproduce proofs in my office, encouraging students to sketch merely the idea on their paper, or offering to make copies of students' proofs. Alternative in-class presentation methods are addressed under the chapter, "Frequently Asked Questions."

Play the many roles of the instructor

The instructor must be a bit of a chameleon, able to switch between various roles at any given time. The instructor's role alternates between student, discussion moderator, coach, sage, and guest lecturer. Perhaps the majority of the time the teacher merely participates in the class as a student, asking questions when confused and deferring questions to which he does not know the answer (wink, wink) to the rest of the class for help. Sometimes discussions generate too many questions at once and the student must morph to being the moderator and help the class address one question at a time. Sometimes the discussion may become too animated and the coach must step in and keep the team on task. Occasionally, the professor must serve as a sage, pulling together a body of material to help the students see the forest while they have been working on the trees or perhaps giving some historical perspective to the material. A nice guest lecture on a day when students don't have anything to present can be refreshing and can add spice to the class through the use of nice applications, a related sub-topic, or posing additional questions.

Answer questions with questions

When students ask questions of me, either during class or out of class, my "answers" are almost always questions. The questions will be different for different students asking the same question, because I gauge each student's current mathematical conceptual framework for the problem and ask a question that is difficult enough that he or she will not answer it immediately, but rather will need to think some more, answer the question, and then use the newfound understanding to address the original question. Let's take an example. Suppose in an analysis class we have the definition of continuity that the function f is continuous at $(x, f(x))$ if for every open interval containing $f(x)$ there is an open interval containing x such that if t is in the domain of f and in the open interval containing x, then $f(t)$ is in the open interval containing $f(x)$. Suppose the analysis student is struggling with the question of whether the function $f(n) = n$ is continuous over the set of natural numbers. Applying the definition seems simple to us, but the student is struggling with the misconception laid forth in high school, and often in calculus, that continuity is intuitively defined by the ability to draw the function without "lifting the pencil from the page." If the student were to ask me a question, then likely my response would be. “Hmmm. Suppose we have the function, $g = \{(3, 10)\}$. That is, g is the function defined on the domain, $\{3\}$, and $g(3) = 10$. Is g continuous at (3,10)?” The student is likely to try to eek more information out of me at this point and will shoot back a multitude of good questions indicating the seed of understanding. I might write the questions on the board or on paper and help make the questions precise. But I would likely not answer any further questions with more than encouragement. “You are too close to that problem for me to say more.” I don’t want to rob the students of their potential joy from solving the problems on their own.

Lie to your students!

A proof is on the board and we vote on whether it is correct. I’ll often take a misleading or downright incorrect stance. If there is a typographical or minor error that makes it false, I’ll vote that it is incorrect to mislead the class. If it is seriously flawed, I’ll vote that it is correct and even attempt to argue the point. The students then must use correct logical statements to argue me down by irrefutably discrediting my false statements. By the end of the exercise, they try to corner me and prove that I lied to them. Sometimes I have and sometimes I haven’t. Either way it is a fun and profitable exercise that trains them to accept only that which is true, question that which requires clarification, and confront politely that which is incorrect. The exercise also solves the problem of the students always looking to the instructor for the answer. They must speak carefully and accurately. Sometimes we end in stalemate, I can’t convince them of a falsehood, but they cannot shoot me down. They’ll come back the next day ready to tackle me again and quickly defeat me. Knowing that I will intentionally mislead them once again establishes the democracy that I am not going to play the authority on the subject. That is, when I speak, I am a member of the class – possibly right, possibly wrong. There are times when I will return to authority. If they want a clarification on grading or if a student wants me to readdress a problem that was previously put up, then I will go to the board and help them through it. In doing so, I’ll not give an alternate argument. I’ll simply act as secretary, asking the previous presenter questions like, “Didn’t you do this first?”, or “Was this your notation?” Again, the whole class is involved, but I am not going to let a student get seriously discouraged by not providing leadership when it is needed. Surprisingly, it is rarely needed. Most of the time, the student will go to the author after class or during the presentation and get the problem clarified without such authoritative leadership on my part.

Allow students to make mistakes

I tell my students that I add one point on their final grade average the first time they make a major mistake at the board. This almost completely relieves the pain of the first mistake. When a mistake does occur, the class reminds them that their final grade just went up and it becomes a lighthearted laugh, rather than an embarrassing moment.

Holt is also concerned with our students’ prevailing attitude that it is “bad” to make mistakes. In discussing children’s approach to playing twenty questions to guess a number between 1 and 10,000, Holt (Holt, 1965) writes:

> *They have not learned how to learn from a mistake or even that learning from a mistake is possible. If they say ‘Is the number between 5,000 and 10,000’ and I say ‘yes,’ they cheer, while if I say ‘no’ they groan.*

Clearly students are conditioned at a very early age that "getting the right answer,"—in this case, "yes"—is important for every single question, even when "no" yields the same amount of information. Breaking down the taboos associated with making mistakes is important. I use various means. I add one point to the final grade of each student the first time he or she makes a mistake. Of course, this is merely an ice breaker or pedagogical trickery. The students know it as well; the good student makes no mistakes and gets an *A*, the weaker students make a mistake early on and find their grades boosted. They will not let you forget to add that point! I also use common-sense examples. When students talk of working long hours on a problem, I'll ask, "What problems of consequence have been solved that took only thirty minutes or an hour? The theory of relativity? The invention of the computer? Fermat's last theorem? Were these solved in ten or fifteen minutes?" How many business failures does the average self-made millionaire experience before hitting the one that creates wealth? (Answer: Ten.) Problems worth solving take time and the people who solve them make lots of mistakes along the way. In Holt's words (Holt, 1965), "…we should see that failure is honorable and constructive, rather than humiliating." The culture and the grading must support this notion.

Classroom Techniques to Avoid

Attacks

Avoid using, or letting students at the board use, words like "obvious," "stupid," and "trivial." Avoid letting students attack or intimidate anyone. A proof by intimidation is not a proof. Don't let students attack one another at the board. Emphasize (demand?) a polite and respectful exchange of questions and answers. Over the past fifteen years in only one class did I struggle with students who seemed unable to cordially address classmates' questions. I prepared and presented the "Question and Presentation" guidelines available in Appendix II, which resolved the problem.

Correcting

Avoid correcting the individual, and instead address the work. For example whether posed by a class member or the instructor, asking the question, "Does line four hold for the function $f(x) = \ldots$" is preferable to "I think you are wrong on line four. What about $f(x) = \ldots$"

Hints

Avoid hints. Giving a hint implies that the student is not smart enough to conquer the stated problem. This is rarely the case. You might state a related problem, or ask a question that helps the student resolve the problem, but don't tell the student that the new problem is easier.

Runaways

Avoid letting the "bright" students run the class. If some students are head-and-shoulders above the class, pull them out and let them do an independent study. Or call on them last every time. In his biography of Moore, (J. Parker, 2005) Parker writes that both Gail Young and Burton Jones agreed, "this takes a certain amount of patience and understanding on the part of the brighter students as weaker students struggle to present…."

Better proofs

Avoid showing, or allowing a student to show, a "better" way to do a problem. If a problem goes on the board and there is a quicker or slicker way to do it and a stronger student starts to point this out, I will politely interrupt with something like, "Excuse me, John. The question at hand is whether Mr. So has a solution. Do you have any questions regarding Mr. So's work? If you have an alternate solution, I would very much like to see that after class or you may write it up and turn it in." I want all students to know that I respect *their* way of doing it, but I don't want a discussion on which way is better. On the other hand, if what appears to be a weaker student in the class, a student with lower self-esteem, or a student who has presented less has an alternate solution to one that a stronger or more confident student has put on the board, then I may make an exception to this rule and encourage that student to show the solution. This relates to one of the questions in the FAQ chapter—"does the Moore Method work better with stronger students?" And the answer may be, "it depends on how well you manage the students in the classroom."

Testing

Avoid testing. If you must test, de-emphasize it. My own words cannot express my opposition to rote testing as well as John Holt does in *How Children Fail*. Hence, I include here an excerpt on the subject from his text (Holt, 1965).

> *When I began teaching I thought, naïvely, that the purpose of a test was to test, to find out what the students knew about the course. It didn't take me long to find out that if I gave my students surprise tests, covering the whole material of the course to date, almost everyone flunked. This made me look bad, and posed problems for the school. I learned that the only way to get a respectable percentage of decent or even passing grades was to announce tests well in advance, tell in some detail what material they would cover, and hold plenty of advance practice in the kind of questions that would be asked, which is called review. I later learned that teachers do this everywhere. We know that what we are doing is not really honest, but we dare not be the first to stop, and we try to justify or excuse ourselves by saying that, after all, it does no particular harm. But we are wrong; it does great harm.*
>
> *It does harm, first of all, because it is dishonest and the students know it. … children learn that what most teachers want and reward are not knowledge and understanding but the appearance of them. The smart and able ones, at least, come to look on school as something of a racket, which it is their job to learn how to beat. And learn they do; they become experts at smelling out the unspoken and often unconscious preferences and prejudices of their teachers, and at taking full advantage of them.*
>
> *Not only does the examination racket do harm by making students feel that a search for honest understanding is beside the point; it does further harm by discouraging those few students who go on making that search in spite of everything. The student who will not be satisfied merely to know "right answers" or recipes for getting them will not have an easy time in school, particularly since facts and recipes may be all that his teachers know. They tend to be impatient or even angry with the student who wants to know, not just what happened, but why it happened as it did,*

and not some other way. They rarely have the knowledge to answer such questions, and even more rarely have the time; there is all that material to cover.

Especially in a Moore Method class where the emphasis is on the discovery and presentation of material, testing can disrupt the flow of the class, betray the honest nature of seeking out the mathematics that is the core of the class, and discourage students who are having success in this new format, but who still do not perform well under pressure.

(Parker) A day in a Moore Method classroom. Well, here we are. Class is due to begin in five minutes. I've reviewed what happened last class period, slept on it, and thought about it again. Beavis got a nice example that showed that Problem 15 is not a theorem because of the behavior of the function he defined at the boundary of the open interval. Cassie was working on Problem 21 when we ran out of time last class and, from what she's done so far, it looks like she can make an induction out of what she did when she assumed continuity on two disjoint open intervals. I've read the turn-ins that I collected three class periods ago (and am feeling pretty guilty about not having returned them sooner, but that is the way it goes sometimes), made remarks on the four papers, put firmly in mind the neat alternative to Justin's argument that Bertie found in her proof of Problem 17, and socked away a reminder to look for an opportunity to reinforce the way Justin organized his cases. Ben seems to have had roughly the same idea as Justin, but didn't sort out the cases effectively in his write-up, so if an opportunity occurs that allows us to recall the structure of Justin's proof, it will probably help Ben. I have a new sheet containing the next three problems I intended to hand out after Problem 21 got done, plus two problems that Beavis's example suggests.

SHOWTIME! *I return the papers.*

"Before we get started today, would you mind, Bertie, if I shared with the class an idea from your proof of Theorem 17? It's okay if you say 'no.' You may not be ready to share your idea yet, but it is awfully pretty." I mustn't bully Bertie into giving away her ideas since it may give her an edge in presenting next time her turn comes up. In fact, if I could have thought of a way that she might use what she did to resolve one of the problems that is still unsettled or created a problem that fit the course well on which her idea would work, I wouldn't be revealing what she did, just congratulating her on that beautiful idea different than Justin's that she had used in her proof. Bertie seems to be surprised, but gives her assent, and I give a brief description of her argument. After class, she tells me that she thought, since what she had done was different from what Justin presented, that it must have been wrong.

"Does anybody have a question about Beavis's example from last time or, for that matter, anything else in the course so far?" Harry asks a question about a technical detail in Beavis's example. Since the point of logic involved has surfaced several times, I answer the question, connecting the point to an earlier theorem and pausing several times to ask Beavis if I am being faithful to his argument. Katie Sue asks a question about the beginning of Cassie's argument, which I defer, suggesting that she ask it again when Cassie returns to the board.

"If there are no more questions, I have another sheet for you." I pass out the sheet and make appropriate remarks. In this case, I point out why Problems 25 and 26 are natural conjectures to make in the wake of Beavis's example, and that, if Problem 21 is, indeed, a theorem, then Problems 27, 28, and 29 are reasonable questions to ask.

"Cassie, can you finish?" From here, my job becomes wholly reactive, with due attention paid to keeping my peace.

Now that we've seen an example of what the beginning of a class might look like, let's go back and consider an outline of scenarios that are likely to occur on any given class day, and look at some ways to address them.

Getting started

I prefer to use the beginning of a class period to touch base with what has occurred in the previous class. This may mean repeating the highlights from an argument (either successful or failed) given at that time. If time ran out right at the end of an argument, it may mean doing the recap that usually follows a student presentation. If something has shown up in a write-up of a problem, this is a good time to remark on it. If I have a new sheet for the class, this is typically when I hand it out. New sheets are usually given in response to the class asking for more problems, or in response to the class having solved problems that indicate, in my mind, that they are ready for additional problems, or in response to a question or questions that the class has raised. If there is an aspect of the problems that I want them to deal with naïvely, I will give out the new problems without substantive remarks. More often, I will read through them and underscore any point of logic that has troubled them previously, or give a lesson on "where conjectures come from."

I *always* ask for questions during the beginning time, usually by asking "Does anyone have any question before we begin?" I always try to emphasize that what we are about is making more mathematics; thus, anything else is a preview or a postscript to their presentations. Student questions are typically about problems that have not been settled or how recent work fits together. When a student asks about the specifics of a problem that has been settled, I decide whether I think the class as a whole will benefit from my answering it or whether the student who presented the work about which the question was asked will benefit from answering the question. If neither of these issues seems viable, I point out that the matter has been settled to the class's satisfaction, and invite the student to come by and discuss the question with me one-on-one. This is not a put-off since I have promised, from Day One, to go over with any student, in my office, any problem that has been settled in class as many times as he or she wants to see it. This is the students' safety net for getting a clean write-up of any problem that meets the class's approval, and removes the requirement of careful note taking.

I have a preset order on the class roll which I follow. If a student is at the board at the end of a class, he or she may continue during the next class period. Otherwise, the student following the last student in the preset order who was asked to present gets the first opportunity.

Student presentations

When a student's turn comes, the student may choose to show his or her work on any problem that is currently open. A proof of a theorem or an example that shows that a problem is not a theorem is what I hope for, but a student may also choose to present whatever progress he or she has made on a problem. I usually take the seat that the presenting student has vacated, symbolism for my becoming one of them during the presentation. Creating a proper level of student-student or student-teacher interaction (ideally predominantly student-student) during student presentations is the most difficult aspect of Moore Method teaching for me, probably because I like to be a part of conversations. Nevertheless, the ideal for which I strive is that students will uncover any errors and that my role will be reduced to recapping arguments and finding the best problems to follow up with. But this is utopia, not routine reality, when teaching undergraduates. In particular, early in a Moore Method course in which the students have not previously been exposed to the Moore Method, students are often reluctant to question a fellow student's claims even when they have reservations about the argument, and may even be reluctant to present their own findings for fear of being wrong. In regard to raising questions, a common student conception, nurtured by the experience of "this lecture is being given by an expert, the expert must be right, *so* I will take careful notes and sort things out later" can easily transfer to "there is an expert in our midst, the expert hasn't said anything, thus the presentation must be correct, *so* I will take careful notes and sort things out later." Students may also be afraid of offending a peer. Thus, creating a nurturing and safe classroom culture[6] in which these "social" issues are seen as constructive rather than destructive is of paramount importance.

A proactive ploy to foster student interaction is to appear to be puzzled, but not say anything. There is the possibility that your reaction may reinforce the doubts of someone in the class and that person, thus fortified, will ask the question that needs to be asked. There is also the possibility that the presenter will notice you and ask you "What is the matter?" In this case, you turn to the class and ask, "Is anything the matter?" If someone asks a question, then you have what you want. If no one does, you have a decision to make. One is to respond to the silence with, "Then everything must be okay. Please continue." This saves any issues for the end of the argument. Then, there is the possibility that the argument will crash on another issue, or that this issue will tear a hole in it so large that a student will see what the problem is clearly enough to raise a question. Or, you can raise the issue you had in mind when you acted puzzled. I consider this last option the least attractive since it makes the instructor an active agent instead of a catalyst. If I do choose to voice a concern, I never use an accusatory mode. Rather I direct the problem at myself. An appropriate preface to a remark is, "I do not understand …." Inappropriate prefaces are: "There is a mistake here…" or "Do you mean…" or "It is unclear…"

Some very good things can happen when the presenter responds to "I do not understand …" by clarifying his or her idea or patching a faulty argument. Even when the presenter doesn't understand why you don't understand, the discussion that occurs when you turn the question on the class can be beneficial. On the other hand, there can be bad long-term consequences brought on by taking an overactive presence in class dialogues. Chief among them is that it can become expected behavior for the instructor to intervene when something is wrong, and this is not what we want. What then should be considered when deciding whether or not to interject oneself into a problem?

The best response I can give is that it depends on the class, and it depends on the nature of the error. Since I am already on record as being committed to giving the students the logic, but having them make the mathematics,[7] I am much more inclined to intervene

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6 See Chapter4—On Culture.

7 See Chapter 5—Materials, Development and Selection.

when a question concerns a point of logic. Particularly early on in an introductory course, this may actually be appropriate since the question raised may be used to re-emphasize the point that logic arbitrates arguments in mathematics and thus dictates, at least partially, the form an argument must take. Also, the students may well not yet understand the fact that part of the language that has been given them is for their use in expressing logic, and separating such language from the language of the mathematics can be educational. Even in a second-level class in which the students have already been held, at some level, accountable for proofs in an earlier course, I sometimes find it instructive to raise such issues. Since I find that the students who are most confident in what they are doing often use symbols to express the logic, it can be instructive to have them articulate which words from the active vocabulary of the course are being translated into which symbols. In an advanced class, if a student starts throwing around symbols that have not been defined within the course, I may actually pretend not to understand even when a student is correct, just to heighten the awareness of the expectation that symbols need to be defined. I *am not* saying to always intervene on a point of logic. If, even in an introductory class, a comparable error has been made a sufficient number of times that there is a reasonable expectation that someone in the class (notably someone who has made the error and had it corrected) should see the mistake, then I wait.

Just about any error within the mathematics is best left alone to run its course. Nevertheless, the question of when to sit a student down still remains. When a student is challenged and sees that the proposition that elicited the challenge has merit, I always give the student an opportunity to "fix" the problem. Indeed, at times, I will actually try to clarify the question, usually by prefacing it with a remark to the questioner such as "Would you tell me if I understand your question correctly?" and then making the clarification while facing the questioner. This gives the presenter a few extra moments to collect his or her thoughts and allows me to reinforce the merits of the very act of questioning, since I can end the sidebar with "That is a nice question!" as we return our attention to the presenter. Usually, once a presenter recognizes that the question has identified a problem that he or she cannot surmount, he or she will stop of his or her own volition. However, it may be appropriate to intercede, particularly if the presenter is repeating the error in the same form in which it was originally challenged. Even if an attempt to make a correction is patently wrong, if it is different, I think it is a good idea to let it run its course. Every time a student articulates a new idea, even if it is not pertinent to the question at hand, it is still a new idea. I work really hard at finding problems for subsequent distribution that redeem such ideas. We must remember that often the mistakes that students make can be more instructive than their correct arguments. After all, if all that interested us were correct proofs, we could just have the students read books.

There are no hard-and-fast rules on teacher participation. On the other hand, if you have doubts about interrupting, err on the side of silence. You will always have an opportunity at the end of an argument to raise questions.

Following up

Once a correct argument has been made and I am sure there are no questions from the class, I praise the presenter. Success is what I want, and having just witnessed it, the time to acknowledge that mathematics has been made is at hand. In the wake of the celebration of validation, it is often instructive, particularly in introductory courses, to recap the argument by pointing out how the presenter met the demands of the logic and to gush over any new idea introduced or any clever use of a previously introduced idea. Also, if a technique that had been created before is used, or a previous theorem is invoked, this is a good time to praise, again, the previous work and the student or students who initiated or nurtured it. Remember, no matter how transparent an argument may appear to us, it was not likely transparent to the student. We do not want to diminish in any way the triumph a student has wrought.

Incorrect arguments should also be praised. Find something good about the presentation and point it out.[8] This may be material that was proven before the fatal flaw occurred, but it could be, if a presentation has very little to offer, something as simple as congratulating the student on clarifying what the problem is. If the student has proven something, I will often isolate what was proven, state it as a theorem, and add it to the notes on the next page given out with the name of the presenter attached to it. If the theorem connects directly to the problem whose truth or falsity the student attempted to argue, I will give it a number such as $28 - \frac{1}{2}$ or $28 - .01$, with the fraction subtracted indicating how close I think the student is to succeeding. Also, if the student has shown that, by proving one more theorem, the theorem in evidence together with it would be sufficient to settle the problem in question, I will state that theorem as a numbered and name-attributed problem. If I can see that proving one more theorem would, with the theorem in evidence, settle the problem in question, I will typically state that additional theorem as a problem on the next sheet I hand out without comment, leaving it to the class to figure out its potential worth.

> Question: "Clint, how did you get that line?"
> Answer (Clint): "I got it from Maple."
> Question: "Could you explain why Maple is correct?"
> Answer (Clint): "Can I get back to you on that?"
> Next class period (Clint): "Let me show you why what I said last time isn't right and what is."
> Exchange from a junior-level analysis class

Observing both correct and incorrect arguments gives you opportunities to adjust your notes to fit your class. When a student settles a problem with an argument that you had not anticipated, you may want to state another problem on which such an argument will work. When a student settles a problem with an argument that you had not anticipated, but the argument you anticipated is an important part of the mathematics you hope to teach, you need to find another problem for which the argument you anticipated will work. Particularly with undergraduates, do not make elegance judgments on mathematics that is correct. Let them find what you had in mind as they work on other problems. In an incorrect argument, a student will often make a construction that has merit. We can design problems so that such ideas can be redeemed.

I have had more than a half dozen teachers who used the Moore Method effectively in classes that I have taken. No two of them were remotely alike in how they managed a classroom. I once tried to prod one of them, whom I judge to be the best at the Moore Method that I have experienced, into documenting some of the things that he did. He basically blew

8 Done properly, this can go a long way toward defusing the fear of looking stupid in front of peers.

me off, responding at the beginning of the exchange with the dismissal, "All you have to do is listen to what they say." I do not believe that it is quite that simple, but certainly the primary activity in the classroom is *listen to what they say*.

Day-to-day preparation for being ready for in-the-classroom dynamics

A second level of preparation occurs from day to day during the semester because, just as in a lecture class, pre-class preparation is critical to in-class success. The bulk of this preparation can be check-listed.

1. What problems were addressed in the class and which were solved?
2. Did the presenters forge new ideas or did they apply ideas already in evidence?
3. Was a given presentation the presenter's first presentation? If not, how does it compare with what else he or she has done?
4. Did what transpired create the need for any new problems?
5. Should I follow up with commentary at the beginning of the next class period?

The answers to these questions should provide a basis from which to prepare for the next class. In my courses, I offer myself as a continuing source for any problem once it has been solved. Reviewing successful arguments gives me the chance to decide whether or not I need notes to myself in order to be able to reproduce the argument, as given, in case a student were to come to my office and ask about the problem. Also, I have found that reconstructing the argument sometimes gives me additional insight into how the student put the ideas together, since my main focus during class is usually, "Is this correct?" Thus, it may give me other things I want to reinforce beyond those I may have highlighted when I recapped the argument on the spot. It may also encourage me to repeat, at the beginning of the next class period, some points I highlighted in the recap.

For arguments that are begun, but interrupted by the end of the class period, we have an improved context for making an intelligent guess at what will likely happen next and thus to be prepared to exploit such occurrences were the guess to be correct. We will already have found a way to praise something about a failed argument after it failed, but the leisure to think about it away from the excitement of the class can uncover additional ideas that might be nurtured. This is how I look for ideas, create problems that I had not originally intended for the notes, and make decisions on whether or not it is time for more problems. Reflecting on the work of individuals against what they have done previously should provide prompts that allow us to encourage them as individuals and perhaps accelerate any momentum they may have created. Sometimes a problem is added because the class is ready for it, but a problem can also be added because you think some one person will be able to get it and sustain his or her success. Even if someone other than the intended target solves the problem, you will have likely made an occasion in which you can reinforce the intended student's work by making a link to it in the recap.

To give a concrete example of what I have described above, let's consider an actual course that I give periodically at James Madison, the first semester of our real analysis sequence. In the current incarnation of this course, the notes with which I begin the semester have as first-level goal theorems the intermediate value theorem and the extreme value theorem for functions continuous on an interval. The completeness axiom that the students are given is the Dedekind cut axiom. In the *pre*pared notes are problems that guarantee that the students will dissect the definitions of function and order, address particular functions and the continuity of those functions, and address the continuity of arbitrary functions formed by using the algebra of $+$, $*$, and $\circ$.[9] I am prepared, on the basis of how the students handle these problems, to state more problems about particular functions, or, if things happen like I hope they will, to use their arguments to indicate why the structural problems that are precursors to the goal theorems are plausible.

Fall before last, my class developed a myopia biased toward the algebra of numbers that was so strong that, given $f(p)$ in (a,b), they would try to show that both a and b were in the range of f and then claim that choosing the open interval between the pre-images for a and b(!) guaranteed the continuity. In all of my previous experience in the course, trying to make the estimates that guarantee the continuity of I^2 [10] at 0 had shown this to be a doomed strategy since, for $\varepsilon > 0$, $-\varepsilon$ isn't in the range of I^2. But it didn't register with this group. So, consequent to a remark a student made in working on I^2, I pulled out the square root function[11], and through the investigation of its continuity and domain, they were forced to deal with the question "Why must each positive number have a square root?" (a fact I am usually willing to grant as an assumed technical lemma) and use the consequences of the Dedekind cut property to answer the questions. This particular class eventually put the lub/glb theorem in place using the nested-interval theorem, and used the lub/glb theorem to get both the intermediate value theorem and the extreme value theorem. I had thought, from an idea that appeared in an early proof attempt, that the Bolzano-Weierstrass Theorem would be the direction in which the class would go, but that line of inquiry died. What's more, the student whose work initiated the "need" for the problem wasn't the one who made the ideas work.[12] Other times when I have taught this class, B-W thrived and became the big tool, or the Heine-Borel Theorem appeared. One memorable semester, the class got so good with the Dedekind cut axiom that they used it directly to get all of the goal theorems and, during the second semester, even after they had B-W and H-B available, continued to make their arguments directly from cuts.

To recap, preparation for a class includes:

1. Thinking about what the students have done in the previous class(es) and deciding whether or not following up on the reactions made at that time (either yours or the students') might have the capacity to enhance the learning experience.
2. Thinking about whether what the students have done presents ideas that merit inclusion of a problem or problems that, if solved, would allow the ideas observed to comment on the problems at hand or take on a life of their own.
3. Thinking about what the students have done in order to anticipate what may come next relative to the problems that you had planned to state and to decide whether or not the time is right to give out more problems.

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9 The idea is that the problems on the definition of function and order will give me a gauge of their command of logic, and the students will develop familiarity with the definition of continuous through functions that they have, at least, manipulated in calculus. These functions can then be used as examples on which to try out sum, product, and composition estimates as they progress towards working with arbitrary continuous functions.

10 $I^2 = I*I = \{(x,y) : x$ is a number and $y = x*x\}$.

11 $\{(x,y) : x$ is a number that has a square root and $y = \sqrt{x}\ \}$.

12 I am able to "recall" these particulars because I kept a course diary, but similar situations, although not necessarily this extreme, occur on a routine basis.

We always begin a class with a plan, but we also begin having the knowledge that we can bend the plan to give impetus to the ideas the students develop. We adjust our grand plan as we prepare day-to-day.

A day in which no one has anything to present

From time to time there may be a day in which no student has anything to present. My most common response is to suggest that we work on a problem as a group. I ask if anyone has a problem that he or she wants to see the group work on, and give all students veto power. Thus, if a student thinks that he or she is near a solution or has some other reason for keeping the problem pristine, that student can veto the class working collectively on the problem. Once a problem has been decided upon, I declare the discussion to follow to be a "no-notes" period. This way, if anything is uncovered, each student will have to at least recover the thinking that produced the discovery without notes. I then proceed through the guideline for working on problems,[13] and have the students identify any definitions or theorems they consider relevant and discuss possible connections among these definitions, the proofs of the theorems, and the problem on which we are working. At this stage, if no attack on the problem has surfaced (if an attack emerges, I stop and go on to another problem), we proceed to try to make an example that exhibits the premise of the problem and see if we can get the conclusion for that example. If we can, I ask what is special about the example. If we can identify such a quality, we check to see if we used it in our proof. If we did, we try to state a problem that includes that quality in the premise before trying to make another example without the quality we have identified. At any time, if I think the students have seen enough so that they should be able to solve the problem, I will call a halt to the proceeding, tell them that they have seen enough, and go on to another problem if there is time.

Another response is to use the "dead day" to present, with the discoverer's permission, any technique that you may have seen on a write-up but that the class as a whole has not seen. In some classes there may be theorems that the class has been granted as true; this is a good time to make proofs for such theorems. For instance, in my analysis course, I often grant the class "technical lemmas" which the students may use as if they were true, but which I do not declare to be axioms. Also, in two-semester sequences, I often have theorems from the first semester that don't get proven that I grant as true at the beginning of the second semester. Again, whenever I present, there is a "no notes" policy.

A third response, that can only be used once in a given course, is to act hurt (not angry —the effect I am aiming for is that I believed in them and they didn't meet my expectations and have thus "let me down") and dismiss class. There are two scenarios for which I believe this to be effective. One is very early in a semester. At this time, the students may still believe that they can force me to "teach" them. Since they do not know that I won't do it again, at least a renewed effort is usually forthcoming. In my experience, the beginning of the next class period is usually full of questions and my office traffic is usually busy on the intervening day. The other scenario is when I perceive that a class is slacking. In either case, if another dead day follows soon thereafter, when I lead the class discussion, when something is uncovered, I will make a remark such as, "That sure seemed to be there for the

[13] See Appendix IV, "An Introduction to Doing Mathematics."

finding. I wonder if I would have thought of that." I never accuse them of not working, but try to give every indication that someone who was working would have almost certainly stumbled onto such an idea.

The first day

This day is atypical since it is teacher-dominated. I have discussed, in my essay on culture in the classroom, how I use it to try to fertilize the culture that I want to develop.

Summary

We can use the dynamics of the classroom to inform us of where each individual student is and what he or she might be thinking. We can use what we hear to accelerate the class's momentum by tailoring at least some of the problems to the ways the students are thinking. We can reinforce positive things that students do and make positive recognition of even slight progress. We will listen to what they say and think about it so that we come to class prepared for things that might happen and realign or enhance the notes appropriately. If we are unsure of what to do, we err on the side of inaction.

(May) The best way for me to explain what goes on in a typical classroom for which I am responsible is to present several pairs of my lesson plans and diary entries. All are taken from my course, "Introduction to Abstract Mathematics." The course has a double purpose. For majors in mathematics who want more preparation as they cross the bridge from the computational experiences of the freshman and sophomore years to the abstract ones of the junior and senior years, the course is a first experience with all-the-time abstraction. For students concentrating in mathematics education, and those minoring in mathematics, the course serves as a capstone to their program.

It might be helpful if I say something about the diary. This is the best means that I have found for evaluating my students' performances in class. To each meeting of the class I take a pad and pen. With them, I attempt to record, in my personal shorthand, every significant event that occurs, and the person or people responsible. As soon as class is over (I ask for a schedule that contains no back-to-back classes), I return to my office and, with a word-processor, convert the shorthand into a full-fledged diary entry of an average length of one page. The diary has proved to be an invaluable tool for keeping track of students' progress. It also forces me to pay close attention to everything that goes on in class, conjectures raised and other comments made by students in their seats as well as the argument proceeding at the board. Finally, producing each entry of the diary aids me in preparing the lesson for the next meeting of the class.

> "Independent learning—there was a track that we followed but the students controlled how to get to the end."
>
> Anonymous answer to the question, "What did you find most valuable about this course?", on an evaluation of Introduction to Abstract Mathematics.

The selections that follow span the term. The first lesson-plan/diary-entry pair is from Day One; the final one deals with the last meeting of the class; and the second and third pairs cover typical days in between.

Lesson 1

I. Introductions

A. The course itself: MATH 300-001, "Introduction to Abstract Mathematics"

B. Me (name, office-hours)

C. Students: Call roll. (Operate on a first-name basis?)

D. Overview and policies of the course

1. If the overview, syllabus, and evaluation-form haven't already been handed out, hand them out now.
2. Go over them briefly. Highlight the passages dealing with my expectations of them, especially the centrality of "homework" to the course and the amount of time that I expect the students to spend on homework.
3. Entertain any question or suggested modification of the policies. (A final decision about the policies will be made by the class and me during the next meeting.)

II. "Lecture" : Answer questions on Chapter 1 of the notes. If it seems wise to do so, lead the class through the following development.

Definition. If each of a and b is an integer, then the statement, "a divides b", means that there is an integer m such that $b = ma$.

Example. 4 divides 12. 5 divides 25. 7 does not divide 10.

Theorem NT.1.[14] If each of a, b, and c is an integer, then the following statements are true.

(a) If a divides b and b divides c, then a divides c.

(b) If a divides b and a divides c , then a divides $(mb + nc)$ for each integer-pair $\{m,n\}$.

(c) If a divides b and b divides a, then $a = b$ or $a = -b$.

If there's time, allow everyone to mull over Theorem NT.1(a) before asking someone from the class to present a proof, or leading everyone through a proof yourself. Do the same with parts b and c of theorem NT.1. If the opportunity presents itself, mention the three-fold process consisting of clarifying, deciding, and proving or denying.[15]

III. Homework for next time

A. Attempt to prove Theorem 1.1 of the notes.

B. Suppose that n is an integer not less than 1, you've been able to prove Theorem 1.n, and you still have some time that you're willing to invest in the course. Then attempt to prove Theorem 1.$(n + 1)$.

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[14] NT stands for "Number Theory."

[15] "The three-fold process of clarifying, deciding, and proving or denying" is what I use when studying any theorem or conjecture that is new to me. First, I make sure that I understand the statement of the theorem or conjecture. As I do so, I arrive at an initial judgment as to whether the statement is true. If I decide that it is, I begin to try to prove it. If I decide otherwise, I begin to attempt to construct a counterexample to the statement, or otherwise attempt to disprove it. I mention this process to my students.

Diary Entry 1

Theorem NT.1(a). If each of a, b, and c is an integer and $a|b$ and $b|c$, then $a|c$.

AC presented a correct and pretty-clear proof of this. She is the first member of the class to go to the board. **KR** asked a good question.

Lesson 7

I. **Discussion and presentations** (People whom I'd like to see at the board: **EB**, **HC**, **EF**, **EG**, **JW**)
   A. Theorem 1.7, *redux*: **CC**
   B. Theorems 1.8 through 1.12
   C. The exercises on Pages 3 and 4 (Procedure regarding exercises: we'll look at people's work on them after we've looked at all of the proofs of theorems that people have to present in a given period.)

II. **Homework:** Suppose that n is a nonnegative integer and you've been able to prove Theorem 1.n. Then try to prove Theorem 1.(n + 1).

Diary Entry 7

This was another good class. (It actually began before the period officially started, with **JW** seeking help with her argument on Theorem 1.10. She appears to be a good theorem prover. Even her difficulties are sophisticated.)

Theorem 1.7. If each of A, B, and C is a set and there is a point common to each of A, B, and C, then $A \cap (B \cap C) = (A \cap B) \cap C$.

CC presented a correct, if laborious and redundant, argument. He needs work in the construction as well as presentation of proofs. Helpful comments were offered by **EB**, **JW**, and **AC**.

Theorem 1.8. If each of A, B, and C is a set and A intersects each of B and C, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

EG, in arguing that the left-hand side (LHS) is a subset of the right-hand side (RHS), picked her "typical" point from $A \cap B \cap C$. **JW** caught her mistake. Her proof that RHS $\supseteq$ LHS was excellent. Then, when I questioned her about her over-assumption on the first half of her proof, she did a beautiful job of thinking and adjusting on her feet, correcting her argument on the spot. It was a virtuoso performance. Helpful questions and comments were offered by **CC**, **JW**, **AC**, and **EB**.

After class, **HC** showed me very good proofs of Theorems 1.10 and 1.11, and **BP** displayed a sound modification of his argument on Theorem 1.7.

Lesson 30

I. Get a report on how Friday's class went. (I was at the fall meeting of the Maryland-District of Columbia-Virginia Section of the MAA.)[16]

[16] On this day when I was away from campus, I, as I usually do in my Moore Method classes, left the class in the charge of the two best students. (Sometimes, I will choose only one stand-in.) They called on people to go to the board and moderated the discussion.

II. Discussion and presentations (People to call on: **HG**, **JW**, **DJ**, **DN**, **AC**, **BP**, **EB**, **KR**, **EG**)

A. Within the sequence of theorems:

1. Any leftover from Chapter 2? (For example, an "easier" proof of Theorem 2.3 from **HC**)
2. Chapter 3
   a. Theorem 3.1
   b. Theorem 3.2
   c. Theorem 3.3
   d. The lemma to Theorem 3.3
   e. Theorem 3.4

B. Outside the sequence:

1. Any exercise
2. **HC's One-to-Oneness Conjecture.** If the function f is one-to-one, then its domain contains the same number of elements as its range.
3. **Conjecture.** If A is a set, then A is not an element of A.

III. Homework: Suppose that n is a nonnegative integer and you've been able to prove Theorem 3.n. Then try to prove Theorem 3.$(n + 1)$.

Diary Entry 30

Almost from the start of today's class, it became obvious that the class, almost to a person, was as comfortable with the real-number system as it had been uncomfortable about the concept of *function*. I asked for suggestions as to what I could do to increase students' comfort-level about functions. "Modify the definition of *function*," came back the immediate reply. The modification desired seemed to be to replace the existing definition of *function* with this: "a function is a set of ordered pairs no two of which have the same first term." I shall do this before the beginning of next semester.

Theorem 3.2. A number set that is bounded below has a unique greatest lower bound. **CC** and **JW** both volunteered to prove this. I chose **JW** because of her having been to the board less frequently than **CC**.

JW began a proof-by-contradiction that yielded four cases, where each of x and y is a greatest lower bound of the same set A and z is a lower bound of A: (1) $x = z$ and $y = z$; (2) $x > z$ and $y = z$; (3) $x = z$ and $y > z$; (4) $x > z$ and $y > z$. After obtaining a contradiction in Case 1, she stopped, concluding, as apparently had been done on Friday in Theorem 3.1, that deriving a contradiction in one case was enough to demonstrate that the assumption, "$x \neq y$," was false. After I stated that this simply eliminated Case 1 from the list of possibilities, **JW** said that she would try to deal with Cases 2 through 4 for Wednesday.

AC then put up the essentials of the Book Proof[17] for both Theorems 3.1 and 3.2: if each of b and b' is a greatest lower (least upper) bound of A, then $b \leq b'$ and $b' \leq b$, so $b =$

[17] The "Book Proof" to a theorem is, in the view of the late, great Paul Erdös, the best—that is, the clearest and most elegant—one. See (Hoffman, 1998).

b'. Bravo! **AC** is really coming along well.

A very worthwhile and fascinating discussion then ensued. **HC** asked to see an example of a number set that was not bounded both above and below. She also asked for a definition of *ordered quadruple*. I asked for a definition of *ordered triple*, to get us going. **HC** then popped to the front of the room and produced, first, the Book one for *ordered triple* and then essentially the Book one for *ordered quadruple*: an ordered pair consisting of an ordered triple and a point. **HG** introduced another, that of an ordered quadruple as an ordered pair of ordered pairs. I stated my belief that the two definitions were equivalent. I also pointed out my opinion that **HC**'s definition generalized more conveniently than **HG**'s.

DN offered her latest conjecture: the set P of all positive integers is bounded above and below. I'm afraid that it comes from **DN**'s misreading the part of Axiom 3 that describes P, but I continue to applaud her willingness to formulate conjectures. It shows that she is doing a commendable amount of thinking about the course.

Other contributors to the discussion were **EG**, **BP**, **EB**, **CC**, and **KR**.

As class ended, I pointed out that a critical issue of Theorem 3.3 that had not been addressed was the existence of a greatest lower bound for a set that is bounded below.

Lesson 42

I. Discussion and presentations

A. Format. Anyone may present anything that he or she wishes to present. The order in which I would like people to present is as follows: **AC** (Theorem 3.7 or something related), **JW**, **DN**, **BP**, **EB**, **EG**, **DJ**.

B. Topics

1. Within the sequence of theorems:
   a. Theorem 3.5: **DN**?
   b. Theorem 3.6
2. Outside the sequence:
   a. Any exercise?
   b. **HC's One-to-Oneness Conjecture.** If the function f is one-to-one, then its domain contains the same number of elements as its range.
   c. **Conjecture.** If A is a set, then A is not an element of A.
   d. Any leftover from Chapter 2? (For example, an "easier" proof of Theorem 2.3 from **HC**)
   e. **DN's Latest Conjecture, Revised.** The set of all positive integers is bounded below but not above.
   f. **AC's Conjecture.** Every finite set has a greatest and a smallest element.
   g. **AC's Latest Conjecture.** Denote by -P the set of all negative integers. If the number x is in $-P$ and z is a number such that $x < z < x + 1$, then z is not in $-P$.

II. Lecture: A wrap-up of the course

Diary Entry 42

Today...I witnessed effort and commitment that I hadn't seen for several weeks.

Theorem 3.10. If each of d and k is a positive number, then there is a positive integer n such that $nd > k$.

CC pulled off one of his better efforts in recent memory. He considered three cases: $d > k$, $d = k$, and $d < k$. On each of the first two cases, he provided an acceptable argument. On the "$d < k$" case, however, he displayed the traits that have haunted him all semester. He said, in effect, that, if there is a positive integer q such that $qd = k$ and there is a positive integer m such that $md > k$, then we're done, and stopped. He never realized that it was his responsibility to show that there was a number such as q.

The rest of the class didn't do a whole lot better in reacting to **CC**'s argument. **EG**, **DN**, **EB**, and **BP** all raised questions about or minor objections to **CC**'s argument, **DN** and **EB** most effectively; but no one came out and said, "Your argument hasn't convinced me, and this is why..." I mildly chided the class for its lack of zeal or aggressiveness (I mistakenly used the term *aggression*) in pursuing the truth of mathematics. **EB** then presented her argument about this same theorem. It wasn't much more advanced than **CC**'s. **BP** then held forth. With my prompting, he provided 2 as an "n" in the second case, and, surprisingly, with more difficulty produced 1 in the first. For the final case, he accepted the truth of **DN**'s conjecture that P is not bounded above.

Theorem 3.13. If each of x and y is a number, then

(a) $|x + y| \leq |x| + |y|$

and

(b) $|x * y| = |x| * |y|$.

JW presented a terrific proof. Instead of bogging herself down in the usual case-proof, she cut right to the chase, employing only the two cases, "$x + y \geq 0$" and "$x + y < 0$." She handled each one with skill and aplomb. My mathematical heart aches to think what she might have contributed to the class and to her own mathematical development had she put in more time and effort between her sterling performance early in the semester and this one today.

Everybody present contributed to the discussion, even if only hesitantly to voice an opinion about the correctness of someone's argument...

The material above provides, I hope, some insight into what occurs, on a daily basis, in one of my Moore Method classes. As with most of life, however, there is no substitute for experience. The best way to imbibe the spirit and technique of Moore Method teaching is to take such a course. The second-best way is to teach one under the mentoring and guidance of a Moore Method practitioner.

(Coppin) The methods I use vary greatly from class to class, even with the same course number and sometimes depend on the particular group of students I have in front of me. Therefore, in that spirit, instead of attempting to give a generic description of how I would run a class, I have chosen to describe a particular class I taught in the spring of 2001. A general description, a one size fits all, will not communicate to you, the reader, as much as an instantiation of the specific. For example, when I studied the computer programming language Java, I understood very little from the written discursive material *until* I saw a particular example of a Java program. *And*, I didn't understand the concept thoroughly until I had actually implemented a Java program of my own creation. By analogy, you, the reader, if you choose to teach a course by the Moore Method, must implement your own Moore Method class and not emulate anyone else's. I can't do that for you anymore than the author of the Java book could create and run my program for me in such a way that I learned Java deeply. However, I can give you an example, the most useful thing I can do at this point. In particular, this was a class in Synthetic Geometry taught to liberal arts students. I call this class ENE (Euclidean and Non-Euclidean Geometries). I chose a liberal arts class because, in the mind of some, the students of such classes are the hardest to teach mathematics, that is, authentic mathematics; therefore, you might learn something useful for any course you might want to teach. The word "notes" refers to a set of notes I have given individuals who desire to teach this course. In Appendix I, you will find a diary I wrote while teaching this class. Clearly, as you will see, the method I use is the Moore Method. I have changed the names of the "innocent" in the diary for their protection.

ENE Teaching Notes

Imagine you have been assigned this class to teach and I am to assist you in getting started. Here are some preliminary things I believe you should think about.

The notes

Refer to Appendix I for a sample of the notes that I use in ENE. One highly effective technique is the use of worksheets, some of which you will find in the appendix. I found that worksheets were very helpful in managing the classes, especially if courses were larger than desired. Moreover, and most importantly, the worksheets handed out at each class period focused the students from day-to-day: each student was clear as to what was required from one class period to the next. Moreover, the students had the option of turning in their worksheets for grading. They could receive presentation credit for written work although more credit would be received for making a presentation at the board. I made comments on each worksheet, assigning a grade to each problem on the worksheet. If students were not satisfied with their grade, I allowed them to turn in an "improved" version for re-grading. If a worksheet is recycled two times without an acceptable grade, I recommended that the student involved present to me in my office where we can have a productive one-on-one session.

Format

The following is the routine in most of my classes of ENE during the semester. Before we start the worksheets, I pass around the roll sheet and any new handouts. I make preliminary announcements, such as an examination time, colloquium speaker, etc. Generally, I ask if there are questions from the last class period. If there is unfinished business, I will review the last class and tie things together. Sometimes, I set up the next worksheet by explaining new axioms, definitions, and theorems. If there are problems that arise during class, I then give additional explanations.

When we start the worksheet for a particular day, I may allow students to work individually on a worksheet during a portion of some class. Most likely, this would be at a time when students are having significant common problems with the material and, thus, need extra help. However, normally, in most class sessions, I take up Worksheet *X*, and ask who has Problem *Y*? I write those names on my copy of the worksheet and pick one or two people to go to the board to write their responses. When they finish, I ask them to "walk" us through their responses on the board by reading and explaining what they have written sentence by sentence. This tends to focus our attention. Without divulging my own judgment as to the correctness, I ask the class to critique or ask questions for clarification. This is most important; teaching students to critique presentations of their classmates. Usually, students do a respectable job of focusing on those areas of the presentation that have problems. If there is an error, they may not know exactly what is wrong but they know something is wrong. It is up to me to direct the discussion so that it is fruitful and that it informs both the presenter and the students. Students, however, are to give no direct aid. After some discussion, I may make my opinion public; however, I do not divulge the key to the proof, problem or question at hand, if the presenter has not displayed the key. To do so would be counter to the Moore Method. Finally, everyone, especially the presenter, must be encouraged even if he or she is not successful. I attempt to find something positive to say. This is normally not difficult.

A change of pace such as a small lecture on some aspect of history or a mathematical sidelight may cut through the day-to-day routine. As much as possible, I try to make the class fun *and* mathematical. Students need to enjoy themselves!

The following are some notes that you may find helpful:

1. With a course like ENE, you actually have a great deal of freedom because of its sparse syllabus. There is only one dictum: Get each student to the point where he or she can prove a significant number of theorems completely on his or her own without recourse to outside help. In time, many students will have a transformative experience.
2. Discuss with others who have taught this course. You may consult them from time to time. Anyone who has taught the course will be most happy to assist you in getting started.
3. I personally use a modified form of the Moore Method. A copy of *Challenge in the Classroom,* a film about Moore and his method, is available from the Educational Advancement Foundation.
4. Coverage is not of paramount importance. In the past, students have presented approximately fifty-nine problems and theorems in an ENE course; however, this is a high number. The development of mathematical maturity is more important than coverage. Personally, I always attempt to produce "budding" mathematicians even if there are no mathematics majors in the class. I identify mathematics more with its processes than its information. We are seeking to inculcate in our students that essence of mathematics that is invariant throughout the ages. For example, I believe that such individuals as Archimedes, Rene Descartes, and Isaac Newton were math-

ematicians even though they could not be expected to pass the mathematics qualifiers in a typical contemporary doctoral program. That is, even though they did not know subjects such as modern algebra or functional analysis, we believe they were mathematicians, even of a high order.

5. One exercise that has been successful is to set aside a class period for a review of proofs after we have covered several worksheets. This is done just before an exam. The format is as follows: One of the students volunteers to be the stenographer for the class. He or she writes a line of a proof on the board that is posed by other students. I call on the students one by one, to give the first line of a proof or the next line of the proof. When we finish the proof, each participant earns some credit, either check plus or check. The stenographer earns credit as well. This drill is good for class chemistry and gives the class a change of pace. This assures that students have at least one "correct" proof in their notes. It also allows discussions on the finer points of writing proofs. Students are allowed to pass without participation and without penalty. I do not want to put undue pressure on students. Students are not allowed to look at their notes during the exercise.
6. The purpose of memory exams is to firmly plant the definitions and statements of theorems in the students' minds that might otherwise be just dim impressions. The exams are intentionally easy. This is fine for the very reason that the meat of the course is the presentations. I may put proofs on the exam that they have already seen. Thus, the memory aspect also includes proofs or examples students have already seen.
7. During the last two weeks of class, it would be good to cover the discoveries of Bolyai, Gauss and Lobachevski. This could start with a discussion of Euclid's Fifth Postulate. Discuss why, historically, individuals believed that the Fifth Postulate could be proved from the preceding four postulates. I point out that false proofs were believed to be true proofs for a very long time. I usually make the point that this effort was an example of mathematics done for its own sake rather than utility. This is an opportunity to make a point that truly great discoveries in mathematics are often definitional in nature and are made strictly out of intellectual curiosity or aesthetics. It was Bolyai, Gauss and Lobachevski who realized that the Fifth Postulate was independent of the preceding four and that a simple negation of the Fifth Postulate actually led to two other noteworthy geometries. At any rate, this is a great time to bring closure to the course and to make several pedagogical points. There is a lot that can be said along these lines throughout the course as well.
8. Years ago, I had a colleague sit in on one of my ENE classes. As he watched students attempting proofs with no obvious success or very little success, he said he could not stand the agony the students were going through as a class and individually at the board. Contrariwise, I believe the students were better prepared for this sort of course than he was prepared for learning. This is not a method for the faint hearted. I believe this sort of experience for students is essential for authentic learning, especially in an age when so many teachers are "pulling their punches."

Diary of select classes

When colleagues or visitors ask me, if they can sit in on my class for the purpose of learning how the course is taught, I give them the following words of caution. Because of the way the class is taught, sitting in on 3–4 classes, most likely, will not give them even a snippet of the actual course they expect to observe. It would be like viewing a two-hour movie for just eight minutes. Could you say you know the movie? The class is evolving, changing and building each day. On some days very little progress is made but on other days the class may explode with new proofs or solutions. Instead, I invite colleagues to sit in on all or most of the classes, which some have done. For this reason, I will not describe just 3–4 days of class. Instead, from an actual course, I have selected a diary of some representative classes. For the sake of brevity, I have placed them in Appendix I. In most cases, I mention only that which is noteworthy or unusual, but instructive. At all other times, I followed the approach described under "Format" earlier in this essay. For each of these, I have included the lesson bullet points as a footnote. The first two classes in the diary are very different from the others. Normally, I would use only one class for motivation but I was having trouble with my voice the first day and had to shorten that class period and use a second class period to continue motivation. I have found that it is worth taking the time to set the stage for the course because of its unusual nature.

As far as the in-class presentation portion of the class is concerned, for the most part, I am a purist in following the "rules" of a Moore Method class as presented in the chapter titled, "Moore's Moore Method." These are rules that I learned under the tutelage of the master, Moore. I have tried groups and collaborative learning; however, for me, the Moore Method works best. From the diary, you will get a hint of the tone I attempted to develop. As you can see, I am continually "selling" the method and running the class so as to minimize negative emotions. I find it useful to at least attempt to tear down the "dogma" students bring to class concerning learning, the nature of mathematical thinking, and creativity. Then, the students are more open to learning in a new way.

To flesh out the day-to-day class, read my complete diary of Math 1301 Euclidean and Non-Euclidean Geometries (two classes) taught spring, 2001 at the University of Dallas. Similarly, I suggest that you read Lee May's diary as well.

"The men who try to do something and fail are infinitely better than those who try to do nothing and succeed."[1]
— Martin Lloyd Jones, Welsh Theologian, 1899–1981

7 Grading

Each of the coauthors places a high emphasis on the presentation of material, with the lowest percentage listed by any author being twenty-five percent of the final grade.[2] While each author outlines a grading scheme that appears, on the surface, to be distinct from the others, all the formulae ensure that those students who create a significant amount of original (to the students) mathematics receive an *A*. Those students who do not create mathematics, but successfully reproduce the mathematics presented in class by the end of the course, receive at least a *D*. All strive to allow the students the maximum time to achieve the goal of producing mathematics; in each case this is at least to the end of the class meetings and in several cases to the end of the final examination. It seems as if they would like it to extend even farther, but the timeline for grades associated with semesters or quarters prevents a longer wait. Each system evaluates students' written work and each system asks students to demonstrate, at the very least, the ability to write correct mathematics on at least one examination. Each author recognizes the subjective nature of grading in a global sense and strives to be objective in assessing student performance by considering numerous factors such as difficulty of the problems attempted, quality of the presentations, facility in responding to questions about the work, and the posing of additional conjectures. In brief, we assess, at the very least, difficulty, quality, style, defense, and number of presentations. The grading schemes and syllabi[3] are further constructed to assure that students cannot "work" the system by meeting some "minimum" standard in order to receive a grade.

(Parker) Grading a Moore Method course contains as much potential tension as any aspect of Moore Method teaching. On the one hand, at the end of any such course, you will likely *know* what the right grades are since you will have been witness to what each student has done as he or she has presented, written, and/or interacted as a member of the class. On the other hand, expectations from the class are probably for periodic testing that "shows" what the student has "accomplished" and for some sort of "point system" that quantifies

[1] http://www.famousquotessite.com/famous-quotes-14752-lloyd-jones.html

[2] The system counting presentations as only twenty-five percent is referred to as modified Moore Method.

[3] Each author's appendix begins with a sample syllabus from one course.

the student's "success." I strive for a grading system that allows me the flexibility to document the grades that I am sure of and inform me if the grade I am considering is too low. At the same time, I do not want my "grade gathering" to interfere with the development of whatever version of a Moore classroom culture that might emerge in a particular class. Thus, the grade gathering must be sufficiently different from typical student expectations to encourage the idea that the mathematics they make is what achievement in the course is all about, yet bear enough resemblance to traditional grading schemes to give the students some semblance of comfort, and provide for me a frame of reference against which to defend myself if my methods are challenged. When I was younger, I didn't even worry about such things, and just assumed that I would find a way to deal with whatever came up. I was content in the knowledge (or perhaps naïveté!) that I would see what I needed to see in order to know what the right grades were by the time the course was over. My standards were simple: the students could make their grades during the course by presenting their solutions to previously unsolved problems or make them at the end of the course by solving problems on the exam. The highest grade could still be made by solving problems on the exam that had not been solved during the course. I gave credit only for being first with a problem or for making solutions on the exam given at the end of the semester. I reasoned that, as long as everyone had equal access to the opportunity to present, this was fair. Certainly it was consistent with a culture based on the making of the mathematics as the main objective of the course. At that time, creating a context for authentic competition was my main affective goal.

Experience has led me incrementally to make substantial changes in my Moore Method grading schemes. For instance, students who showed excellent acuity dissecting arguments of their peers, but never broke through with proofs of their own, used to go unrewarded on the grade book. I had assumed that such behavior in a student would be a precursor to that student proving something, but this turned out not to be a universal phenomenon. One particular student caused multiple changes in my outlook. Five separate times during an algebraic structures course, this student had a single problem ready on the day his name came up, a student in front of him in the call-on order did the problem, and thus he had nothing to present. Perhaps having such a thing happen once might have created incentive, but multiple occurrences could have been stifling. Fortunately, this student had a sturdy psyche and survived and I found a way to adapt the exam that semester to make sure that he got credit for what he had done. Doing so initiated the idea for an exam format that I now typically use. In addition, in the semesters that followed and in direct response to what had happened to him, I adopted a policy that guaranteed the opportunity to reward achievement of that sort. But the policy I was using then might have squashed him. Over time, I also came to value the capacity of students to reproduce correctly the work of others. I offer the above remarks in the following spirit. I make no apologies for how I originally graded; I was fair and it allowed me to assess what I thought needed to be assessed and, to the best of my knowledge at the time, dovetailed with my expectations of how the course should go. But I no longer grade that way, nor should I. The lesson offered is that you should be committed to what you do because you have thought about it, but also let your experience inform you of what you might do to make it better. I can think of no reason why the standards for a course should change from year to year, but I can think of reasons to adjust how I measure how well those standards may have been approached or attained.

Having just tried to make a case that you should devise your own grading system and let it evolve as you continue to mature professionally, it must seem a tad presumptuous to offer my own (current) system. But here it comes! My expectation is not that the system itself should to be imitated (although you are welcome to; I certainly believe in it), but that the system, together with the explanations of why I do what I do, will be food for thought when you formulate your own Moore Method grading scheme.

Grading goes part and parcel with how the class is conducted. Students accumulate "resumés" that constitute evidence that their accomplishments justify the grade they get. On Day One, when I pass out the syllabus for the course and discuss with the class how the course will be conducted, I also discuss grading. I spell out that the student expectations for the course are: making mathematics; exercising quality control on the contributions of classmates by making sure that what is presented is correct; and confirming the correctness of what they have done or seen done by making a careful written argument in their notes. I refer to this last item as "writing the book for the course." I act as if I believe that all of them can make the mathematics themselves, but emphasize that, once a presentation is made and declared correct, it becomes class property and that I consider it the responsibility of all students to prepare write-ups with which each is individually satisfied. I also offer myself as a reference for any theorem or example that has been judged by the class to have been correctly argued so that students who are unsure about their write-ups can talk one-on-one with me about their uncertainties. This not only creates opportunities for additional interaction with students, but also clearly offers support for preparation of materials for which I give a tangible payoff at the end of the course.

To document progress towards a grade, I offer two credits whenever a student successfully argues a problem in front of the class. On a day that a problem is completed, any student who has that problem at the beginning of the class period may submit his or her written version at the end of that class period for one credit with the credit granted if the argument for it is correct. I am well aware that it is far more difficult to create a correct written argument than it is to present orally and make corrections as objections are raised and that these ostensible rewards are backwards. Nevertheless, it is important to have incentives in place to persuade students to present their work. Moreover, I know who has done what, and I am, as the dispenser of final grades, in a position to assure that good work is not undervalued. In response to this, someone always asks a question such as "How many problems do I have to solve to get an *A*, or a *B*?" I never answer this question directly, preferring instead to use the opportunity to suggest that the goal is always to solve another problem, regardless of how many one has already gotten. I also indicate that some of the problems in the course have pedigrees and that I want to have the flexibility to give extra rewards to students who do things that, had they done them 100 years ago, would likely have made them famous. In courses where the content of the course is considered by the department as an important component of the course, I have minimum standards that the class, as a body, must achieve in order for anyone in the class to get the highest grade. I seldom share this with the class unless I consider it to have possibilities as a motivating factor deep into the course. I never share this with the class at the beginning of a course since it is seldom a factor. To make clear that I am not completely avoiding the question, I promise that, at any time during the course, I will give them my appraisal of how their work, up to that point, projects towards a final grade. Students often take me up on this, particularly as the end of the semester approaches.

I require a take-home final examination at the end of the course on which I have a section of problems that have been solved during the course and a section or sections of problems that have not yet been solved or have not been addressed. A student may keep the grade he or she has going in to the exam by reproducing correctly what has been done in the course. Or, if the work during the semester has not been at a level justifying the grade desired, the student may make the desired grade by producing arguments on the exam for problems not previously done by the class. In a case in which the work during the semester has not warranted a *C*, a student can make a *C* by successfully reproducing that work I ask for which has been presented during the course. Students are allowed to use their notes while taking the exam, so even those students who have not broken through with proofs of their own, but who have done the rest of what has been asked of them, can literally copy a *C* from their notes so long as they are savvy enough to recognize that the problems on the first section of the exam have been solved during the course. I also guarantee that no student whom I judge to have made a good-faith effort on the exam will lose more than a letter grade because of the exam. I justify the possibility of losing a grade by espousing a standard that students who make *A*'s should be able to communicate mathematics through their writing.

With these policies articulated for the class, it remains for me to keep records sufficient to remind me of the substance of each presentation and hand-in, to document any particularly insightful classroom observations made by students, and to write a final examination that will tell me what I still need to know at the end of the semester. I use the grade book to record presentations and hand-ins and my course folder for comments for which there is not sufficient room in the grade book. The final examination is the flexible tool for assessment.

I design the final examination to document the extent to which the students have command of the mathematics that the class has made, and to provide one more opportunity for the students to do something on their own. This documentation is for me and may be for the department. At James Madison University, we are subject to a university-wide assessment mandate. In our department, this assessment is done by embedding, for each core course in the major, questions about common goals on the final examination. The way we do it has sufficient flexibility so that it is not particularly intrusive on the teaching process. I give the examination in a take-home format and try to guarantee at least five calendar days in which the students can work on the examination. If the examination period is scheduled early during examination week, this may entail sacrificing a class day during the last week of classes, but I believe the sacrifice is warranted to ensure that the students have ample time to consider some of the "new" problems and to make sure that they can give our exam the time it deserves without impinging on their preparations for their examinations in other courses.

The choice of questions on the examination must be broad enough to make it likely that I see important proof techniques and, at the very least, substantial components of the curricular content of the course. But within this structure, the problems I choose to accomplish this objective are highly dependent on what the class has done and how the class has done it. In a class with a broad base of participation, I give lots of choices, taking pains to make sure that there is at least one theorem or example on the examination from each student who originally presented a "significant" problem. If there is a problem for which the class has made a sprawling argument that extended through several lemmas of their own making or several class periods, I will often make it the only choice in a question to see how well the students have cleaned up or solidified the ideas driving it. In a class that has proven lots of theorems, I will have fewer questions on the "reproduce" section, with each question likely to have several options. In a class that has barely reached my minimum goals or, worse yet, fallen short of them, there is likely to be a larger percentage of the students who are taking the examination to make their grades. In this situation, I will have more questions on the "reproduce" section with fewer options for each question so that I can get a clearer view at how the students have sorted out the curricular substance and the proof techniques of the course.

On the part of the examination where students can extend their grade or atone for an error on the "reproduce" section, I typically provide two sections of problems, each containing several questions. My goal is that any student who is ripe to make a breakthrough can find a problem about which he or she can profitably think. When I distribute the exam, I make it clear that I know that there are *lots* of problems on the non-reproduce section and that the intention is not that any student will do them all, but rather that each student have the opportunity to find something that he or she can profitably think about. In one of the sections, I give them problems designed to extend what has been done in class. One source is problems that were stated during the course, but not solved. If I think what the class has done is sufficient to prepare them to solve the problem, I will just restate it. A more common approach is to state a problem or two to point the way to the problem I want, and state the problem as the last part of the question. Another source is problems that would have come next had the course continued. When I have students who appear to have worked hard and have been active in class and/or following up on class, but have not been able to make proofs of their own, I will simply state a new definition and a new problem that is a direct consequence of the definition. This approach gives such students a platform for a "breakthrough" experience. In the non-reproduce section, I state problems where the ideas they have already developed can be used to solve the problems. Examples to which the theorems apply are appropriate here.

When reading through the exam with the class, I will tell them that if they can do the last question in any problem, I don't care whether or not they do the others. I encourage them to turn in any progress they make towards any problem, but also make it clear that the complete solution to any problem is more highly valued in the grade book than lots of partial solutions. Two final examinations, one for a course in analysis for which there are specific departmental expectations and the other for a course on the real number system that was being used as a bridge course at the time it was given, can be found in Appendix IV to serve as examples of what is described above.

In grading the exams, I expect that all problems in the reproduce section will be done and be correct. That is the standard for making a *C* or keeping the grade that the student brought into the exam. I am more lenient with arguments from the other sections, using the degree of clarity in "essentially correct" arguments to separate the *A*'s from the *B*'s. After reading the examinations and reviewing the record from the semester, I am routinely sure of what the correct grade is for a student. In the rare occasion when I am uncertain, I always grade up; if I am going to make a mistake, I want it to be in the student's favor. In conclusion, I remind the reader that one of the benefits of teaching with the Moore Method is that you will know your students' work well by the end of a semester, and this knowledge will inform grading.

(Mahavier) From freshman- to graduate-level courses, I send students to the board and count productivity there as a significant portion of their grades. And I admit that this grade is subjective, even as other practitioners codify their grading through complex rubrics.[4] In my class, one student may put only a few problems on the board and earn an *A*, while another may put an abundance of problems on the board and earn a *B*. Everything matters at the board—difficulty of the problem, creativity in solving the problem, quality of presentations, quantity of presentations, how other students' questions are addressed, whether the presentation is done line-by-line bringing the audience along or is written and then read. Students don't hear all of these criteria on the first day—it would be overwhelming. On the first day, I say things such as "you'll get some credit every time you go to the board and there is no minimum number of times you must go to earn a good grade—still, more is better. For now, just go to the board as often as you can and do your best. But whatever you do, *go to the board, even if just to show us an idea you had that did not work.*" I encourage students to drop by my office and ask how they are doing on their grade and mandate appointments if I don't see all my students regularly. More often, I simply drop by and catch my students in the math lab. By far the best place to open communication lines with students is on their turf.[5]

Throughout my courses, students do ask questions about the grading in class, which I interpret as a good sign that they are relaxing and willing to talk about anything. At these times, I add some of what I said above about how I arrive at a grade. If a student were to prove the chain rule in calculus in a way that I had never seen and that I thought was worthy of publishing in a student journal, then I would make a big deal out of that in class and individually to the student and we would be off on an undergraduate research project to be presented at the next regional conference accepting undergraduate talks. I would give that student an *A* for board work even if he or she presented nothing else. That student is potentially bound for great things, mathematically speaking. Usually, I see all my students enough and encourage them and tell them how well they are doing and what they might do to improve. This board work may take as little as twenty-five percent of class time or as much as ninety-five percent of class time. Generally, the percentage of time a class spends at the board increases with the level of the class, the exception being when the class has an applied aspect such as numerical analysis. For example, a three-hour-a-week college algebra class might spend only one hour a week at the board, a three-hour graduate numerical analysis class might spend about half of its class time at the board, and an undergraduate introductory analysis class might spend virtually all of its class time at the board. Still, I have taught both calculus and trigonometry to students where essentially all class time went to presentations. Classes that maximize student participation are by far my favorite classes and are the classes that, based on my experience, create the greatest impact on the students' perception of and appreciation for mathematics.

[4] One such example is Jonathon K. Hodge at Grand Valley State University, author of *Voting Made Easy*.

[5] In his article on designing the perfect mathematics building (Birgen, 2005) asserts the importance of having the student work areas adjacent to faculty work areas so faculty will have just this sort of chance encounters with students on a regular basis.

> "He is an awesome teacher. In fact, he is the best math teacher I've ever had. He really cares about his students and he likes to see us excel. I have learned more in this math course than I have learned in any other math course. It is comforting to know that my professor is anxious to help the students and that he gets to know us personally."
>
> Anonymous evaluation from Moore Method Trigonometry (1998)

Because grading of a Moore Method course is inextricable from the structure of the course, I will first discuss two grading rubrics and the course structure associated with each rubric. Then I will address certain specific goals that these grading methods are intended to support. I refer the reader to Appendix II, which includes handouts that I modify only slightly for each Moore Method class I teach. In some classes, I also hand out Ed Parker's material in Appendix IV, entitled, "An introduction to doing mathematics."

The first method, which I call the modified Moore Method[6] is one that I encourage new practitioners to employ and that I often implement in lower-level courses. This method works very well for getting new practitioners accustomed to sending students to the board and for adjusting to the dynamics that such an act creates. It also works well where there is an assigned text book or a mandated syllabus that requires certain topics be covered. It takes no more time to teach than a traditional lecture course because you can use the departmental text and syllabus and even predetermined tests and assignments. The modified Moore Method entails less risk because the students really love it and because it is close enough to traditional methods to avoid controversy. It offers a simple way to test the waters of student presentations. You get great student evaluations, you actively involve the students, and you easily cover the full departmental syllabus with deep understanding on the part of the student. If this method is so successful, why discuss a second method? Because this method is *not* the Moore Method. It is perhaps what Coppin refers to in other chapters as *neo*-Moore Method and what I often call a *problem-based* method. Still, it is student-centered and does have value as a transition to the Moore Method.

The second method, which I call the pure Moore Method is harder on the students and harder on the instructor. Successfully implemented, it forces students to grapple independently with the material and enables them to make a fundamental transition from working a few problems on their own to tackling mathematical concepts completely on their own and mastering those concepts. Such an experience, and the transition it represents, is beautifully highlighted in "Christmas at Big Lake" (Sam Young, 1998). Hence the second method truly embodies the pure Moore Method as opposed to a somewhat watered-down version. In the "Extreme Moore Method" (McNicholl, 2006), Timothy McNicholl warns of the danger of watering down the method. In its purest form, with students working and presenting a majority of the time, you create an environment that has the maximum potential to create fundamental transitions in students' approaches to learning and to students' understanding of their own talents. You maximize comprehension of and appreciation for mathematics. While it is a formula for such success, improperly implemented, it can also be a formula for undermining students' confidence. It is my opinion that poorly implemented examples of the Moore Method and Moore's own provocative personality are two prominent reasons that resistance to the Method is still prevalent today. Hence, the pure Moore Method is the

[6] Although we arrived at it independently, Parker uses a similar modification. See (Parker, 2004).

one to admire and strive toward. Still, the modified Moore Method is the one that I began with some eighteen years ago and it was a slow transition for me to move to the more valuable pure Moore Method. Due to time constraints, I oscillate between the two.

Modified Moore Method

The student averages come from three tests (50%), homework presentations in class (25%), and a comprehensive final (25%). I promise students that I will not give a grade *lower* than their average and that I reserve the right to modify the weights in the event that doing so will *increase* their grade for the course. Each problem done by a student in my class, whether board work, written work, a quiz problem, or a test problem, receives a grade of 0-4 where 0 = wrong, 1 = mostly wrong, 2 = half right, 3 = mostly right, and 4 = completely right. I like this grading system since the student who gets every problem on every test mostly right and receives an *A* (95) on presentations has a *B/C* (80) average. Clearly students strive for better than "mostly right" on their work. Grading flexibility is important, and I admit to students that high presentation averages or a high comprehensive final exam grades can raise their final grade above their computed average. At the end of the semester there are inevitably students with *D* averages and *A's* on presentations. Such students receive a *C.* Also there are students who were bright enough that, even though they never came to class for presentation days, had an *A* average on the tests and final. These students receive *A'*s, despite an *F* on presentations. I have very specific rules for myself as to when and how to adjust the weightings and I hold these rules consistent from semester to semester and student to student. Grading flexibility is one of the goals of the method; having interacted with the students all semester, there are inevitably students whose grades do not reflect their mathematical accomplishments. Perhaps they spent too much time exploring alternate ways of solving problems (researching) and not enough time practicing mundane problems. Without the interaction, such students go unnoticed. It has been said (Steen, 1988) that calculus should be a pump, not a filter. This method supports the idea that we should be spending our time catching the good students rather than filtering out the (presumably) bad.

Structure. I lecture on Monday and Wednesday, assigning problems as I go, and the class presents these problems on Friday. (For the sake of brevity, I'll stick with the MWF analogy; if the class meets two days a week, then the beginning of each class goes to presentations followed by lecture over the next topics.) The word "lecture" is misused here —even my lectures are highly interactive discussions. A normal class period has lots of questions because students know that this is the only source, other than my office hours or the text (if there is one), that can help them prepare for the homework I am assigning. There are two definite benefits of assigning problems as I lecture. First, it encourages the students to look over their notes (they must in order to find the homework problems); and second, it assures that I have assigned problems on all the important aspects of a topic. In addition to these problems, I often assign problems from a text. If a student can do every problem I assign at the board, then there is no need to open the text; however, few students find this to be the case. Most of them need practice and they find that practice in the problems I assign from the book. Since I rarely discuss the book, they must decipher or question me about other notations. This encourages using the book as a reference rather than the authoritative word on the subject. The students need to understand my problems, and the book is a possible source of help in doing so. Since I rarely write out my problems beforehand, students must interact with one another to find homework assignments from days they missed. As a consequence, I have good attendance in my classes and good student-to-student communication. An added benefit is the ability to create problems in real time that address questions from the class or that are tailored to address the needs of individual students.

TGIF! It is Friday and we are ready to go to the board and earn our presentation grades. As I call on students and assign problems for them to present, the first students immediately go the board and put their work up simultaneously. I request a room with at least two full boards to maximize the number of problems we can put up simultaneously. We typically present six to ten problems at once and easily two sets of ten during one class period. Excluding assignments from the book, every problem I assign during class makes it on the board and I generally assign only about twenty or thirty problems a week. They are allowed (expected) to take their notes and use them. Each student receives a grade based on the scale previously mentioned. In truth, I am quite soft on homework grading in this method. For example, if a student does not have the problem they are called on to present, I will let him or her do another problem of his or her choice for one point less. Also, I record a grade of *4* as I call on them and rarely mark them down if they make mistakes. The exception is when I note a student consistently going to the board unprepared or consistently making glaring errors. Orally, I encourage and offer support, but in my grade book, I knock the *4* down to a *2* for these problems. I'll note with a *4** the problems that are either particularly hard or well-presented to remind me to catch these students later and encourage them to study more mathematics. These grades can also affect the final grade.

While the students are putting the problems on the board, I answer questions from the students at their seats. I wander the room asking for questions and encouraging them to discuss among themselves the mathematics they have done and the mathematics on the board. After the problems are on the board, any students who have questions on the material are encouraged to speak up. I generally ask if they agree with the answers and, if I don't get a consensus, I ask for a vote or let students point out mistakes. I don't check every step if the class agrees the problem is correct and there are no questions. In fact, I go so far as to tell them that I will not check every step, so they need to do so in order to verify what is wrong. This way, students with differing answers will say, "I didn't get that," and we will look more carefully at the problem. Then, using a different color marker, I will grade and correct as I would on a test so that they may see what I consider to be valid mathematics. Note the self-adjusting nature of the course. If only a few students have the homework, then I am teaching over their heads, while if almost all the students have the work, then I can assign harder problems and move faster. In one precalculus class years ago, we were well into a topic not on the syllabus (financial mathematics) by the end of the semester, having spent numerous days on this extracurricular topic because the students were covering the material so efficiently.

This method is just a small step away from a pure lecture-and-test method and is easy to implement without requiring development of materials or challenging departmental norms. It is a great way to test your ego and see if it is strong enough to allow students to take credit for presenting and explaining material instead of presenting yourself as the "sage on the stage" and all-powerful authority.

Pure Moore Method

In this method, students' grades will be not less than the average of three grades resulting from board work, written work, and the average of their midterm and final. As in the previous method, the board work grade is available upon request, is usually communicated to the students during office hours on a regular basis, and is subjective based on the quality and quantity of presentations. Responses to grade inquiries are always positive: "You have a *C* now, heading for a *B* and there's no reason why you can't earn an *A*. Have you solved Problem 59?" The written-work grade comes from work that I collect at the end of each week and return at the next class meeting. Each student may resubmit a written piece that received a grade of *C* or lower and I'll give the better of the two grades. Students resubmitting papers are still responsible for turning in a new problem each week. They are to write, on every problem they turn in, either "original," "new," or "resubmit." "Original" means this is a problem that no one has presented yet. "New" means this is a problem that is not original, but is one that they have not turned in yet. "Resubmit" means they are resubmitting last week's problem. On rare occasions, a student who has resubmitted something and is very close to a deeper understanding will be encouraged to resubmit a second time. Written work receives the following grades and the students are given this list:

A – This is correct and well-written mathematics!

B – This is a good piece of work, yet there are some mathematical errors or some writing errors that need addressing. Remember, *B*'s can't be resubmitted.

C – There is some good intuition here, but there is at least one serious flaw.

D – I don't understand this, but I see that you have worked on it; come see me!

F – I believe you have not worked on this problem enough.

> In Fall 2007, I returned a paper with a grade of *B,* and the student cried out—"Please, change this *B* to a *C*!" I told her I was sorry, but the paper was just too good for a *C*.
>
> Mahavier

I tell them to expect *C*'s, *D*'s, and *F*'s in the beginning. In truth, students often get these grades randomly sprinkled throughout the semester, although the class average improves weekly. I also tell them that I did not expect them to come to my class as mathematicians; I only expect that they will leave the class as mathematicians, so I will grade on improvement. If they get *D*'s on the first five written problems, even after resubmits, *C*'s on the next five, and *B*'s on the last five, I'll call that a *B* for written work. They have to believe that I *want* to give them the best possible grade for every portion of the course. They also have to believe that I will help them attain the necessary skills that define that grade, *if* they give me their personal best. And lastly, they have to believe me when I say, "I have done this with countless classes before you. I know that what I am asking of you is harder than what others may have asked. And I know that, like those before you, you too can do what I ask. Stick with me, work hard, and you will do fine." And they must know that while I will help them every time they come to me, I won't give them a proof and the burden of creating mathematics is on their shoulders.

At the beginning of the course, the weekly written submissions, which are intended to train the students to think carefully and write carefully, are over any problem at all. It may have already been presented in class. Usually about three weeks into a fifteen-week course, I tell the students that the write-ups have become so good and their board work so good that I want them to start writing up original work. That is, each written problem at the end of the week must be an "original" problem that no one has put on the board yet. This rule accelerates productivity in the following way. To get a write-up ready for the end of the week, they must do an original problem. Once done, they can present it for more credit than they would get by turning it in. Once they present it, they can't turn it in so they must do another original. Once they get another original, they can present it, and so on. This forces the issue of attempting to create mathematics and I have found it to be very effective. By starting out allowing them to write up another's work, they have time to warm to the unique mode of instruction. They become stronger writers as they build confidence in their ability to understand and convey an argument that may not have had quite enough detail during presentation. Once that skill is established, the next skill is to create and write up *their* mathematics.

The midterm and final are intended to be low stress for the students who are presenting regularly. The class votes on the format: in-class, take-home, or a combination. I don't give them a break from the write-ups because I want to emphasize that *doing* mathematics is more important than studying for and showing off their mathematics. And I tell them that I will de-emphasize this grade if their written and board work is in good shape, thus further encouraging them to *do* some mathematics rather than relying on the test grades for help. These tests are given for two reasons, which I tell the students. The first reason is for their continued education because I feel that reviewing all the material will build confidence and understanding. They will see that the early problems are now "easy" and will see how far they have progressed. Because they are always working on problems that are just above their level, they don't realize that their ability has increased. Also, they will review the material in its entirety, and this will help them see the forest as well as the trees. They will be able to show off their new-found ability to solve problems, as I promise that no problem on the midterm or final will be one they have seen before. (I do make exceptions for statements of definitions and for any problem that someone presented incorrectly and the class let go by.) The second reason for the midterm and final is to be able to issue feedback to those who don't present. While it is a rare event, nonpresenting students occasionally suggest that they "understand it all" after it is presented, but just can't get any of the problems "quickly enough" and therefore have low written and presentation grades. Giving an exam exhibits whether this is the case. I've not yet found a case where a student failed to present and turn in written work and still had a command of the subject, but I don't doubt that this could occur.

I employ this method primarily in upper-level courses, although I have made exceptions in both calculus and trigonometry as previously discussed. I, and more importantly my students, have had great satisfaction with both schemes. If I were a teacher with plenty of time on my hands then I would always use the pure Moore Method, developing my own notes as I went and refining them in subsequent semesters so as to maximize the benefit to my students. With numerous research and mathematics education projects underway, I alternate between the two described methods.

Having described two methods, here are a few of the goals that these methods are intended to support:

- promoting learning over earning,
- allowing multiple ways for students to succeed,
- rewarding attempts to do mathematics, and
- driving all students to their maximum potential.

Learning over earning

In the chapter entitled "On Culture," Lee May wrote on moving his students from "earning to learning." He discusses just how he does this through grading and creating a democratic setting that de-emphasizes the teacher's authoritative role. He goes so far as to admit his own failure to maintain this mindset during much of his undergraduate career. An illustrative example of students' tendencies toward earning is addressed by Holt (Holt, 1965). In the example Holt discusses, the teacher locks a balance beam in the horizontal position and then places weights on one side. The students are provided with weights to place on the other side. Before the beam is unlocked to see if it will balance, groups of students are asked to discuss and vote on whether it will balance. Each group receives a point for each student in the group who votes correctly. Holt records one student stating, "It might teeter a little, then balance, but not really." After most of one group has voted that it won't balance, the last student in the group adds, "I'll say it will, just in case it does, so that we won't get too low a score." The first student is attempting to maximize points by covering all possibilities. The second is trying to optimize the group score. These students are concerned with "earning," not "learning." Both are attempting to cover all the bases for getting either a right or a wrong answer.

Students are adept at optimizing grades and most college students await the syllabus on the first day of class so as to assess the grading strategy of the teacher in preparation for maximizing their grades for the semester. With internet sites recording individual student evaluations of faculty and social networking sites addressing individual faculty at individual schools,[7] many students begin this process before choosing classes and teachers. Hence, the grading must support a method that encourages actual learning by creating an environment in which learning is rewarded and there is no "game" to play—they simply try to learn the mathematics and learn how to do mathematics and this alone is the way in which they can earn their grades. So "earning" only occurs if they are "learning" and there is no way to "beat" or "game" the system.

Allowing multiple ways for students to succeed

I and faculty I have mentored in the method have had first-rate students drop our classes in the first week. One student, when questioned about her reasons, responded to me, "It sounds like fun, but it is not clear that I can guarantee an *A* in this class." Early in my career, I was astonished by such remarks but, after fifteen years of teaching, I am less surprised. This is a good student. She knows that she is a good test taker and that she can absorb and repeat well-presented information at a rapid rate. She has perfected a skill that assures *A*'s in traditionally taught courses. When confronted with a course that will grade her upon demonstrating original work, work that she produces rather than absorbs and repeats, she avoids the course for fear of failure. In her mind, any grade of less than *A* is a failure and

[7] See www.pick-a-prof.com, www.rate-your-prof.com, www.facebook.com and www.myspace.com.

risks affecting her entrance to medical school. Amazingly, this is a student who says that she *loves* mathematics. Now, because of the grading system I had in place at the time, she may never know just how good she might be at *doing* mathematics. By adjusting my discussions on the first day and by altering my grading system (to the systems outlined above), I have avoided losing talented students since that experience. Occasionally, such students turn out not to be so talented in their ability to create mathematics, but still earn *A*'s by creating some material and mastering the remainder of the material, much as they would in a lecture course, by proving themselves on exams. More often, these students turn out to be the stars we expect them to be and the method pushes them beyond their comfort level to perform at a high level. At a minimum these students earn an appreciation for the difference between comprehending mathematics and creating mathematics. That alone is a distinction that every undergraduate student deserves and the distinction that should facilitate the decision to attend graduate school. Hence, grading and the discussion regarding grading during the first days[8] of class are critical aspects of the method worthy of attention.

Rewarding attempts to *do* mathematics

The grading algorithm should reward with *A*'s only those students who *do* significant mathematics and *produce* original work, yet should not intimidate any student in the beginning. The relaxed atmosphere that each author discussed in the chapter on culture needs to be established through the grading, through the first days of class, and through the aura and culture of the classroom. Since the goal is to develop the students' ability to do mathematics and since many undergraduates do not possess this ability even upon graduation (Smith, 2005, p.6), the grading must give points for students *attempting* to improve their ability to do mathematics. The ability to resubmit work described earlier rewards repeated effort on a problem. The grading and the notes must account for the fact that not all students start at the same place with respect to that elusive trait, mathematical maturity.

Driving all students to their maximum potential

In a traditional grading scenario, weighting three tests and a final, the student earning an *A* on the first test knows that based on the point grading system, he or she may perform lower on subsequent tests. The student earning an *F* must raise his or her grades sometime in the next weeks (presumably on future tests weeks away) to survive; and survival excludes the possibility of an *A*. Teachers have devised many ways to counter this issue such as weighting the final more heavily to reward false starts or dropping the lowest test grade.

Consider a student in a Moore Method class making his or her first presentation. Rather than lowering the need to perform for the remainder of the course, a brilliant performance on this problem raises professor's expectations for this student. In order to present again, the student will need to have a presentation that no one else has since all other students will be selected first for presentations. Hence, this student must now work on the challenging problems. By example, this presentation has raised the bar for the entire class. On the other hand, if the presentation is going badly, the student has the opportunity to repair this by resolving the problem at the board or prior to the next class period. Both successful and unsuccessful presentations motivate the respective student to perform at a higher level im-

[8] I do not discuss grading on the first day as outlined in detail in Chapter 6—In the Classroom.

mediately due to the continuous feedback system. Because *every* student in the class will have both successful and unsuccessful presentations, the culture of exploring the correctness of an argument becomes the focus, not the grade, style, sex, ethnicity, or wardrobe of the presenter. In my experience, a struggling student is almost always given the most emotional support from other members of the class and a strong student is typically scrutinized very carefully. For each student, this treatment is likely optimal for their mathematical and emotional development.

Conclusion

Teaching a class using the modified Moore Method is a nice way to test the waters. Do you like sending students to the board? Does it work for you? If so, the pure Moore Method can be tackled later. Of course, there are a large number of mathematicians whom I (and others) have mentored via email, phone calls, or visitations in using the pure Moore Method. These have dived directly into the pure Moore Method with good results using others' materials, or writing their own. To the undecided, I quote James Dean, "Live fast, die young, and leave a good looking corpse." Take a chance! Push the envelope! Seriously, you'll do no harm to yourself or your students (and possibly great good) if you simply give it your best shot. For myself, I can say without hesitation that the reward of the successes that my students have had that I (and more importantly they) attribute to the method far outweighs any risks I have taken.

(Coppin) H. S. Wall, a colleague of Moore, said in class one day that to measure one's intelligence with an I.Q. test is like measuring one side of the classroom with the most precise device available, measure an adjacent side by stepping it off, and conclude that the product of the two numbers is the area of the floor to the precision of the most precise device. Of course, he meant that measuring something as complex and subtle as human intelligence is futile. As we all know, measuring student performance can be just as difficult. I find that evaluation of my students' performances in my classes is onerous, frustrating and almost futile. However, I have found that it is an essential part of my teaching. This became abundantly clear during a year I was a visiting professor at another university. There, we had common exams that counted as fifteen percent of the total grade making it hard to motivate my students to do anything more than focus on the common exam. Take away my ability to administer grades and you have removed my ability to teach. As feeble as my grading schemes are, they still serve as a means to communicate to my students how far they have advanced and what they need to do to improve.

Ultimately, because my students' welfare is always on my mind, grading is a day-to-day proposition, continuously occurring 24/7. I have a moral duty to do a thorough and fair job. I think about what needs to happen in the course for them to make good progress. The evaluation of a human being's development in mathematics is very complex and most difficult. It has many layers, some of which are so deep as to be subliminal. However, the top levels are describable and testable. I use any appropriate tools at my disposal. At the outset, let us agree that the evaluation of the holistic progress of the students is subjective. As far as I know, human beings create all tests and all examinations and are, therefore, fallible. The emotional canvas on which we learn is critical to motivate students. As a result, I attempt to make my exams fun and educational, at least in some small way. We laugh and joke and learn. In all this, I am aware of my students' overall progress, increase in the mastery of mathematics, increase in imagination and creativity, and ability to use language increasingly well, both oral and written. I become a student of my students intellectual personalities.

Presentations

The student presentations are, by far, the most important component of a Moore Method course. I track the results students say they have ready to present, what they attempt and what they successfully present. Grades are based on the overall quality of the presentations, as well as the number and difficulty of their presentations. At times, but not always, I use a scoring sheet such as used in bowling, with a spare denoting that a student claims to have a result and a strike denoting that this student has successfully presented the result. See the sample below.

Name	Th. 1	Pr. 1	Th. 2	Th. 3	Qn. 1	Th. 4	Th. 5
Joe Smith	/	/	X		X	/	X
Sally Brown	X	/	/	X	/	/	/

It has been my experience that the top students and those students with positive, enthusiastic attitudes toward proving theorems, settling conjectures, or answering questions are identical groups of people. Below, I give the two different approaches I use in calculating the overall presentation grade.

Option 1. I do not give the students a number of results they must present; nor do I give a point value to each theorem, problem or question. If I were to be so specific, students would begin to calculate just what they need to do to make a particular grade. It is very difficult to predict how much material will be covered. Thus, for example, a student may earn an *A* some weeks before the end of the course. That student might essentially quit working and thereby lose the opportunity to grow to his or her potential. The following are my definitions of grades *A*, *C*, and *B*.

Grade *A*. Ones who earn this grade have mastered the material and processes of the course. They will most likely have settled a good number of theorems including one or more of what I call *A*-caliber theorems. They could teach the class if I were to be late to class. They demonstrate authority.

Grade *C*. Ones who earn this grade have demonstrated industry and initiative. They have shown that they have insight and most likely have proved a small number of theorems on their own. Performance on a written examination is important for a *C* student. The student who earns this grade has some command of information but has had real difficulty with proofs.

Grade *B*. Ones who earn this grade are not *A* students but are better than *C* students, i.e., $C < B < A$.

Option 2. This one is more structured. Each presentation will earn one of the following marks:

√–: Some engagement, but substantial questions remain
√ : A well-reasoned argument, but some questions remain
√+: A complete and thorough explanation, no further questions

A presentation grade is computed as follows:

For a *D*, a student must eventually get at least a √ on 50% of the presentations.
For a *C*, a student must eventually get at least a √ on all of the presentations.
For a *B*, a student must eventually get at least a √ on all the presentations, and a √+ on 40% of the presentations.
For an *A*, a student must eventually get at least a √ on all the presentations, and a √+ on 80% of the presentations.
Note: A √– is cancelled by a √+ (so one √– and one √+ is the same as two √s). Each in-class presentation is worth an extra +.

The final examination

This is comprehensive. The examination will be split between material they have seen before and completely new results; however, the new results are appropriate to the time allotted for the exam. Unlike Moore, I may not really challenge the students, especially at a time when they have to handle the stress of other examinations. I prefer to work the students hard during the semester but not at final examination time.

Final course grades

I have used the following four evaluation schemes over the years with, of course, varying amounts of success.

1. The course grade is the maximum of the presentation grade and the final examination grade if there is no more than one letter grade difference between the presentation and exam grades. Otherwise, the final course grade is the average of the two.
2. The course grade is based on:
   a. Presentations 90%
   b. Final 10%
3. The course grade is based on:
   a. Memory Exams 40%
   b. Presentations 50%
   c. Final 10%
4. The course grade is based on:
   a. Attendance (.5 points per each of 30 classes attended) 15%
   b. Write-ups, quizzes, etc. 15%
   c. Presentations/Major Exams 60%
   d. Final 10%

Summary

In the final analysis, you must find a method of evaluation that fits your style and your students best. There are two principles to keep in mind:

- Remember that student expectations of how they are to be evaluated must be taken into account. They have been trained to expect homework, quizzes, examinations, and a final examination. In using the Moore Method, you are already asking the students to adjust to a radically different teaching method from what they have become accustomed to. Will you ask them to adjust to a radically different grading scheme as well?
- Your grading will affect how students learn. It can be used constructively as a motivator but can inhibit learning if it is too complex and too distracting. You will have to strike a balance that works for the level of academic maturity of your students. The more immature the students, the more you have to invest in grading tools such as quizzes and homework and, of course, your students will not learn as much as they would have with a less complex grading structure. The more mature the students, the more you can rely on presentations to generate grades and, thus, the greater opportunity for learning. You have to strive for an optimal grading scheme that maximizes learning for the audience you have.

(May) The technical aspects of my grading are well-defined; my grading philosophy is not. Let me expound on the former first.

Techniques

I begin to practice my first grading technique on the first day of class. As I hand out and go over my policies for the course, I point the students to the syllabus, which contains a paragraph on evaluation. One for the course "Introduction to Abstract Mathematics" looks like this and is typical of my Moore Method courses:

EVALUATION

Portfolio	10–30%
Presentations	30–70%
Midterm Examination	0–25%
Final Examination	0–25%

The evaluation-scheme is accompanied by an explanation, which serves as part of the policies document. A typical explanation follows.

Evaluation of your work

Your performance in the course falls into four categories: portfolio, oral presentations of your work, and midterm and final examinations. The portfolio of your written work must contain your own write-ups of proofs of five theorems from the notes. It may also contain exercises from the notes that we have not covered in class. It may contain one or more conjectures and work that you have done—examples, proofs, or commentary, for example —to try to decide whether the conjectures are true or false. The portfolio may also contain journal- or diary-like comments by you on your work and what goes on in class.

There is no maximum length to the portfolio. The minimum length is five (not-necessarily-full) pages, one for each theorem that you include. With the exception of mathematical symbols and illustrative diagrams, the portfolio should be typed in at least 12-point font with double spacing. The spelling and grammar should be error-free. Mathematical symbols and any diagram that you choose to include may be hand-written or drawn in the document. (For examples of a style in which to compose your portfolio, consult *Mathematics Magazine*, *The College Mathematics Journal*, or *The American Mathematical Monthly*.) The portfolio will count as p percent of your final average, where p is a number chosen by you so as to satisfy the conditions to be presented below.

My evaluation of your presentations at the board or in my office, and of other comments that you make in class or in my office, will count as b percent of your final average, where b, like p, satisfies the conditions to be presented below.

The midterm and final examinations will count as m percent and f percent, respectively, of your final average. One or both of these, should you choose to do them (note the percentage-ranges), might be of the take-home variety.

The numbers p, b, m, and f must satisfy the following conditions:

$$\text{each of } p, b, m, \text{ and } f \text{ is an integer;}$$
$$p \text{ is in } [10,30];\ b \text{ is in } [30,70], \text{ and each of } m \text{ and } f \text{ is in } [0,25];$$
$$p + b + m + f = 100.$$

Let me know your choices for p, b, m, and f not later than the end of the second week of the term. You may change your choices until one week after the middle of the term.

Allowing the students to choose, within the restrictions specified, their own grade-weights (percentages) is an attempt on my part to promote the sense of responsibility that I am seeking from them (for more about this, see my essay in the "In the Classroom" chapter). Under it, they are encouraged to consider their own strengths and weaknesses and structure their participation in the course to optimize their strengths. Once I have entered the students' chosen weights into the Excel spreadsheet that I use as my grade-book, keeping track of their progress is no more difficult than it would be if every student had been assigned the same weights.

The best means that I have found for grading presentations at the board is the diary. To each meeting of class I take a pad and pen. With them I record, in my personal shorthand, every significant event that occurs, and the person or people responsible. As soon as class is over (I ask for a schedule that contains no back-to-back classes), I return to my office and, with my word-processor, convert the shorthand into a full-fledged diary-entry of approximately a page in length. At the middle or end of a term, or whenever a student wants to check on my evaluation of his or her presentations, I use the "find" feature of the word-processor to select every diary-entry containing that person's name and create a record of his or her performance in the class. The diary has proved to be an invaluable tool for assigning presentation-grades, particularly in borderline cases. Samples of entries from one of my diaries appeared with my "In the Classroom" essay.

Appendix III contains an example of the type of examination that I assign to those students who choose to take one. The exam is typically a take-home experience.

Philosophy

There are four levels of mathematical prowess or accomplishment. The lowest is the ability to understand a proof that one reads or has seen presented by someone else. The second is the ability to prove a theorem. The third is the ability to test a conjecture, producing a proof when the conjecture is true and a counterexample when it is false. The highest level of mathematical achievement is the ability to formulate conjectures. (I am assuming here that someone at Level n, $n > 1$, possesses also the ability of Level $n - 1$.)

A student who finishes one of my Moore Method courses no higher than Level 1 can expect a grade of *C* or below in the "presentations at the board" grading category. Anyone who proves at least one theorem during the course can expect to earn at least a *C*; and anyone who proves theorems on a regular basis, even one per month, will earn at least a *B*. Everyone who has attained to Level 4 by the end of a term will, of course, have earned a grade of *A* in presentations.

Much subjectivity goes into assigning presentation-grades. I am like the United States Supreme Court with regard to pornography: confident that I can recognize an *A*-, *B*-, *C*-, *D*-, or *F*-performance when I see it, even though I can define it no better than I have. Fundamentally, what I am looking for in each student is an indication that he or she has, by the end of the term, learned how to "play the game of doing mathematics," at least within the confines of the present course.

The only deadline for learning to play the game is the end of the term. From my experiences as both a student and a teacher, I know (and tell the class often) that it is possible for someone to appear to be floundering at or below Level 1 for much of a term and then, by one appearance at the board, to reveal to everyone present that he or she has been developing all along, hoping and striving to "crack the code," to "have the light come on." When this epiphany occurs, even during the last week of class, it can raise a student's grade from an *F* to an *A*.

There is a more radical means for de-emphasizing grades. On the first day of class, ask each student to hand in, on a piece of paper, the grade that she or he hopes to achieve in the course. At the beginning of the next meeting of the class, return the pieces of paper and

> I teach a course entitled Liberal-Arts Mathematics: Statistics through Baseball. Like most liberal-arts math courses, this offering is aimed at math-phobics who are looking to satisfy my university's mathematics requirement in the least-scary and -painful way. I have taught two sections of this course for each of the last four semesters. In the fall of 2006, when, during the first day of class, I asked for amendments to any of the class policies, some quick thinker proposed that the percentage-range for each of the grading categories—homework and class participation, project, and midterm and final examinations—be changed to [0,100]. This would allow an individual to "put all of his eggs into one basket"—for example, to invest all of his effort for the course into his project and, theoretically, never attend class or take an exam. The rest of the class, somewhat incredulous, jumped onto the bandwagon; and the proposal passed unanimously. Although class attendance and participation continued to be surprisingly good after this change to the grading policy, some students did attempt to pass the course solely on the basis of one grading component—with mixed results. Word apparently got around; for, in the four sections of the course that I have taught since this event, the phenomenon of the class's voting in 0%-100% grade-ranges in all of my grading categories has failed to occur in only one section. (Ironically, even in that section I, by now so used to the ploy, mistakenly imposed it on the class. When the students and I discovered what I had done, they quickly forgave me and adopted the plan for themselves.)
>
> May

announce that you guarantee that each student will end the course with a grade no lower than the letter that she (or he) handed in, provided that she fulfills the requirements of the course in terms of attendance and effort put forth. Then say something such as, "I hope that that puts the issue of grades to rest. Now, let's learn some mathematics." Although I have fantasized about trying this procedure, I have never yet actually done so.

“…no thought, no idea, can possibly be conveyed as an idea from one person to another. When it is told, it is, to the one to whom it is told, another given fact, not an idea. The communication may stimulate the other person to realize the question for himself and to think out a like idea, or it may smother his intellectual interest and suppress his dawning effort at thought. But what he *directly* gets cannot be an idea. Only by wrestling with the conditions of the problem at first hand, seeking and finding his own way out, does he think.”[1]

— John Dewey

8 Why Use the Moore Method?

Recurring in each author's essay is the theme that the Moore Method is a natural way to teach any subject and that, as such, it has the potential to offer considerable benefits to students—benefits that might be expected to extend beyond those associated with successful lecturing. If one were to seek training for a skill outside the walls of the Ivory Tower such as horseback riding, painting, pottery making, investing, programming, or sewing, odds are good that it would be taught through a hands-on, apprenticeship-style model. Participation breeds interest, something we all wish to instill in our students. Consistent with this model, the authors intertwine their personal beliefs and experiences with an exposition of the benefits of a minds-on, student-as-apprentice-mathematician model. In all four essays, the authors suggest that the method fosters increased possibilities for active learning in which the dominant mode has the student, in addition to being a processor and user of ideas, becoming an active agent in the making of ideas.

May ties his own life experiences as a father and award-winning teacher to the method, stressing the benefits of immersing students in the art of *doing* mathematics by having them create examples, formulate and settle conjectures, and prove theorems. Beginning with a tour of his own experiences as a student, Parker touts benefits to the students that include group learning from individual students' mistakes, learning to read mathematics critically, eliminating reliance on the authority of the teacher, training in the art of inquiry, and arguably most importantly, allowing each student to learn within the mode that is most natural to his or her innate thought processes. Coppin's piece includes his childhood experiences of being naturally “driven to know” alongside references to established practitioners of inquiry-based methods. He reiterates our belief that it is most natural to teach a subject just as it is learned by those who practice it, by forming and testing conjectures. He emphasizes that doing so encourages students to become independent learners. Mahavier reflects on his beliefs through reference to the literature on learning theory, literature that consistently supports inquiry-based learning over alternate, more prevalent, modes of instruction.

(May) I teach by the Moore Method for a number of reasons. Primarily, I use it because it best expresses who I am, as a teacher and a human being. As far back as I can remember,

[1] *Democracy and Education*, Kessinger Publishing, 2004, p. 130.

I have believed that the vast majority of my students have had within them the ability to do mathematics in its essential form: formulating and settling conjectures by means of proving theorems and constructing examples. Consequently I have viewed my role as teacher as that of coach, advisor, and even cheerleader, one who draws from his students abilities that already reside within them. (The Latin root of *educate* means "draw out of.") The Moore Method, better than any other, enables me to play this role. The method is enough of my philosophical and emotional makeup that my wife and I raised our two children, both daughters, by it, although much of the time we were unaware of what we were doing.

As an example, one time our older daughter, who was less than one year old, was playing with one of her toys, a "Busy Box." It was composed of several devices from everyday life: a drawer, a mirror, a door, a windmill turned by a crank. Only she, instead of turning the windmill with the crank, was turning the crank by means of the windmill. My wife—who is also a mathematician, as well as the most-outstanding software engineer and best all-around person whom I have ever known—saw what our daughter was doing, came up, and began to "correct" her. I objected, saying, "Who's to say that the way she's doing it isn't just as good as the 'prescribed' way?" Both daughters are grown now and married, the "Busy Box" one raising two daughters of her own (she is also a part-time graphic designer with a degree in architecture), the younger one working as a Ph.D. in the field of learning disabilities. At the least, I can claim that my wife's and my parenting has done no harm. And, actually, I believe that our daughters' effectiveness and joy in living spring in large part from the fact that they have discovered so much of life for themselves and shared it with us, rather than the other way around. In this, the most important arena of my life, the Moore Method has worked.

May

Observing my daughters grow up has given me another reason for preferring the Moore Method. As I noted in the chapter entitled "What is the Moore Method?", the most important skills that are learned in life—walking, talking, running, reading, and writing—are developed by the Moore Method or similar teaching traditions of inquiry-based learning. That is, with each of these incredibly difficult challenges, the child teaches herself. She "knows," almost by instinct, when the time is right to begin the curriculum for each development; and at that time, she begins to learn, simply by trying. If she is fortunate enough to have good parents, she receives encouragement and praise for even trying, empathy and support when she fails, and hugs and cheers when she succeeds. If the Moore Method works so well in these most difficult of all of life's learning-tasks, why shouldn't it work at least as well in others?

Doing mathematics is itself a skill. No harder than walking, it is perhaps more like swimming—not necessary for a full life, but certainly life-enhancing. An instructor would not teach swimming by doing laps for fifty minutes at a time, three times per week, while his students sat on the side of the pool and merely watched. By the same token, mathematics is learned and mastered only by doing. The Moore Method gets students into the "mathematical pool" immediately, at a depth that they can handle. It has them constantly "swimming," affording them a better opportunity to master the subject than does any other method.

Although I very much enjoy lecturing myself, I have never learned mathematics well by having someone else lecture to me. I seldom derive much pleasure from watching another mathematician prove a theorem, unless he or she is a truly outstanding expositor or a student in one of my classes. By the same token, I have never felt the satisfaction from reading someone else's proof that I obtain from constructing a proof of my own. This is partly because I am slow. It takes me a long time to digest what someone else is attempting to teach me. I always feel the need for some preparatory work on the important ideas myself. I seriously doubt that, without the Method, I would ever have learned to enjoy mathematics as I do, much less earn a doctorate and publish my research. I believe that I would have floundered and ultimately drowned under the lecture method. For that reason, and for the fact that nothing in life brings me greater joy than to see and be a part of students' developing their mathematical abilities and self-confidence, I enthusiastically practice the Moore Method. And the method seems to have been beneficial to my students and to me. I have received outstanding-teacher awards from the Maryland Council of Teachers of Mathematics and the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America. I have sent at least four students on to earn their doctorates, one in mathematics, two in statistics, and one in oceanography. Many other of my students have gone on to earn their master's degrees in mathematics. Finally, I have produced many graduates who are decorated teachers at the K–12 level.

The Moore Method is particularly useful in today's teaching environment. Today's students have grown up with video games and other sensory delights, stimulating experiences in which they at least think they are actively involved. They are probably the most restless group of people in recent history. With the Moore Method, the students never sit still; they lean forward in their seats, trying to understand what one of their peers is explaining to them, ready to question and perhaps challenge that peer. The students are eager to illustrate a point or present a counterexample or a proof to a theorem at the board. It has them engaged in the enterprise of understanding and creating mathematics. From the experiences of Dr. B. Dale Daniel of Lamar University (Daniel, 2001), it seems also that the method may be the best way to teach a mathematics course on-line.

> "It was very rewarding to understand a proof and be able to present it to fellow students. I feel that the Moore Method was very beneficial to this course and could be applied to other courses as well. Finally, I felt that I received a better understanding of the course because of the Moore Method of teaching — Thank you for an excellent course!"
>
> Anonymous evaluation of Discovering Affine Transformations.

I believe that no one who is going to teach mathematics to others should do so without having proved at least one theorem all by herself. I am talking here not just about people who are working toward their doctoral degrees with an eye to becoming college professors; I am talking about everyone from Ph.D. hopefuls to the prospective kindergarten teacher, who will be perhaps the first person formally to instruct her students in mathematics. The theorem proved need not be original; it need not even be "hard." It could be along the lines of "$A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$." What is important is the experience of looking at a conjecture until one understands it, decides that it is true, and then thrashes through the task of proving it, to the satisfaction of oneself and the concurrence of others. Furthermore, I believe that anyone who cannot do so should seriously ask herself whether she has chosen the correct career. Finally, I believe that almost everyone who wants to teach mathematics can prove many theorems, even some "hard" ones.

Paul Halmos, in his book *I Want to be a Mathematician: An Automathography* (Halmos, 1985) states,

> *Some say that the only possible effect of the Moore Method is to produce research mathematicians, but I don't agree. The Moore Method is, I am convinced, the right way to teach anything and everything. It produces students who can understand and use what they have learned. It does, to be sure, instill the research attitude in the student—the attitude of questioning everything and wanting to learn answers actively —but that's a good thing in every human endeavor, not only in mathematical research. There is an old Chinese proverb that I learned from Moore himself: I hear, I forget; I see, I remember; I do, I understand.*

I disagree with Halmos only slightly. There are outstanding teachers, inside mathematics and out, who do a beautiful job of lecturing or practicing some other non-Moore Method. I imagine that they would be less effective with the Moore Method than they are with what they currently practice. I also believe that they are at least as effective at inspiring and instructing students as I am. I know, however, that I would be neither true to myself nor as effective as I am if I were to teach by any approach other than my brand of the Moore Method.

(Parker) When I was a freshman at Guilford College, I found myself in a course given by J. R. Boyd. He had become my advisor by default when, during orientation, I hastily abandoned my plans to study biology when I learned that you were required to attend labs instead of baseball practice if there were a conflict. I chose mathematics not because I was interested in it, but because I thought that the slow speed at which I read (comprehension had never been a problem) would be a handicapping condition in the possible majors in which I was interested. Besides, as I compared myself to many of my classmates, I had been "good" at mathematics in high school. I had been afraid when Mr. Boyd placed me in calculus; I had protested that I had only had four weeks of trigonometry in high school. He reassured me that if I needed to know more, I'd learn it when the time came. On the other hand, I was naïve enough not to worry about the second mathematics course since it was Mr. Boyd's course and I didn't even know what Linear Point-Set Topology meant. As I experienced the way the course was being taught, this didn't bother me either. Since I was new to college, for all I knew this might have been the way people did things in college. About half way through the semester, I was in a death struggle with Problem 21 trying to show that our continuum was the range of a sequence (have you ever tried to prove something that isn't true?) and the problem was the topic of discussion after the person I still believe to have been the ablest student in the class (it was not he, but a mortal whom I didn't think was even as good as I who eventually got it) had made a failed attempt at a proof. As the discussion, in which I was just a bystander, wore on and the ideas I had built crashed one by one, I broke my silence when I blurted out, "Mr. Boyd, why don't you just show us how to do it?" Mind you, this was not a polite inquiry, but a disruptive outburst spontaneously driven by frustration. He turned in the same rhythm in which he was leading the discussion, and, with apparent sincerity, immediately responded, "Mr. Parker, why would I want to limit you to what I know?"

At the time I was impressed by, and grateful for, his handling the situation in such an unruffled way and not further embarrassing me beyond the way I had embarrassed myself.

As I matured as a mathematician and a teacher, I became especially grateful that the incident stuck in my memory because Mr. Boyd really communicated an important lesson. As teachers, we are likely experts on the things about which we teach. Students also recognize us as experts. Under these circumstances, most students are at least as likely to defer to the teacher's authority as an expert as they are to the authority of the argument that the expert gives. We want logic to arbitrate arguments, not prestige. Furthermore, if the students do actually turn out to be more able than we, they will not be limited by the way we think about the mathematics. To illustrate, consider the following. Suppose that the task were to teach a left-handed child to throw a baseball. Would we force the child to throw right-handed because that was the way we threw best? If we did, we would likely conclude that the child had little or no ability. On the other hand, suppose that we gave the child a baseball and put the task to the child of getting the ball to the catcher without leaving the little hill on which the child stood and watched what the child did. Wouldn't we then be in a better position to judge how best the child could be molded into an effective pitcher?

I suspect that training a brain is even more complicated than training muscle memory. To me, lecturing seems to carry with it the likely possibility of being analogous to forcing everybody to throw right-handed. A lecture must be prepared, in its entirety, in advance, and thus is almost certain to be predicated on how the professor would "do" the mathematics. Even though the professor may have access (if he or she actually grades his or her students' papers) to how the students have responded to earlier lectures and could use these responses to inform how alternative approaches might be employed, these responses are likely to be strongly influenced by what the students were shown in the lecture. Certainly, some in the audience will be able to identify with the lecturer and imitate the lecturer's thought mode and, if they are brighter than the lecturer or in a position to commit more time, perhaps create things that the presenting thinker couldn't. But I consider it more likely that the students will just try to imitate what they have seen the professor do. In addition, feedback comes after the fact and, for the lecture that has already been given, its impact is shown, at best, the next time the lecture is given and the audience is then a different set of students.

If we were to compare preparation of a lecture with preparation of a problem sequence in a course being taught by professors for the first time, we would likely see that both preparers start from the same place—asking themselves what a student needs to understand in order to follow arguments that they have in mind for the theorem in question. The lecturer then puts these pieces together and makes the connections for the students during the lecture. It is not unusual for the lecturer to pause and question at important junctures to assure him/herself that the students are following. For further assurance, the lecturer will assign follow-up assignments to see if the students can use, at some level of mastery, the ideas from the presentation. In Moore Method, the observation of the students' reactions to the ideas is put up front rather than back-loaded. By presenting the ideas as questions, we see the students dealing with them and have a gauge for the adequacy of their preparations. We also give ourselves the flexibility of asking follow-up questions that can seize upon *their* insights since they do not know, in advance, what we had in mind for a proof. Whereas a lecturer will get access to information through homework and exams that can improve the lecture for the next time the course is given, the Moore Method instructor gets this feedback in an ongoing stream as a part of the classroom dynamic and can use it to adapt the course notes to address deficiencies or accelerate progress with the students from which it comes.

One recurring phenomenon that I have observed over the thirty years in which I have used the method is that, when a student gets a theorem on her or his own, it tends to be a transforming experience. For students experiencing Moore-style pedagogy for the first time, the first problem that a student gets seems always to be the hardest; almost without exception, the second success comes more quickly than the first. What's more, it doesn't seem to make much difference how "easy" (mind you, any problem is hard to someone who doesn't yet have a proof) a first problem is. The students who succeed on their own on some problem tend to go on to "harder" problems. There seems to be something about success that motivates students to want to succeed again. Part of our job in promoting the culture within the classroom is to try to make sure that students who, at any point, have not yet succeeded in creating mathematics, recognize that they may be able to do what their peers are doing and to have problems available to enable breakthrough successes.

My own experience as a mathematician has been that the things I have done myself, either as a classroom student or a continuing learner preparing courses or in pursuit of research results, tend to stick with me. I have complete command of the ideas I developed myself regardless of how long ago I made them. In contrast, I typically remember where I read a result or heard a proof, but when I need such an idea, I usually have to go back to the sources and renew my command of the arguments I have borrowed. I suspect that this phenomenon may be pervasive in persons who have become independent makers of mathematics, and the Moore Method classroom certainly both nurtures and rewards those students who become independent thinkers.[2]

A Moore Method classroom also provides opportunities for a traditional learning experience for students. Those students who are not presenting are listening to a lecture, but with the authority of the expert removed. It is their responsibility to arbitrate the correctness of arguments; their acceptance of an argument offered signifies their admission that they understand it. Contrast that with a lecture by an expert where an *a priori* expectation exists that the argument is correct, and the task is to follow a line of reasoning presumed to be valid. The perceived possibility that a peer's argument might be incorrect fuels extra involvement. Yet, if an argument is correct, a student has duplicated the lecture experience. Even then, there are meaningful contrasts. If listening students already have arguments of their own, only good things can happen. If an argument given duplicates an argument in hand, the product of the thinking is reinforced. If an argument given is different from an argument in hand, the listeners have picked up extra perspective on the problem. Even when students have worked on a problem, but not achieved an argument, they are better prepared to critique the argument, since the presenter is almost sure to travel ground on which they have trod. Hearing someone else articulate a way to bridge a gap that had stymied them can provide effective learning experiences.

There are at least two additional fringe benefits that come in the Moore Method classroom. One is that students make good mistakes. No student goes to the board with the goal of being wrong. Thus the mistakes that occur are sincere and often typical. This allows other students having the same misconception to be educated when the mistake is uncov-

[2] Such phenomena could be objects of collegiate mathematics education research. See the chapter on assessment for information on research currently underway.

ered. It also allows the professor an insight into what the students are thinking as he or she experiences the students grappling with the mistakes and coming up with alternative attacks. Along with unexpected arguments for problems, these experiences provide the information that allows a professor to tailor subsequent problems to meet or remediate the needs of the class.

Second, by learning to construct meaning in the wake of reading definitions or theorem statements and in responding to the need to present or write clearly, students nurture the skills that are needed to read mathematics. I view it as ironic that, by stranding the students from prepackaged arguments, they are given an enhanced capacity to assimilate prepackaged arguments. Perhaps it is because they develop their ability to recognize those instances in which understanding eludes them, and, as a result, focus more intensely on the arguments of others that attempt to address the source of their confusion. Perhaps they feel as if they have been given servants to aid them in their work. Perhaps they see that details matter, particularly in dictating the logic of the statements. Whatever the source, the ability to read mathematics matures.

Mind you, should you choose to use the Moore Method, there are some built-in complications. One of the beautiful things about the method is that when students succeed, they have no one to credit but themselves[3] because they are clearly the ones who have done

I should confess that, when I graduated from Guilford, I would never have guessed that I would become a proponent of the Moore Method. I had been in Moore-style courses for most of the major at Guilford and thoroughly enjoyed the experience of making mathematics, but I had also enjoyed the four mathematics courses I had taken that had been given by lecture. My outlook, at that time, was altered when I took the advanced test in mathematics on the Graduate Record Exam. I answered the question on topology and answered the question on groups and then spent the rest of the three hours trying to figure out what the words and symbols in the other questions might mean. I rushed to the conclusion that I had been shortchanged in my mathematical education. After all, I had been a successful student in mathematics at Guilford, yet I couldn't even read questions that, apparently, math majors were expected to be able to answer and answer quickly. Nevertheless, even with this aversion, some of the method had stuck. In my first job out of college, a high school mathematics teaching position, I found myself trying different things in different sections of the same course to try to get comparisons on which ideas made the best basis from which to think. I also turned loose students whom I had identified as particularly able on problems without showing them how to do them. But it was as a graduate student that I was fully converted to the Moore Method. When I compared my experience to those of my fellow students, I saw that we were all committing every waking hour of every day to mathematics. But my colleagues had to learn what it is like to go a week without a result because someone had always shown them how to do things. Meanwhile, I just had to learn lessons like how to multiply matrices; I already knew how to deal with the fact that some problems come more easily than others. The contrast was a life-changing experience in the effect it had on my teaching. It was clear that I was no abler than they, but that my mathematics education had prepared me to use those faculties that my mind had. Many of my colleagues' educations seemed to have given them a beautiful toolbox, but no instructions on how to use those tools on tasks they had not previously rehearsed.

Parker

[3] We can forgive them for not realizing the pains we have probably gone through to put them in a position where doing it themselves is both possible and likely.

the work. The flip-side of this is that when students do not succeed, they will blame you because you have not taught them in the manner they expected to be taught. If one is to choose to put oneself in the position of deferring credit and being the focus of blame, such a person had darn well better have the strength of his or her conviction that there is a bigger payoff for the student than from using traditional pedagogy. After thirty years of teaching in the cauldron, I remain convinced.

(Coppin) Herein is my perspective on why I continue to use the Moore Method style of teaching in many of my classes. If you are reading this piece, you are most likely dissatisfied with your own teaching, you are curious or you want to improve your teaching. I present six reasons why one should at least attempt to use the Moore Method. They are the following: the Moore Method as a pedagogical hyperbole, as a means of optimizing learning, as a context in which to engage students in authentic mathematics, as a moral imperative and as part of a strong tradition. As a sixth consideration, I give a personal perspective with which the reader may identify.

As Pedagogical Hyperbole

Any good teacher is an idealist and a romantic, and as a result seeks to connect, to awaken a joy, and to play a significant role in the maturation of students. All teachers, especially beginning teachers, should experiment with different teaching styles. If for no other reason than as a pedagogical hyperbole, consider teaching a course by the Moore Method. As a means of communicating with your teacher's soul, you owe yourself this experience. Pick the right time, the right place, the right course and a good set of notes. Teach a Moore Method course at least twice. This experience alone will do more to develop your teaching skills than any other single thing you can do!

To Optimize Learning

One summer thirty years ago, still somewhat inexperienced as a teacher, I took the time to reflect on what separated good teaching from average teaching. I came to just one very clear conclusion. For students to learn a thing deeply, the teacher must lead the students to the edge of a cognitive gap. It may take the form of a problem to solve, a question to answer, or a theorem to prove. The teacher expects the students to try hard and successfully close the gap a good percentage of the time. At that time, I called this situation, "an intellectual gap." Later on, a colleague informed me that what I called an "intellectual gap" was what Lev Vygotsky, a proponent of constructivism,[4] called the Zone of Proximal Development (ZPD).[5] Those students who are successful will learn if their teacher gives them many ZPDs. This would be true of any good course taught by lecture, seminar, lecture-discussion or the Moore Method. The only difference among the many teaching styles would be the size of the ZPD, assuming, for the moment, that we can quantify ZPD. For example, in a lecture course with 200 students, the ZPD would likely be quite small. A small class of

[4] www.funderstanding.com. Constructivism is a philosophy of learning founded on the premise that, by reflecting on our experiences, we construct our own understanding of the world we live in. Each of us generates our own "rules" and "mental models," which we use to make sense of our experiences. Learning, therefore, is simply the process of adjusting our mental models to accommodate new experiences.

[5] ZPD was referenced and defined in Axiom 2 of my essay in Chapter 5—Development and Selection of Materials.

fewer than twenty-five might have the luxury of allowing a somewhat large ZPD. Thus, I concluded that the Moore Method maximizes the ZPD for each student, thus optimizing the opportunity for significant intellectual growth in each student.

To Experience Authentic Mathematics

As teachers of mathematics, we want to teach the genuine article. We must continually ask if we are teaching the essence of mathematics or if we are teaching a subject that is a knock-off of the original. After all my years of reflection on the nature of mathematics, what characterizes good mathematics, and what drives mathematicians to discover the great seminal mathematics, I have come to what seems to me to be an inevitable conclusion. We should teach mathematics courses so that our students are expected to do mathematical research appropriate to their level, as part of their course work. In fact, in *Master of the Game* (EAF, 1999) no less a mathematician than Paul Halmos stated, “I think students should do research in their subject at their level.” The Moore Method courses I have taught all have a significant element of research. I believe this to be true of any Moore Method course. To carry this point to a higher level, in his book, *The Art of Mathematics*, Jerry King[6] (King, 1993) quotes Henri Poincaré who makes a statement that struck to the core of my being. Poincaré asserts that the reason mathematicians do significant mathematics, the seminal mathematics, is that *they are driven by aesthetics*. *Great mathematics comes from a search for beauty*. How often have we made a statement after a presentation of a proof, “That was a beautiful proof.” King further emphasizes this point by quoting Keats,

> “‘Beauty is truth, truth beauty,’ — that is all
> Ye know on earth, and all ye need to know.”

Seymour Papert, a pioneer in artificial intelligence (Papert, 1993) and a professor at MIT, believes that Poincairé’s idea should be extended *and* “brought down to earth and made applicable to teaching and the learning of mathematics.” King conjectures that *the* missing component in our mathematics courses *is* aesthetics. And, I emphasize here that the Moore Method, properly applied, *does teach aesthetics* by *teaching research* that is appropriate to the emotional and mathematical maturity of the students in the class. The triad of aesthetics, research, and mathematics are so intrinsically tied together that to remove one destroys the others.

It is best to teach subject matter as it is practiced. As an example, art, itself, is taught in such a way that students actually *create their art as they learn art*. It has long been understood that art is very closely related to mathematics. Mathematics should imitate its spiritual kin. Moreover, music is taught by doing. In the 19th century, physicians learned their art as they practiced it, which, to some extent, still occurs today in some medical schools (Oh, 2003), (Birnbache, 1999). Nobody learns how to paint by listening to a lecture nor does one learn to play a violin by listening to someone expound on the finer points of the violin. Based on lectures alone, will any students be able to paint or play the violin after a year? Because of its nature, mathematics is best taught by doing mathematics; that is, by doing mathematical research that is appropriate to the students.

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[6] *The Art of Mathematics* should be required reading for all mathematicians, students of mathematics and especially teachers of mathematics.

As a Moral Imperative

I also see a moral imperative. Is it mathematics if students learn rote techniques or formulaic thinking? Adam Robinson (Robinson, 1993), co-founder of *The Princeton Review* stated,

> *Our entire school system is based on the notion of passive students who must be 'taught' if they are to learn…. Our country spends tens of billions of dollars each year not just giving students a second-rate education, but at the same time actively preventing them from getting an education on their own. And I'm angry at how schools produce submissive students with battered egos. Most students have no idea of the true joys of learning, and of how much they can actually achieve on their own.*

Frankly, we all seek to affect student transformation. The bread-and-butter courses of most departments of mathematics are College Algebra, Precalculus, and the Calculus sequence. These are hard courses in which to enable transformative experiences. The advice that I give young faculty is to use the Moore Method to teach a transition course, an upper-level mathematics major course, or a liberal arts course. Then, there is a very good probability of success. Only then will you have the following experience:

> *It is the supreme art of a teacher to awaken joy in creative expression and knowledge.*
>
> Albert Einstein

Part of Tradition

The Moore Method is an example of inquiry-based learning. Historically, such learning started with Socrates and his practice of drawing knowledge from his pupils by means of well-thought-out questions. In recent times, it has been spurred on by the ideas of John Dewey and the constructivist movement in its more recent incarnations. During the 1960s, there were many attempts to reform the educational system. The goal behind the reforms was to produce problem solvers; however, apparently that goal was unmet. We leave it to others to solve that riddle. However, these reforms had many good results; that is, they led to awareness of the importance of teaching inquiry-based courses, which, in turn, led to new attempts at other reforms, new textbook designs, and initiatives sponsored by NSF.

Many consider the Socratic method to be the epitome of great teaching. In one of his dialogues, *the Meno*[7], Plato presents Socrates as questioning an uneducated slave boy, a blank slate, so as to draw from the slave boy knowledge he was never taught but knows *a priori*. With the recent resurgence of interest in inquiry-based learning, there is much interest in the Moore Method, which predates the afore-mentioned reforms of the 1960s. Socrates lived during the 5th century B.C. Moore, a latter day Socrates, lived most of his life in the 20th century. Of course, the Moore Method officially started when Moore began his experimentation with inquiry-based learning (Moore would not have known it by that name) in 1911 on joining the faculty at the University of Pennsylvania.

In the ten-minute video, *Master of the Game* (EAF, 1999), Paul R. Halmos, Professor Emeritus at the University of California Santa Clara, and James W. McClendon, a theologian at Fuller Theological Seminary, make appearances. Professor Halmos discusses his own conversion to the Moore Method decades ago. Halmos had met Moore and applied

[7] The reader might find a more modern "Meno" of interest. See Alfred Renyi's *Dialogues on Mathematics.*

the method in his own teaching. Halmos intoned (Halmos, 1975) that the method "worked like a charm." He believes that it can be used in all subjects and at all levels. Professor McClendon, a former student of Moore, did not become a career mathematician but became a theologian instead. He states that Moore made you feel that when it came to mathematics you could do it, maybe not always at a high level but you could at least make significant progress.

The numbers of academics who support teaching philosophies that resonate with the likes of the Moore Method are legion. We list some who are well known and others less known. Alfred North Whitehead, mathematician and philosopher, stated, "Your learning is useless to you till you have lost your text-books, burnt your lecture notes, and forgotten the minutiae which you learnt by heart for the examination." Maria Montessori, creator of the well-known and widely used Montessori method, believed, "One test of correctness of educational procedure is the happiness of the child." Anyone familiar with Einstein's academic career would understand that he would have benefited from an education based on a method like the Moore Method or a method akin to it. Papert, co-developer of the programming language LOGO, also wrote *Mindstorms: Children, Computers and Powerful Ideas* (Papert, 1980) that advocated developing classroom environments in which students would learn mathematics completely on their own. LOGO was to be an instrument by which students would explore and learn. However, Papert's ideas were never widely applied to develop a new learning culture. Instead teachers and authors of books who were of the teacher-centered classroom culture used the turtle geometry associated with LOGO to develop "cool" pictures. Papert's ideas have great merit today.

As a summary and a sixth perspective, allow a personal note, which may resonate with your own experience. Over my career, I have heard well-meaning colleagues (not proponents of the Moore Method) state that the Moore Method would be more appropriate for college seniors or first-year graduate students. My response, after I recovered from the shock, was this: The Moore Method, if used at all, is much more appropriate for students who are beginners, raw neophytes. I believe that to learn *anything* one must have some sort of schema, some cultural, cognitive scaffolding by which one can receive new ideas and methods. We do not learn in a vacuum. When a student has gained some significant command of mathematical processes, then that student can listen to someone lecture on a mathematical topic *and* actually learn. In other words, begin with the Moore Method. *Then* when a student has matured mathematically to a sufficient degree, that student may profitably sit in on a lecture *and* actually learn.

Just as mathematics courses should be taught as mathematics, we teachers *are* mathematicians. I was first a mathematician. Then, I sought to share my mathematics, my joy in real mathematics, with others. I think that is the proper order. I teach by the Moore Method for reasons of DNA and culture. I grew up in the 1950s in a small town with little outside stimuli. There were none of the "mandatory" extracurricular activities such as soccer, cross-country and martial arts. School was very boring. It would appear that my environment was an intellectual and creative desert. And, it was almost so. However, I was very curious. I always had a project. Among several examples, I built a microphone using the carbon rods from an old D-size battery and pencil lead. I constructed an electric motor from nails and copper wire that happened to be around. My sources were old Life magazines and the 1923 World Book Encyclopedia. I wanted to know why!

After receiving a baccalaureate in mathematics, I found myself in graduate school studying mathematics under H. S. Wall and Moore. When I first met Dr. Wall, he spoke of mathematics as an art. I found a culture of mathematics in place that reminds me of what I have read of Göttingen before 1940 (Mac Lane, 1995, Reid, 1986). Because of this culture, I became a mathematician!

My background was deficient. However, in a way, I was fortunate. My deficits became my assets. Frankly, I am saddened by the thought that millions of children are being inoculated against a species of learning that is grounded in deep ideas and creativity, the lifeblood of an intellectual or creative life. Harry L. Grace, sociologist, agrees in *Learning the Student Role: Kindergarten as Academic Boot Camp* (Grace, 1975). He states, "While children's perceptions of the world and opportunities for genuine spontaneity and creativity are being systematically eliminated from the kindergarten, unquestioned obedience to authority and rote learning of meaningless material are being encouraged." Gore Vidal stated that he had never seen a stupid seven year old. What he meant was that our educational system suffocates the creativity that children are born with. I had a college algebra student tell me after class one day that in middle school he used to ask why in his mathematics classes. His teachers would tell him to quit asking why and just do the math. This student, in his first year of college, told me that he now no longer wants to know why. Too many schools drum creativity and curiosity out of our children.

In the final analysis, I was driven to know. What I learned excited me. It was profoundly deep. It brought me great joy and power. Passionately, I wanted to share all this with others. As a result, I became a teacher of mathematics. It is the Moore Method experience that I had as a student and as a teacher that has been my vehicle. I was transformed into a mathematician and impassioned to share with others; that is why I use the Moore Method.

What do you say?

(Mahavier) At the most elementary level of instruction, consider teaching a student to ride a bicycle. Would you consider beginning with a series of lectures? How many hours of lecture would you estimate would prepare a student to cycle properly upon leaving the classroom? The answer is almost too obvious to state: You can't teach a child to *ride* a bicycle by lecturing and you can't teach a student to *do* mathematics by lecturing. That is not to say that you cannot teach a student *about* cycling by lecturing. You can teach the student about the operation of the brake levers and shifters, or how one pound of weight on the rim of a wheel equates to a much greater amount of weight on the seat as measured in work. And you can teach a student *about* mathematics by lecturing. Furthermore, such lecturing (or reading) is appropriate to provide breadth. Still, lecturing about cycling or mathematics is best done after the student has done some cycling or some mathematics. And for this reason, most Moore Method practitioners whom I know do lecture at appropriate times to summarize results or to provide historical perspective and mathematical breadth. But the focus of the method is on training students to *do* mathematics and the lecturing is a result in response to what the students have done and where they are within the material under study. Why use the Moore Method? Because, no matter how much mathematics our students *know,* if they don't know how to *do* mathematics, they won't have success upon graduation in the schools, academia, industry, or government where the problems will *always* be different from those that were taught.

Having made a brief personal case that the method is the natural strategy for instruction and the best preparation for student success upon graduation, I'll now take a more objective tack addressing three themes: there is considerable literature supporting a move toward student-centered learning; the Moore Method meets with the approval of numerous well-established mathematicians; and the method supports the "best practices" for teaching as outlined by the mathematics education community. I'll close with an argument based on my personal perspective as to the future of education given the rapid growth of on-line education.

In addressing the literature, one caveat is in order. Throughout this book, the authors have tried to describe and support a method of instruction that we hold dear and believe to be effective based on our own experiences. We have done our best not to criticize other methods and to support the underlying theme that each teacher must find the method that best parallels his or her own psyche. The literature on traditional methods and especially the lecture method is not so kind, and to answer the question "Why Use the Moore Method?" without addressing the growing body of literature suggesting a move away from the most prevalent form of instruction would be remiss. Mind you, I studied under and know of lecturers who are amazing teachers and accomplish all that one would hope with their students, yet the research does not bear out the overall effectiveness of the method.

In *Our Underachieving Universities* (Bok, 2006), past president of Harvard Derek Bok writes of university faculty:

> *Though trained in research themselves, they continue to ignore the accumulating body of experimental work suggesting that forms of teaching that engage students actively in the learning process do significantly better than conventional methods in achieving goals, such as critical thinking and problem-solving, that faculties everywhere hold dear.*

and

> *Quietly but steadily, the ground is being prepared for an eventual shift in American colleges away from a teacher-oriented system featuring lectures delivered to passive audiences to a more learner-centered process in which students become more actively involved in their own education and professors adapt their teaching in accordance with more complex understandings of human learning.*

The lecture method *is* the most prevalent method of instruction, with an average of seventy-three percent of faculty using this method (Blackburn, 1980) and with the average faculty in the classroom lecturing eighty percent of the time (Chickering, 1991).

In his introduction to John McLeish's *The Lecture Method* (McLeish, 1968, p. vii) Robert H. Thouless writes,

> *The lecture has had a long history as the central method of university instruction. Its usefulness for this purpose has been so much taken for granted that it may seem to many to be almost blasphemous to doubt its value. If one is a university lecturer, the main competence on which one prides oneself may be skill in delivering a lecture in such a way that students are both interested and informed, and have their intellectual difficulties resolved. For those whose 'job satisfaction' derives largely from their use of this skill, it is an uncomfortable thought that perhaps the skill may not be such a useful one as they had supposed… .*

In discussing the Northern Polytechnic Experiments (McLeish, 1968, p14) McLeish confirms what every student who has ever skipped a lecture class already knows. The experiment tests one group of students (the "auditors") directly after a lecture and another group (the "readers") directly after reading the same lecture for the same amount of time. Based on the test scores he concludes, "It is also clear that reading the text is more effective than listening to the lecture, if, as in this case, equal time is available for the "readers" as for the "auditors," and that minimal context for relevance is provided." He further points out in discussing the Norwich Experiment (McLeish, 1968), "Students listening to an uninterrupted discourse within their range of understanding and taking notes in their normal fashion, carry away something on the order of forty percent of the factual data, the theoretical principles stated, and the general applications referred to by the lecture. A week later they have forgotten at least half of this material."

On the other hand, Bok summarizes nicely the growing body of experimental work on memory, learning, and cognitive psychology when he writes (Bok, 2006) "interests, values, and cognitive skills are all likely to last longer as are the concepts and knowledge that the students have acquired not by passively reading or listening to lectures, but through their own mental efforts."

There is further evidence that a move away from the lecture method is worthy of consideration. In *What's the Use of Lectures?* (Bligh, 1972), Bligh's research indicates that lectures are "not especially effective, even for conveying content [i.e., facts] – largely because a good deal of the content presented by the instructor is not attended to by students and what is attended to may be distorted." In "When Good Teaching Leads to Bad Results," Schoenfeld (Schoenfeld, 1988) expounds on a problem that he refers to as systemic.

> *…the issues raised in this article are general, and the causes of the behavior discussed here are systemic. Mathematics curricula have been chopped into small pieces, with the focus on the mastery of algorithmic procedures and isolated skills. Most textbooks present 'problems' that can be solved without thinking about the underlying mathematics but by blindly applying the procedures that have just been studied. Indeed, typical classroom instruction subverts understanding even further by providing methods for solving problems that allow students to answer problems correctly, without making an attempt to understand them*.

He goes on to explain the results of such teaching when he writes, "the students developed perspectives regarding the nature of mathematics that were not only inaccurate, but were likely to impede their acquisition and use of other mathematical knowledge." And, "There was little sense of exploration, or of the possibility that the students could make sense of the mathematics for themselves. Instead, the students were presented the material in bite-sized pieces so that it would be easy for them to master."

What does work?

In "Open and Closed Mathematics: Student Experiences and Understanding" (Boaler, 1998), J. Boaler presents a large-scale, three-year case study of two United Kingdom schools: one using the traditional direct-instruction approach under a well-respected teacher and one using open-ended activities exclusively. While this experiment addresses middle-school students, the powerful statement that time for discovery results in improved test scores is something that this author (having taught students from middle-school through graduate school) believes to be age- and grade independent. Boaler's observations at the middle school level parallel Smith's observations at the university level as outlined in Chapter 9. Boaler concludes that the students in the direct-instruction school, "developed an inert, procedural knowledge that was of limited use to them in anything other than textbook situations." To the contrary the students in the open-activity environment "had been enculturated into a system of working and thinking that appeared to be advantageous to them in new and unusual settings." These students not only out-performed their direct-instruction counterparts on the United Kingdom's assessment exam, they also surpassed the national average. This is particularly interesting given Boaler's statements that these students "did not know more mathematics," that some of these students "spent much of their time not working," and that these students "had not encountered all areas of the mathematics that were assessed."

Boaler attributes their success to their attitudes that mathematics is flexible and adaptable to new and different techniques. In essence, these students learned to investigate and think on their own and this ability surpassed ability obtained through high-quality direct instruction and rote drill in preparing students for the examination. Such an approach has clear drawbacks that Boaler addresses: students not working, the full curriculum for the examinations not being fully covered, and a sometimes chaotic environment in the classroom. Still, despite these flaws, the students performed well. In the Moore Method the materials are carefully developed with open-ended problem sets and open-ended activities. The method differs from the open classroom model, because the class is directed along a path that covers the necessary curriculum while still allowing students time to develop and construct their own understanding of the material. Hence, the Moore Method offers a more structured environment than that described in Boaler's study.

Consider the following quote regarding the Moore Method and note how it resonates in light of Boaler's experiment. Edwin E. Moise (Moise, 1965) who taught using the Moore Method had this to say about the competition and compromise between the goal of process and information:

> *… I believe Moore's work proves something of broad significance…that sheer knowledge does not play the crucial role in mathematical development that most people suppose. The amount of knowledge that a small class can acquire, struggling at every stage to produce its own proofs, is quite small. The resulting ignorance ought to be a hopeless handicap but it isn't. The only way I can see to resolve this paradox is to conclude that mathematics is capable of being learned as an activity and that knowledge which is acquired in this way has a power which is out of all proportion to its quantity.*

In (Selden & Selden, 1993), Annie and John Selden write of the evolution of mathematics education at the collegiate level, "while the exact nature of the coming evolution in the teaching of collegiate mathematics cannot be predicted, it will surely be substantial, with lectures playing a diminished role." They proceed in the same paper to address constructivism, the philosophy most closely aligned to the Moore Method as, "the most influential and widely accepted philosophical perspective in mathematics education today."

They define constructivism much as I would describe the Moore Method, writing "the learner is seen as an active participant, not as a blank slate upon which we write or as an empty vessel which we fill." They go on to make two noteworthy remarks. First they remind us that Begle's study (Wilson, 1972) showed (surprisingly) "there is little connection between teachers' knowledge of mathematics and the performance of their pupils." Thus they conclude that simply providing more mathematical education to the teachers will not necessarily translate to improved performance of the students. Hence, even if lectures were effective, adding *more* lectures and *more* material would not necessarily prepare a better cadre of teachers and researchers to educate the next generation. Second, they remind us of D. A. Grouws' work (Grouws, 1992) that showed us that "despite their methods courses, in-service teachers overwhelmingly tend to mimic the teaching style of the mathematics courses they themselves took." If we hypothesize that this last quotation extends to mathematicians mimicking their teachers and we accept the implications of Anderson, Bok, and McLeish that the lecture method is not as effective as it is popular, then the question of how to increase the mathematical knowledge of the next generation would beg the answer to teach using some method other than the lecture method.

If some alternative method of instruction might be appropriate, then what should we seek in an alternative method? Perhaps we should look for methods that meet with the support of well-established mathematicians and with established pedagogical guidelines. Coppin and May have quoted to Halmos' statements supporting of the method from *I Want to be a Mathematician: An Automathography*, published in 1985. By this time, he was not a new recruit to the method. The transcript of his talk *The Problem of Learning to Teach* (Halmos, 1974), opens with, "The best way to learn is to do; the worst way to teach is to talk." In his speech, "What is Teaching?" (Halmos, 1994) he stated that "the problem method is the way to teach everything" and "if we could teach every teacher to teach every course as a problem course, then one generation from now, in twenty-five years say, there would be no need for talks like this. All we have to do is find out how to do that, and we can adjourn."

In *Devlin's Angle* (Devlin, 1999), Keith Devlin labels Moore as "The Greatest Math Teacher Ever" and gives credence to the method as measured by the only truly significant measure—the product. While he addresses Moore's doctoral students, many of these students were introduced to the method as undergraduates.

> *If you measure teaching quality in terms of the product—the successful students—Moore has little competition for the title of the greatest ever math teacher. (Incidentally, he would hate to be described as a 'math teacher'; he always insisted on using the word 'mathematics' rather than the nowadays more prevalent abbreviation 'math'.) During his long career as a professor of mathematics—64 years, the last 49 of them at the University of Texas—Moore supervised fifty successful doctoral students. Of those fifty Ph.D.'s, three went on to become presidents of the AMS (R. L. Wilder, G. T. Whyburn, R H Bing)—a position Moore also held—and three others vice-presidents (E. Moise, R. D. Anderson, M. E. Rudin), and five became presidents of the MAA (R. D. Anderson, E. Moise, G. S. Young, R H Bing, R. L. Wilder). Many more pursued highly successful careers in mathematics, achieving influential positions in the AMS and the MAA, producing successful Ph.D. students of their*

own—mathematical grandchildren of Moore—and helping shape the development of American mathematics as it rose to its present-day dominant position. That's quite a record!

In 1931 Moore was elected to membership of the National Academy of Sciences. Three of his students were also so honored: G. T. Whyburn in 1951, R. L. Wilder in 1963, and R H Bing in 1965.

In 1967, the American Mathematical Monthly *published the results of a national survey giving the average number of publications of doctorates in mathematics who graduated between 1950 and 1959. The three highest figures were 6.3 publications per doctorate from Tulane University, 5.44 from Harvard, and 4.96 from the University of Chicago. During that same period, Moore's students averaged 7.1. What makes this figure the more remarkable is that Moore had reached the official retirement age in 1952, close to the start of the period in question.*

Devlin and Halmos are not the only respected mathematicians to support the method. Peter L. Renz (FOCUS, 1999) writes, "Is it reasonable to hope that every mathematics department might offer its students at least one modified Moore Method course? Given what the method has achieved, I would hope so." David E. Zitarelli writes in the *American Mathematical Monthly* (Zitarelli, 2004), "The Moore Method is the best known —and arguably the most successful—way to train students to become creative research mathematicians."

Do the features of the method parallel features assessed as effective by the mathematics education community? Consider the features listed by Gardiner in his report on redesigning higher education. Upon a review of the literature, he concludes (Gardiner, 1994) with a summary of "Conditions for Educational Quality." These conditions are:

1. *Challenge. Students need to be provided with activities aimed just above their current levels of cognitive development…*[8]
2. *A supportive environment. Both intellectual assistance in comprehending and emotional support when reflecting are required from teachers and peers alike…*[9]
3. *Sustained, diverse, and appropriate active involvement in learning. Students should be kept busy reading, writing, solving, designing, and interacting cooperatively with peers and professors, both in class and outside of class…*[10]
4. *High expectations. Expectations for quality of educational outcomes should be high. Considerable research has demonstrated that hard, challenging goals can substantially increase one's productivity…*[11]
5. *Clearly defined outcomes, frequent assessment, and prompt feedback. Knowing clearly what desired outcomes should be and having specific and timely knowledge of actual results achieved contribute powerfully to improving performance.*[12]

[8] See Coppin's comments on Zygotsky's ZPD in Chapter 5—Development and Selection of Materials.

[9] See Mahavier's defining features, including "Attitude," in Chapter 3—What is the Moore Method?

[10] The Moore Method deviates here since students interact in class, but work independently out of class.

[11] See discussion on maximizing the ZPD in Chapter 5—Development and Selection of Materials.

[12] Students receive daily feedback during presentations.

The Moore Method, as described in this book, meets each of these conditions. Challenges are laid before the students through carefully prepared materials that allow every level of student to achieve at his or her personal best. A support mechanism is in place through extensive student-teacher interaction (both in and out of class) and from peers (within class) when either success or failure occurs. Value is gleaned from both success and failure. Students are active in their own learning while reading and writing the mathematics of the course and in many instances, upon mastering a particular subject, outside reading is assigned.[13] Expectations are high (Mahavier, L., 1999). As you have seen in the chapter on grading and in the syllabi provided in the appendices, the authors all delineate very clearly what is expected of the students and provide frequent feedback to students in addition to the feedback that is inherent to the method—the feedback of their peers on a daily basis.

Of course, the Moore Method is antithetical to the algorithmic teaching described by Selden and Selden in the chapter "On Culture" and by Schoenfeld earlier in this essay, as students work through one continuous problem sequence that weaves together the materials so that students may see how it interconnects and how one problem may be solved in several ways by different students. Rather than breaking the material into "bite-sized pieces," large gaps are left for students to discover as they construct their own conceptual framework. Rather than emphasizing a grade based on a test, which is based on a set of problems, which is based on a review, based on algorithmic and compartmentalized subject matter, grades are based on the mathematics that the students create and on their presentation of this mathematics to their peers.

While the *National Council of Teachers of Mathematics Standards* (NCTM, 2004) target primarily K–12 education in their standards, many of the standards cross-cut the rapidly blurring lines between high school and collegiate mathematics. Although a review of all the NCTM standards would be excessive, we list below the standards for grades nine through twelve associated with problem solving, reasoning and proof, and communication. Of course the choice of words for the standards themselves, *problem solving, reasoning and proof, and communication* resonate of the Moore Method.

NCTM Standards for Grades 9–12 (NCTM, 2004)

Problem Solving

1. build new mathematical knowledge through problem solving;
2. solve problems that arise in mathematics and in other contexts;
3. apply and adapt a variety of appropriate strategies to solve problems;
4. monitor and reflect on the process of mathematical problem solving.

Reasoning and Proof

1. recognize reasoning and proof as fundamental aspects of mathematics;
2. make and investigate mathematical conjectures;

[13] Moore did not allow his undergraduate students to read mathematical texts. W. S. Mahavier wrote to me on this issue, "He (Moore) once said to me something like this: 'People think I don't want my students to read mathematics. That is not true. I don't want them to read at an early stage. The problem is that by the time I want them to read they won't do it.'" Howard Cook verifies that no outside reading was allowed for undergraduates. On the other hand, Albert Lewis wrote to me that some students have reported that in early years the class used the Ettlinger and Porter calculus book mainly as a source for problems.

3. develop and evaluate mathematical arguments and proofs;
4. select and use various types of reasoning and methods of proof.

Communication

1. organize and consolidate their mathematical thinking through communication;
2. communicate their mathematical thinking coherently and clearly to peers, teachers, and others;
3. analyze and evaluate the mathematical thinking and strategies of others;
4. use the language of mathematics to express mathematical ideas precisely.

Having argued why one might give thought to using the method based on natural learning strategies, the literature, the support of well-established mathematicians, and the best practices defined by mathematics educators, I'll conclude with a practical reason based on my personal perspective of the modern evolution of the mathematical teaching model. While it may not occur quickly enough to affect our careers as educators, I see two paradigms emerging for the teaching of not only mathematics, but of most disciplines. In the first model, lecture-style content will be developed and delivered via the next generation of technology. Numerous examples already exist including defensive driving, equal opportunity training, and a continually expanding selection of freshman courses.[14] Given the improvements in the quality and the decrease in expense of delivering such content en masse, lecturing will no longer be the localized process wherein at any given time hundreds if not thousands of faculty are lecturing on calculus at the same time on different campuses throughout the world. Rather, the lectures that are determined to be "best" by the publishing industry will be recorded and delivered via technology to the students.[15] The instructor on record, web-based homework,[16] web-based support systems, and on-campus support will be available to generate a distributed-learning environment in an effort to increase the retention and success rates of such courses. Automated testing will determine mastery of the delivered content. As the cognitive psychologists continue to make advances, it is likely that such testing can better address problem-solving strategies and ability to apply material in addition to mastery of rote techniques. Just as massive sections of calculus are prevalent at large state institutions due to the economy of scale, so too will large-scale delivery of online courses become prevalent. The second model will be the traditional classroom setting wherein smaller groups of students meet with a faculty member at predetermined times. Because of the mass availability of the first paradigm, students in these classes will be those either preferring the more personalized format for instruction and those able to afford the more costly education. Because they have opted out of the lecture format course, these will be the students seeking direct interaction from their instructors. Hence the demand for alternative pedagogies in the classroom is likely to increase, rather than decrease over the next generation.

In summary, the method offers a natural alternative to traditional lecture that meets with the approval of established mathematicians and the established best practices as defined by mathematics educators. While these may provide logical arguments for using the meth-

[14] See Academic Learning Systems or ALEKS for examples.

[15] See Thinkwell's Great Lectures: *Calculus I with Edward Burger*, 2001.

[16] See WebAssign or WeBWork for examples.

od, one should not ignore the emotional impact. Based on the efforts of the Educational Advancement Foundation, proponents have written hundreds of pages of personal perspectives on their own experiences as both students and practitioners of the method. Many of these are available on-line.[17] Why would so many people write so many pages on the subject of instruction when virtually all of them are already well-established in their fields as research mathematicians, award-winning teachers, and successful business people? A common thread through all of these pages is the incredible sense of personal satisfaction that each has garnered from both sides of the experience. As students, they were empowered by the method and went on to successful and productive careers in teaching, industry, government, and academia. As practitioners of the method, they have experienced deep emotional satisfaction as they saw the method guide and affect the lives of their own students. This emotional impact and sense of satisfaction comes from the direct student-teacher interaction that results from the method and is by far the best reason to consider the method. When we love what we do and our students love what they do, ultimately we all benefit.

[17] See *Legacy of R. L. Moore* website.

"All the instruments have been tried save one, the only one precisely that can succeed: well-regulated freedom."[1]

— Jean-Jacques Rousseau

9

Evaluation and Assessment: Effectiveness of the Method[2]

In this chapter, we discuss the results of educational research that has been conducted evaluating the impact of the Moore Method and describe the results of educational research on similar teaching approaches, draw parallels to the Moore Method, and make suggestions for further assessment of the effectiveness of the Moore Method for interested readers.

Introduction

Whenever an innovation emerges in teaching, there are important questions to be asked. Does the innovation actually work? That is, does it truly improve students' learning, or does it just give the appearance of doing so? Is it feasible that the innovation can be implemented by other instructors in other educational settings, or does it rely upon the talent of a few gifted instructors in a particular context? Is the innovation appropriate for all learners, or just for a particular subgroup? Though the proponents of such innovations are often quite enthusiastic in their efforts to describe and promote the innovation, they are often only able to provide anecdotal evidence for its success. While important and frequently convincing, such evidence is not sufficient to answer the questions posed above. Rather, careful, systematic research should be done to establish the effectiveness of the innovation.

In this chapter, we give an overview of what research exists that supports the Moore Method of teaching. We will do this by presenting ideas from relevant theories of learning, results of research on similar teaching approaches in mathematics, and the results of our own research on one implementation of the Moore Method in an undergraduate number-theory course at a large state university.

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[1] *Emile* or *On Education*, Basic Books, 1979, p. 92

[2] This chapter was contributed by Jennifer Christian Smith, Sera Yoo, and Stephanie Ryan Nichols of the University of Texas at Austin.

Support From Theories of Learning

This section is not intended to be a comprehensive and complete overview of theories of mathematical learning. Rather, we will describe the ways in which learning theory supports particular aspects of the Moore Method. There are several major theories of learning that are applicable to mathematics education. The two perspectives one hears about most frequently are constructivism, based on the work of Piaget and others, and socio-culturalism, based on the work of Vygotsky. There is a growing trend towards a perspective that draws from both of these theories, known as social constructivism.

Constructivism posits that all learning takes place through active effort by an individual to organize her thoughts and assimilate new information into existing frameworks in her mind. Piaget believed that for young children, manipulation of concrete objects was an important part of this process. He also theorized that children pass through distinct stages of mental development, though not all constructivists adhere to this notion today (von Glasersfeld, 1990). Important tenets of constructivism include active engagement, assimilation, and accommodation. Assimilation occurs when an individual fits new knowledge about a concept into his or her existing mental framework (or schema) for that concept, while accommodation requires that the individual reorganize his or her mental schema in order to fit the new information in. This is a difficult process that requires active thought on the part of the individual; often individuals will simply ignore information that does not fit into their current schema.

Socio-cultural approaches to learning regard it as a product of participation in the activity of a particular community of practice (Forman, 2003). New knowledge is not a commodity to be acquired, but rather is a set of skills and concepts that the individual learns how to use by interaction with experts. For example, individuals learn how to do mathematics by solving problems and discussing their work with more knowledgeable experts —their instructors, other students, or other mathematicians via textbooks or the internet. The use of "tools" to mediate the learning activity is a critical part of this perspective on learning. In mathematics, those tools are generally considered to be the symbols and syntax of mathematical language, specific concepts and processes, and relevant technology. An important tenet of socio-cultural theories of learning is that knowledge is not considered a commodity to be acquired, but rather is something that individuals can access by using the "tools of the trade" of a particular discipline. Thus, students demonstrate their understanding by *doing*, by participating in a practice typical of the community, rather than by retrieving previously stored information from their minds out of context.

The third perspective mentioned above, social constructivism, called the emergent perspective by some, though there is a bit of debate about how different those two approaches actually are, is something of a blend of the two. Social constructivists tend to think of individuals' learning as a product of both individual constructive activity and participation in a community of practice (Cobb & Yackel, 1996). Both are necessary for meaningful learning to take place, under this perspective. Individuals can learn in isolation, but the learning will not be as rich as it would be if they were interacting with other knowledgeable people and communicating their ideas. Similarly, individuals can learn through participating in a community of practice, but there is a good deal of work that they must do actively on their own. Social constructivists assert that mathematics learning is maximized in classroom environments in which students have opportunities to actively construct their own ideas, share these with their instructor and peers, and then reflect on the feedback from others to refine their ideas. Assessment should take context into consideration, but knowledge is regarded as something that exists independently of the participatory activity.

At this point, we hope that aspects of each of the presented theories strike the readers of this book as supportive of the type of learning that occurs in Moore Method courses. In our opinion, social constructivism is particularly supportive of the Moore Method. In the remainder of this section, we will describe the specific ways in which this particular theory of learning supports typical enacted variations of the Moore Method.

D. C. Cohen (Cohen, 1982) states the basic principles of the Moore Method as follows:

- "Students understand better and remember longer what they discover themselves than what is told to them."
- "People master an idea most thoroughly when they teach it to someone else."
- "Effective writing and clear thinking are inextricably linked."

The first principle is in accordance with Bruner's (Bruner, 1961) work on the benefits of discovery learning. It is also aligned with social constructivist ideas about learning through active engagement in authentic mathematical tasks, both individually (when students solve problems or prove statements on their own) and collectively (when they listen to and evaluate the solutions of others). The second statement above reflects current perspectives on the importance of classroom discourse and social interaction in developing students' mathematical thinking (Cobb & Yackel, 1996). The last principle is supported by research demonstrating the effectiveness of writing activity in mathematics courses (Williams, 2003) and emphasizes the communicative role of mathematical proof in the discipline.

While many adaptations and variations of Moore's original instructional method have been made by Moore Method proponents over the last fifty years, their teaching has retained several of the basic tenets of Moore's method. For example, Moore Method instructors carefully construct a list of definitions, theorems, and problems based on the structure of the subject. Though some instructors base this curriculum development on their anecdotal knowledge of the ways students develop mathematically, there is no evidence that research about learning or theories of curriculum development have played a role in this process. However, the students' roles as active participants in their own learning and the instructor's role as a facilitator (guide) in the class are aligned with current reform efforts in mathematics education (NCTM, 1991; 2001).

In the preceding chapters, the four authors of this book described their approaches to teaching in ways that reflect much of what learning theory tells us ought to take place in mathematics classrooms. Several features of their philosophies of teaching and learning are strikingly consistent across this book.

All four of the authors approach the design and implementation of their courses in a *student-centered* manner. Student-centered instruction is built upon and responds to the thinking of the students, rather than the thinking of the teacher or of the larger discipline (Bransford, Brown, & Cocking, 2000). That is not to say that the structure of the discipline and the expertise of the instructor are irrelevant; in fact, these are crucial factors. Rather, the structure of the course and the pace at which it proceeds depends upon the progress of the students in the class. This emphasis on student thinking is central to the Moore Method and is supported by constructivist theories of learning.

The authors universally emphasize that in a Moore Method course, the students must be actively engaged in learning and doing substantial mathematics. The importance given to students' presentations of solutions and proofs for peer critique is an important feature of this teaching method. It reflects social constructivist perspectives on learning in that students' critique and discussion of each others' work is the vehicle by which a great deal of learning takes place in the course. In order to learn mathematics in a Moore Method course, students must work individually to solve problems, and then must share their solutions within the classroom community. The process requires both individual constructive activity and social interaction in the classroom, which is supported by social constructivist theories of learning.

Finally, each of the authors notes the importance of developing a classroom community that is friendly, supportive, relaxed, and just competitive enough to encourage excellence. Without a community in which the sharing of ideas is valued, the taking of risks is rewarded, and the making of mistakes is considered a springboard for progress, the Moore Method can fail. The instructor not only plays a vital role in guiding the development of this classroom community to promote learning, but she also serves as a mathematical mentor for the students. She demonstrates and models for the students what behavior in the class is appropriate and expected, and she guides the mathematical discussion as an expert. These roles of the community and of the instructor in a Moore Method course are supported by socio-cultural theories of learning.

Findings of Relevant Research

It may surprise proponents of the Moore Method to learn that Moore-like teaching practices are present and gaining acceptance and acclaim in the K-12 teaching community. These differ from the Moore Method in that they are generally research-based; they were developed and disseminated by mathematics education researchers through years of working with teachers and students and applying the tenets of various theories of learning in classroom settings. In the first part of this section, we will give an overview of Moore-like teaching approaches that have been demonstrated to be successful in field-based research. In the second half of this section, we will give an overview of the results of our research on the effectiveness of one instantiation of the Moore Method.

What can we learn from research on similar teaching approaches? The Moore Method can be considered an example of a type of mathematics instruction that is more generally known as inquiry-based or problem-based instruction. Such teaching approaches are characterized by the following:

1. The primary vehicle for mathematical content is a series of carefully sequenced and constructed mathematical tasks, rather than a scope and sequence of topics to be lectured upon.
2. Students work individually or in groups to solve problems without explicit instruction or significant input from the teacher.
3. The teacher facilitates small-group or whole-group discussion of the students' various solutions, encouraging them to listen to and critically evaluate each other's ideas.

4. The students are responsible for determining the correctness of solutions and the effectiveness and efficiency of strategies, rather then relying on the teacher or the textbook as the ultimate authority.

Examples of this approach to teaching are numerous and widespread. In the well-known TIMSS video study of the 1990s, Japanese mathematics teachers were shown to take this approach to teaching in far greater numbers than western teachers. Japanese students' higher scores on measures of mathematics achievement were largely credited to their schools' problem-based approach to mathematics teaching that emphasizes conceptual understanding over rote memorization of facts and procedures (Stigler & Hiebert, 1999). The videotaped teaching of researcher Deborah Ball has been studied and cited extensively as an example of the richness of children's discourse about mathematics (Ball, 1993). The work of Hans Freudenthal[3] in The Netherlands in the middle of the 20th century resulted in the teaching style known as Realistic Mathematics Education (RME), which asserts that "students should not be considered as passive recipients of ready-made mathematics, but rather that education should guide the students towards using opportunities to reinvent mathematics by doing it themselves." RME has had a large influence on the way mathematics is taught in primary school in Europe and around the world.

A technique called Cognitively Guided Instruction (CGI) has been shown to be effective in helping children develop mathematical thinking, computation, and problem-solving skills (Carpenter, Fennema, Franke, Levi, & Empson, 1999). An important feature of CGI is that students develop their own strategies for solving problems with little or no direct instruction from the teacher. Children communicate their mathematical ideas and solution strategies to each other, and the role of the teacher is to listen to and understand how the children are thinking, and then to push their thinking forward through questioning and discussion. New problems are chosen based on the students' current ways of thinking, rather than on a predetermined curriculum.

At the secondary and undergraduate levels, problem-based mathematics courses have shown to be effective in developing students' mathematical problem-solving and communication skills. Goos (Goos, 2004) reported on the development of a learning community in a secondary mathematics class in Australia. The experienced mathematics teacher in her study modeled mathematical thinking and sense-making, pushed students to clarify, validate, and elaborate on their own work, and waited to introduce mathematical conventions and symbolism when necessary. The students in his course reported that they appreciated opportunities to solve interesting problems and valued the role of explanation and proof in their mathematical work. The teacher's actions facilitated the development of a *classroom community of inquiry*, which Goos defines as one in which "students learn to speak and act mathematically by participating in mathematical discussion and solving new or unfamiliar problems."

Stephan & Rasmussen (Stephan, 2002) documented the development of students' collective understanding of differential equations concepts in a course taught in an inquiry-based style. They found that these classroom mathematical practices developed in nonsequential ways, with regard to both time and mathematical structure. The instructor played a key role in the development of these practices by facilitating discussion and argumentation

[3] *Freudenthal Institute: Research Group on Mathematics Education*, www.fi.uu.nl

in the classroom. Yackel (Yackel, 2002) also emphasizes the role of the teacher in such classrooms, noting that it is important for teachers to have deep understandings both of the content and of "how their students make sense of mathematical ideas, that is, they must be able to make judgments, based on their interactions in the classroom, about what individual children are and are not capable of making sense of." Many researchers in mathematics education (Wood, 1999) and science education (Newton, 1999) have described the critical role of the teacher in facilitating discussion and argumentation. Not only is it important to ask the right questions, but teachers must also "back away strategically when communication and reasoning flourish, [to allow] students to play more active roles in their own and each other's learning" (Martino & Maher, 1999).

Overview of Our Research Findings

For the last few years, we have followed the experiences of students and instructors in undergraduate number-theory courses at a large southern state university. Several sections of this course are taught every semester; at least two are typically taught using an instantiation of the Moore Method. (The other sections are usually taught in a traditional lectured-based style.) In Smith (Smith, 2006), we reported that this instructional approach appears to encourage students to *make sense* of the mathematics when attempting to construct proofs. This study examined the differences in the approach to proof by students enrolled in Moore Method and lecture-based sections of a number-theory course. The most striking differences were seen in the strategies the students employed when presented with a statement to prove and in their approaches to validating proofs of others. In general, the Moore Method students demonstrated a tendency to make sense of the mathematical ideas, while the traditional students were more concerned with finding quick and easy ways to complete tasks. This suggests that the structure of the Moore Method course can provide students with an opportunity to learn mathematics in a meaningful way. In this section, we present an overview of our findings about the role of the instructor and the impact of this teaching approach on students.

The Role and Experience of the Teacher in a Moore Method course

The teacher plays a pivotal role in the development of the classroom learning community. This role can be broken into two general categories of responsibility: the teacher facilitates and directs the mathematical activity of the class, and the teacher guides the development of the culture of the learning community (Bowers, Cobb & McClain, 1999; Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Oliver, and Human, 1997). Research on the development of classroom mathematical practices and socio-mathematical norms, and the literature on the emergent perspective and socio-cultural perspectives on classrooms as communities of practice also informed this study (see for example: Bowers, Cobb, & McCain, 1999; Cobb & Yackel, 1996; Forman, 2003). In this section, we will describe results of studies focusing on novice and expert instructors using the Moore Method.

A small-scale study followed the experience of a mathematician who was attempting to implement the Moore Method in a number-theory course for the first time (Smith, Yoo, &

Marshall; in preparation). In Brown & Borko (Brown, 1992), it was suggested that teachers pass through several stages of concern when attempting to change from a traditional to a more student-centered teaching style: (1) concern for *self*, (2) concern for *task*, and (3) concern for *students*. This change is not easy, and can take as long as five years to occur (Luft, 1999; Hall & Hord, 1987). The instructor's strategies and approaches to the course changed over the semester as she moved from concern for herself as a teacher towards concern for the learning of the students. We examined how her strategies changed over time and suggest that the framework presented in Brown and Borko may be an effective lens for examining the development of post-secondary mathematics instructors, who frequently have little or no pedagogical training. These results and this framework also imply that novice instructors would benefit from different types of support at various points in their first few semesters of teaching with the Moore Method. Support could be designed to help them make the transition from concern about their own role to concern about student learning over time. Without such support or recognition of their concerns, many teachers reject new methods and return to the methods by which they were taught, which tend to be teacher-centered and lecture-based.

A video-based study of an experienced Moore Method instructor teaching a number-theory course revealed that while the instructor played an active role in class discussions throughout the semester, the nature of his comments, questions, and interventions changed over time, according to the needs of the community (Smith, Nichols, Yoo & Oehler, under review). Overall, his actions facilitated the development of social and socio-mathematical norms that may have enabled the students to develop a conception of proof that is quite mature for undergraduates. He made careful choices in directing class discussions, often using his physical position to direct attention to and away from a student presenter. Through his instruction he encouraged a view of mathematics as a human, social activity. The students in the course appeared to change their conceptions of the nature of mathematics and of proof as a result.

From the very first day of class, the instructor did not assume the traditional role of a lecturer at the front of the room. At times, he was the focus of the students' attention and clearly the authority figure in the class, but at other times he was in the background, letting the students direct the flow of the discussion. He appeared to purposefully relinquish and assert his role as leader and mathematical authority during class sessions, both with the whole group and with individual presenters. These shifts in leadership occurred via his position in the classroom, his voice (tone, volume, and silence), and his physical gestures.

These changes in leadership occurred primarily when the instructor was engaged in one of three activities:

1. *Motivating participation* in the developing classroom community by organizing the presentations of proofs and modeling the types of questioning and commentary necessary for understanding.
2. *Facilitating whole-group discussions* of presented proofs by asking students to comment on and discuss a presented proof, as well as directing students to refine their questions and comments.
3. Discussing or questioning students' *strategies for proof*. For example, the instructor would frequently comment on the initial strategies used to begin constructing a

proof, the use of notation, and the use of prior knowledge and experiences. He would emphasize the importance of concrete examples for gaining insight into a statement to be proved, and would occasionally give advice, tips, or comments about general strategies for constructing proofs.

Before we began our analysis, we expected to find that the instructor asserted his leadership most frequently in the beginning of the semester, with incidences tapering off as the students gained more confidence and understanding of their role in the class. However, we found that there were fewer incidents of the instructor asserting his leadership at the beginning and end of the semester than there were in the middle, as can be seen in Figure 1 below.

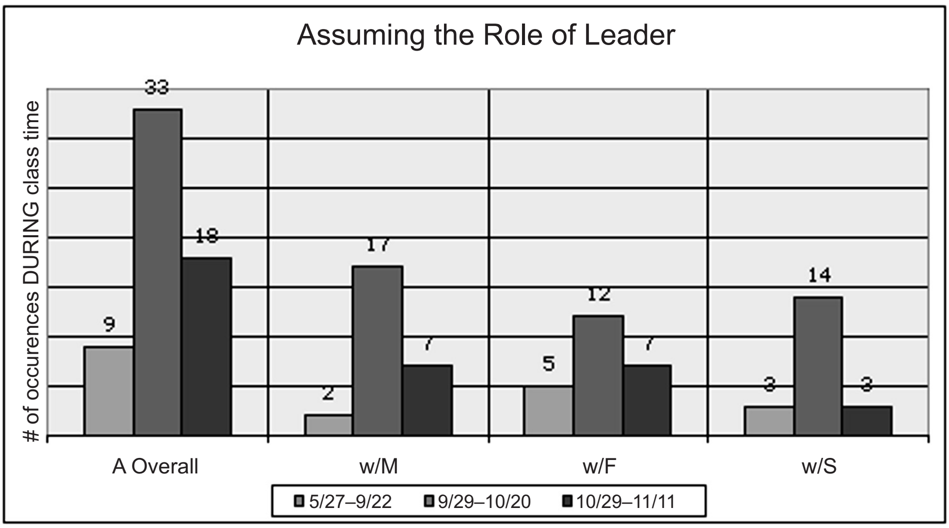

Figure 1

This trend occurred not only in the overall instances of the instructor asserting himself as leader, but was also present when we separated the events into the three categories described above. We kept track of instances where the instructor *reclaimed* leadership of the class discussion after having relinquished it; thus an increase in instances indicates that leadership was being asserted and relinquished more frequently than at other times in the semester. The instructor's choices for when to speak, when to remain silent, and what to say influenced the way students participated in the course. At the beginning of the semester, the instructor's comments were primarily intended to facilitate discussion. He rarely commented on the mathematics being presented; rather, he asked questions that encouraged the students to examine the presented proofs and express their opinions about them. As the semester proceeded, the course material became more difficult, and the instructor began to interrupt discussions and assume leadership of the class more frequently. His comments during this time were more focused on the mathematics being presented and pushed the students to examine and reflect on strategies for constructing proofs. Toward the end of the semester, the students were comfortable with the level of mathematics and with participating in the class, and the frequency of teacher interruptions of discourse was reduced.

By the end of the semester, these undergraduate students appeared not only to be comfortable discussing and presenting their mathematical ideas, but had also developed an appreciation for mathematical proof that was starkly different from what is reported in the literature. We refer to this collective understanding of mathematical proof as a *community concept of proof*. Based on the analysis of our video data, we suggest that the students in this course regarded the following as important features of proof:

- A proof should be written in a clear and concise manner.
- A proof should make sense; it should be understandable by one's intended audience.
- A proof should state what the prover is assuming.
- A "good" proof is one that explains *why* something is true.
- It is up to the prover and her audience jointly to determine if a proof is "good", and therefore, acceptable. (What convinces one person may not convince another, so a consensus must be reached.)

These features became increasingly apparent over the semester in the students' and the instructor's comments and actions. Some were mentioned explicitly at different times, but by the end of the course, all seemed to have become taken-as-shared by the learning community.

It is our belief that the structure of the course and the experienced instructor's careful choices to assume and relinquish leadership of classroom discourse facilitated the development of this unusual perspective of proof. Such classroom environments remain rare at the university level, but research is beginning to demonstrate that they can be effective in helping students develop "mathematically mature" modes of thinking and participating in mathematics courses.

Students' Approaches to Proof after Moore Method Instruction

In (Smith, 2006) we presented the results of a small-scale comparison of the approaches to proof of students enrolled in Moore Method and traditional lecture-based introduction to proof courses. We conducted a series of interviews with a small number of students enrolled in each type of section; in these interviews the students were given several propositions to prove. Analysis of these interviews revealed four primary differences in the ways the students from the Moore Method and traditional sections approached these tasks; these differences are summarized in Table 1 below. We were not particularly interested in what the participants were or were not able to do; rather, we were interested in the strategies each used during the proof construction process.

Despite the small size of the sample, we found marked differences between the students in the Moore Method sections and the students in the traditional section in their approaches to the construction of proofs. As detailed above in Table 1, the differences in approach to proof fell into four categories: use of initial strategies, use of notation, use of prior knowledge and experiences, and use of concrete examples. Taken together, these four attributes can be considered examples of ways in which students attempt to make mathematics per-

Table 1

	Traditional	Moore Method
Use of initial strategies	Began searching for proof techniques	Tried to make sense of the statement
Use of notation	Introduced notation appropriate to proof technique chosen	Introduced notation naturally, in context of making meaning
Use of prior knowledge & experiences	Related to other proof strategies based on surface features	Related to other proof strategies based on the concept
Use of concrete examples	Reluctant to work concrete examples (not a proof, so not helpful)	Worked concrete examples to gain insight into main idea

sonally meaningful. The students in the Moore Method sections were focused on *making sense* when constructing proofs, while the traditional students seemed to be searching for a solution that would be recognized as valid by an external authority.

While the Moore Method students appeared to want to understand the proofs they were constructing and wanted to write them in such a way as to make them clear and meaningful, the traditional students were more concerned about correctly using the appropriate logical structure, such as induction or contradiction. The Moore Method students wrote their proofs conversationally, in a style similar to how they spoke and thought. The traditional students, on the other hand, tended to focus much more on the "proper" form and order of their proofs. Their comments emphasized the process of learning how to "get through" the proof and discovering what one was "supposed to do" in that situation.

This emphasis on making sense of mathematics is striking for several reasons. Much of the research on students' conceptions of proof reports that they hold naïve perspectives on the nature and role of proof in mathematics (Harel & Sowder, 1998; Knuth, 2002). The Moore Method students, however, viewed proof as a means of making sense of mathematics, as a tool for building understanding, and as a way of communicating results to others (Hanna, 1991; Hersh, 1993; Tall, 1992). In some ways, the Moore Method students' approach to proof is reminiscent of Weber & Alcock's (Weber, 2004) notion of a *semantic proof production*: "a proof in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws." The Moore Method students' use of initial strategies, notation, prior experiences, and examples could be considered as such instantiations of mathematical concepts, meaningful ways of thinking about mathematical objects.

Conclusion

A point that should be stressed here is that we do not wish to imply that the results of our studies prove that Moore Method students' understandings of proof or of mathematics are necessarily better or more sophisticated than those of the students in a lecture-based section. Rather, we are interested in the qualitative differences between the approaches to proof exhibited by the students. Though the participants had varied mathematical interests and backgrounds, the differences between the students in the Moore Method sections and the students in the traditional section were marked. Why might the Moore Method course have produced students who approached proof so differently? We are currently investigating this question, but our hypothesis is that the difference is a result of participation in a community of inquiry in which learning was based on solving problems and discussing solutions.

Like other types of problem-based and inquiry-based mathematics instruction, the Moore Method places great importance on individual ideas, discussion, and consensus-building, and these are the main vehicle for the introduction of content. Unlike more traditional teaching approaches, students learn mathematical concepts and processes without being taught specific strategies in advance. Students in Moore Method courses must consider each problem and theorem statement individually and decide for themselves how to best go about solving or proving it. In addition, presenting mathematics to peers and evaluating peers' work is a major course activity in a Moore Method class; it becomes the responsibility of each individual in the group to determine the validity of each solution or proof presented by peers. Sense-making is inherently an important aspect of the course and a requirement for participation.

“The wise man doesn’t give the right answers,
he poses the right questions.”
— Claude Levi-Strauss

10 Frequently Asked Questions

Over the years, each of us has fielded questions related to the Moore Method from audiences during presentations, from colleagues at conferences, and from friends and relatives. We’ve compiled here representative questions, along with responses, alphabetized by author.

1. Does the Moore Method work only for the bright students?
2. Do Moore Method instructors lecture?
3. Does the Moore Method cover less material?
4. Does the Moore Method work best in upper-level and graduate courses?
5. Does the Moore Method make the students do the work so the teacher doesn’t?
6. Does the Moore Method work with cooperative learning?
7. Is there a list of features that define the method?
8. Does the Moore Method foster competition among students?
9. Are there better ways for students to present than writing on the board?
10. Does the Moore Method fail to equip those trained via the method, as students and later as professional mathematicians, with the ability to extract information from textbooks?
11. Are there professional risks associated with teaching a course by the Moore Method?
12. Are the goals for a Moore Method course the same as for a non-Moore Method course?
13. How does one prepare to teach a Moore Method course?
14. Are there lower or upper bounds on the size of a Moore -method class?

1. Does the Moore Method work only for the bright students?

(Coppin) Absolutely not! I have successfully used the Moore Method in classes designed for liberal arts students many of whom are just average mathematics students, math-phobic and even some with 700 SATs (total quantitative and verbal score) although a modified

Moore Method generally works best. If you think about it, the most important things we learn such as speaking and walking are done in self-discovery fashion. Why should mathematics be any different?

(Mahavier) No. On page 217 of John Parker's biography of Moore (J. Parker, 2005), the first-rate mathematician R. D. Anderson is quoted as saying: "his reward system clearly encouraged students, both competitive students and some of the weaker ones, to have confidence they could do things. A large part of his success was in building up the confidence of the individual, and not in the strictly competitive nature of it." Clearly, Anderson felt that the method had the ability to build the confidence of weaker students.

I have seen the method draw out weaker students and have seen them rise beyond what I initially believed they had within them. In the paragraph titled "Time" starting on page 23 of Chapter 3, "What is the Moore Method?", I address just how time for discovery can have a positive impact on weaker students. Techniques for dealing with "weaker" students are addressed on pages 58, 71, and 78 as well.

(May) Absolutely not. It worked for me. I have never considered myself to possess the mathematical brilliance that many mathematicians seem to have. More importantly, at the time when I was first exposed to the Moore Method (in graduate school), I never felt as bright as any of my professors or peers.

(Parker) Several years ago, during a summer in which the course for the U. S. Open Golf Tournament had been prepared in a particularly nasty way, one of the players, in an interview with the press, indicated that he thought that the course had been set up that way in order "to embarrass the best players in the world." An official with the USGA responded that it had not been done "…to embarrass the best players in the world, but rather to find out who they are." Moore Method *does* work for the bright students, and you will be amazed at what they do when they are not chained to the way a textbook writer thinks about a theorem.

But the key word in this question is probably "only." Apparently, there are those who believe that the answer is yes. Consider the following excerpt from the referee's report on my paper, "Getting More from Moore":

> *…two very different animals, as are motivated upper-level mathematics students and the mathematics-injured… such students do need their teachers to be more*[1] *(sic) than a source of problems/questions… . Based both on personal experience and the author's failure to convince me that …Moore's ideas can effectively be implemented in …courses populated by students who have been trained to "learn" in ways that are diametrically opposed to the Moore Method …I suggest that … portion of the article … be … eliminated.*

I may have felt the same way at one time (clearly, by the time I wrote the paper my attitude had changed!), but my attitude changed when I actually tried Moore Method with the supposed "mathematics-injured." In the summer of 1988, I decided, just as a lark, to try Moore Method in "The Nature of Mathematics," the lowest numbered course in the mathematics curriculum at James Madison. I rationalized the experiment in the following way:

[1] If this had been a discussion rather than a one-way directory, I would likely have rejoined at this point, "Being a source of problems/questions allows us to be more."

These students are in summer school. All they are here for is the credit, so everyone will get a passing grade. That way, if the experiment goes bad, the students will at least get what they want, in a pragmatic sense, from the course.

But I didn't tell the students that. I introduced them to axiomatics in the first unit through number algebra and then had them work on a model for the complex numbers in the Cartesian plane. They proved that all of the field axioms, stated in terms of the operations on ordered pairs they had been given, were theorems! The problem had not been that the students could not learn that way, but rather that I hadn't believed that they could learn that way.[2] I have, since 1988, developed additional problem sets on composition and finite geometries and a way to simulate Moore Method in the making of models, and the students in that course continue to make mathematics. Furthermore, the experience with this "lowest" echelon of mathematics students has encouraged me to make Moore pedagogy a bigger part of my instructional strategies in courses that I formerly taught through guided discovery.

The answer to the question may still be yes, but the qualification becomes "yes, because we will find out that, in the majority, the students *are* bright, something that would likely have remained hidden had we insisted that they imitate our thinking." The key to unleashing the brilliance is likely closely tied to two questions treated in the body of the text: finding a way to give them the tools necessary to work within mathematics and pitching problems at a level that contains some reasonable chance of success.[3]

2. Do Moore Method instructors lecture?

(Coppin) Today, most teachers who use the Moore Method do teach portions of their courses by a lecture or lecture-discussion method. A dividend of successfully teaching a course by the Moore Method is that doing so will inform one's lectures. The teaching and the lecturing will be enriched because of experiences with the Moore Method.

Moore did not lecture in the course I took from him. In one year under him, I saw him give one proof, which was delivered with such speed and alacrity that I could not follow it. He did not even repeat the proof! In its purest form, the teacher of a Moore Method course does not lecture per se. For this approach to be successful today, the students must be both emotionally and academically mature, which, in most cases, sadly, is not true. The reality in which most of us reside requires us to lecture to some extent; thus teaching a modified form of the Moore Method. Clearly, in this case, when a lecture is delivered, the teacher must be fully aware of how much information is being delivered so as not cheat the students of real learning opportunities.

(Mahavier) Yes. On days when students have nothing to present, but when I am convinced they have been working hard, I will lecture. One type of lecture might address upcoming material with examples and questions to stimulate discussion and understanding. A second type might be the presentation of a theorem or problem from the notes which no student is working on; I query them to be sure. A third type might be a presentation on a topic related to the subject, but not included in the notes. A fourth type might tie together topics that students have been working on to give them a fuller understanding of how the

[2] Note parallel to Mahavier's comments on "Attitude" in Chapter 3—What is the Moore Method?

[3] This second issue is at the heart of Coppin's discussion of "zones of proximal development."

individual problems and theorems that they have worked fit into the broader mathematical spectrum. This is one of the more ironic aspects of preparing for a Moore Method class on a daily basis. Each day, I prepare not only by assessing each student's mathematical progress and thinking how to best promote his or her mathematical development during class, but each day I prepare a lecture worthy of the course and students. Most of the time, the latter preparation is wasted, as students pull rank and present! So much for the FAQ, "Does it takes less time to prepare for a Moore Method course?"

(May) I do, especially in freshman- and sophomore-level courses. I try, for the most part, to restrict my lecturing to a maximum of twenty minutes per period; and, for example, in a calculus course, many periods will be completely taken up with students putting solutions on the board and the class discussing them. There are times, however – when we come to related rates, the (definite) integral, or volumes of solids of revolution (which I treat as problems in differential equations), for example – when I will give the students an old-fashioned, traditional lecture. Doing so, I believe, helps them to appreciate even more the class-periods when I don't say much.

(Parker) At the 2007 Legacy of R. L. Moore Conference, one of the veterans of Stan Yoshinobu's summer of 2006 seminar on using Moore Method, during a panel discussion in which he reflected on his initial use of Moore Method, reported that he had lectured the last two weeks of the course to guarantee that his students "saw" all of the mathematics on the mandated curriculum guide. So I know that, at least in one instance, the answer is yes. For my own part, I suppose the recaps I do at the conclusions of presentations[4] could be considered, at some level, to be lectures since I give insights that the students may not have made themselves. On the other hand, these "lectures" are given consequent to what the students have done rather than to guide the students to what to do. In courses in which I do not use the Moore Method, Moore Method principles greatly influence the "presentation" style. The "lecture" consists of posing questions that I perceive that the students can answer on the spot if they do, indeed, have command of what has gone on before, that lead to the mathematics I want them to get. If a "lecture" goes as planned, then it ends at a place where, if the students can answer one more question (the first question on the homework), then what has been done together with that answer makes the rest of the questions on the homework "routine." The active dynamic becomes, when they cannot (or do not) do what I expected them to be able to, finding a point of departure from something they can do that allows them to get back to what I had expected them to be able to do. Many of my colleagues do not consider this lecturing.

3. Does the Moore Method cover less material?

(Coppin) This question really depends on what one means by less "material." If one equates material with information, then most Moore Method classes probably cover less material. However, in this case, it is the teacher who covers the material not the students. One caveat I would give is this. If the course is cognitively light then it is possible to cover more material in the real sense of the word. Now, if by "material" we mean both information and process, a Moore Method class most likely will cover more material.

[4] See Chapter 6—In the Classroom.

(Mahavier) Perhaps this debate should focus on the here-to-fore undefined term, "cover." Suppose we have a course that is to cover a block of material of ten chapters which we will refer to as one hundred percent of the material. In the Moore Method, by cutting a path through the material, *students will produce* material from each chapter, perhaps covering seventy-five percent of the material, but including material from every chapter. In a lecture class, the *instructor will lecture* over one hundred percent of the material. If the Moore class has a higher internalization rate on the part of the students, with an average internalization rate of seventy-five percent of what is covered, then that class has covered more than a lecture class where students have an internalization rate of fifty percent. That is, fifty percent of one hundred percent is less than seventy-five percent of seventy-five percent. If there are several Moore-style courses in a curriculum, then coverage ceases to be an issue – the students are accustomed to the method, they hit the ground running, and courses following the first such course cover as much material as a lecture course, with greater internalization of that material.[5]

I also refer the reader to the quote of E. E. Moise appearing in my essay in "Why Use the Moore Method?" In this quote, Moise makes the case that the productivity of the products of the Moore Method indicates that knowledge acquired through this method of active learning seems to have a power "all out of proportion to its quantity." Hence, even if the Moore Method does cover less material, the benefit to the student is greater.

(May) David M. Clark (Clark, 2001) addresses just this issue, asserting that the Moore Method may well cover slightly less material in the first semester of a two-semester course, but will cover considerably more in the second semester. If students have only one Moore Method course in their career, they will likely cover less than the material covered in a lecture class. I agree wholeheartedly with the conclusions of David Clark's article. In my freshman- and sophomore-level calculus courses, for example, my students always cover as much material as those in more-traditionally-taught sections. I cannot offer much evidence from my upper-level experiences, such as real analysis, for I have spent my career in a department where I have had little opportunity to teach two-semester sequences at the upper level.

(Parker) Granted that, when used by a colleague, "covered" usually means "exposed," *I* certainly don't cover nearly as much in a Moore Method course as *I* cover in a non-Moore Method course. What's more, even if the students are exposed to just as much mathematics in a Moore Method course, it is almost certain not to be as tightly organized or to meet a preordained outline as closely as a course given in a non-Moore Method format. But the students that succeed in a Moore Method course come away with enhanced learning power. They are not without options when confronted with a question to which they do not know the answer; they have developed the power to investigate. If they know what they need to know, they can go to books and learn it without having to have someone explain it to them. The proper gauge of successful Moore Method students is not so much the things they can discuss, but what they can do. There may not be "as much covered" at the end of the course, but the capacity to add "coverage" on their own when it is needed is almost certainly enhanced.

[5] See also my discussion on Boaler's comparison of direct instruction to open classrooms in Chapter 8—Why Use the Moore Method.

4. Does the Moore Method work best in upper-level and graduate courses?

(Coppin) I would say that students appreciate the Moore Method at all levels. Because graduate students on average are more mature emotionally and academically than freshmen, a Moore Method course taught at the graduate level is going to be different than one taught at a lower level. One situation is not better than the other.

(Mahavier) I believe that the method works equally well across the spectrum. The following is a list of courses that, with the exception of the first one, I have taught for credit on a university campus. Mathworks Summer Camps (ages 10–13), Remedial Mathematics, Introductory Martial Arts, College Algebra, Discrete Mathematics, Trigonometry, Business Calculus, Calculus I, Calculus II, Calculus III, Differential Equations, Differential Equations II, Topology, Analysis I, Analysis II, Numerical Methods, graduate Numerical Methods I, graduate Numerical Methods II, and graduate Functional Analysis.

(May) The Moore Method works beautifully at any level. Properly executed, it works no less well for freshmen and sophomores—or, for that matter, kindergartners—than it does for juniors, seniors, or granduate students.

(Parker) I'm not quite sure how to take "best" here. Certainly it works in upper-level and graduate courses, and it is less likely to be as difficult to implement in these contexts because finding where the students can begin is a less daunting task, and there is a higher likelihood that such students have some sense of their responsibility as students. In graduate school, admissions requirements typically guarantee the plausibility of some minimal level of mathematical education. For upper-level baccalaureate students, the curricular structures of courses that are prerequisite identify some reasonable expectations of what the students may have seen. Both sets of students are there because they have chosen to study mathematics. But the magnitude of the effect of the "gee whiz, look what I just did" is independent of sophistication. A college algebra student justifying what had previously been sheer procedure has the same motivating effect as a senior beating his or her way to the intermediate value theorem. Realization that "this is true because here is an argument" rather than "this must be true because the authority says so; now I will try to follow the argument" is magnificent whenever, and at whatever level, it happens.

5. Does the Moore Method make the students do the work so the teacher doesn't?

(Coppin) Of course not! You've got to be kidding! I find Moore Method courses much harder to teach than lecture courses. When I am fatigued or ill, I will tend to lecture that day. It takes less energy and concentration. I only have to worry about my delivery and how well I am connecting with the students. In contrast, when I teach a Moore Method class:

1. I must prepare or modify my notes,
2. I must fine tune my lessons for the particular students I have in front of me,
3. I must teach the students via a student at the board,
4. I must be aware of the presenter's use of language, their emotions, and the particular slant he or she will have on the proof or problem at hand, and
5. I must understand all the different ways a concept can be approached.

This is a lot of things to keep on your mind! It is not easy!

(Mahavier) If each of us were the teacher that Halmos was, we would implement the method beautifully the first time and adopt it for all our courses from then on. But as G. H. Hardy wrote, "the harder I work, the luckier I get." Halmos conceded, and I agree, that it is a nontrivial task when he wrote (Halmos, 1994),

> *If you are a teacher and a possible convert to the Moore Method, don't make the mistake that my students made: don't think that you, the teacher, will do less work that way. It takes me a couple of months of hard work to prepare for a Moore course, to prepare the fifty theorems, or whatever takes their place. I have to chop the material into bite-sized pieces, I have to arrange it so that it becomes accessible, and I must visualize the course as a whole—what can I hope that they will have learned when it's over? As the course goes along, I must keep preparing for each meeting: to stay on top of what goes on in class, I myself must be able to prove everything. In class I must stay on my toes every second. I must not only be the moderator of what can easily turn into an unruly debate, but I must understand what is being presented, and when something fishy goes on I must interrupt with a firm, but gentle, "Would you explain that please?—I don't understand."*

The method is more work, but seeing my students' successes makes it worth the extra effort.

(May) To answer this question, try the Moore Method, just once. Even if the experience doesn't convert you to being a Moore Method practitioner, it will provide you with an answer.

(Parker) I can imagine ADA McCoy, from Law and Order, answering this question: "The facts, your honor, are in evidence; the second question goes to motive." For sure, Moore Method provides a context in which the students are asked to do the work and, those who succeed do it. But the motive is hardly to keep the teacher from working. If the work that the teacher supposedly doesn't have to do is the mathematics in the course, all that the teacher is relieved of is the obligation to present the work to the class. Dissecting an argument, as described by May in the chapter on material development, so that the ideas in it can be ranked relative to accessibility, may well be more work intensive than just understanding an argument from a book or one's education well enough to present it. Being willing to temporarily abandon one's own pet way of thinking about a theorem in order to understand what a student might be thinking or, retrospectively, to try to understand what may have suggested that student's argument may well be more work intensive than trying to find ways to lead students to understand the steps in one's own argument. Making decisions about which theorems form a core from which the students can think about the mathematics in a course's curriculum may be more work intensive than mastering the contents of a textbook preparatory to leading the students through it.

My own experience[6] indicates that even if there isn't any greater time commitment, there is a tremendous difference in the intensity of the work. In guided discovery, I lead a class where I want it to go and stay the course unless a student interjects an idea that is just too beautiful to ignore. In a Moore Method course, preparation often doesn't fit reality,

[6] Admittedly, my only real frames of reference are my pseudo-lecture courses that are probably better described as guided discovery.

thus class time is dynamic as the instructor, ostensibly as an observer, constantly reacts and decides, reacts and decides,…, usually keeping quiet all the while. The reactions are to what the students do; the decisions are whether to keep silent and educate through the continuation that will come, or to become a part of the participating audience and educate in the moment—but with questions rather than answers. These are not easy decisions and they must be made in the moment. Continuation consists of thinking back on the class in order to revise the long-term plans and then incorporating the dynamics in the short-term planning. Also, in subsequent offerings of a course, while lectures are likely reasonably static, the Moore classroom, even with the same problem sequence, may head off in different directions. Thus, a second offering may resemble the previous one only in that the same theorems get proven. We are better equipped by experience the second time, but the day-to-day uncertainty and the need to react to, rather than dictate, what happens still necessitates active planning.

Perhaps the Moore Method teacher will spend less time (I doubt it), but the time spent is intense.

To understand why the proposed motive in the question is bogus, one need only experience the passion with which the chapter titled “Why Use the Moore Method?” is written. From that chapter, it is clear that the motive, at least in the minds of the authors, for teaching this way is to enable the students in their development as thinkers, not to avoid work. If one were to find a way to utilize the method *and* avoid work, so much the better. But what will probably happen is that the work will not seem like work. The vicarious thrill of experiencing students’ development that is clearly their own is a delicious high.

6. Does the Moore Method work with cooperative learning?

(Coppin) No, it does not! I have attempted to do both. It does not work. It is not that cooperative learning is bad; much of it is very good. It is just not the Moore Method; it is something else. The Moore Method puts the onus on the individual efforts of each student. Cooperative learning is 180 degrees in the other direction. I have attempted to mix the Moore Method and Cooperative Learning. The students who have adapted to the Moore Method do not mix well in groups.

(Mahavier) There is an aspect of cooperative learning to the method in the sense that, as illustrated by May in the chapter on culture, the entire class is inquiring after the subject as a group. At the same time there will be individual competition when one student desires to obtain a solution before his or her peers. This notion of full-class inquiry is particularly well-described in Chapter 4 of *Teaching with Your Mouth Shut* (Finkel, 1999) where Finkel describes the process of inquiring after the works of Socrates as a class. While this is not a text on mathematical pedagogy, the work parallels the Moore Method in many senses, treating the teacher as a coach, treating the class as a collaborative investigation into a body of material, and of course, promoting teaching outside of the tradition of the “sage on the stage.”

“Cooperative learning” is typically defined by having students work in small groups to achieve some goal. Two sources of information on using the Moore Method with cooperative learning have been written by Jerome Dancis and Neil Davidson.[7] Both are students

[7] Note that in (Parker, 2005, p. 215), John Parker puts a footnote that small-group discovery was in fact a response to the Moore Method and was “intentionally developed as a variant of the Moore Method.”

of R. H. Bing, one of Moore's most productive students. In (Dancis & Davidson, 1970) the two authors discuss the Moore Method and small group discovery. In (Davidson, 1971) the author discusses small group discovery as used to teach calculus.

I too have merged the two learning theories in the classroom and what resulted was an enjoyable class from which the students benefited. Still, I would not have called it a Moore Method class, as the result was more cooperative, project-based, and problem-based, than it was developing individuals' confidence and mathematical ability. I do not believe that the resulting class was as hard to teach as a pure Moore Method class, nor do I feel that it offered the benefit to the students of a pure Moore Method class. It does however allow for a Moore-style experience in larger classes.

(May) As I employ it, the Moore Method encourages co-operative learning inside the classroom and independent learning outside it.

(Parker) In class, Moore Method *is* co-operative learning. As a "group of the whole," the class practices things that are typically identified with co-operative learning. Students (the presenters) come up with ideas germane to what is being studied and the co-operative group questions until it is satisfied that it commands the idea. The obligation of presentation is a group's particular responsibility in that the expectation is that no one person is, on an ongoing basis, responsible for introducing ideas; rather the individuals in the group assume that responsibility as they are able. But outside of class, students are expected to work independently of each other. Nevertheless, the sharing of ideas that, to me, seems the lifeblood of co-operative learning, is done in class and all students have access to what is shown and take it with them into their own individual quests for answers.

I have used a more standard co-operative model in my "Math for People Who Don't Want to Take Math" course. I modify Moore Method in the following way. The students are given problems that they are asked to solve without my showing them how. But instead of every problem being available for every student, I form, by a random process, groups of three or four students and each group is assigned, again by a random process, a problem that is the group's responsibility. If a group cannot get its problem by its presentation date, and there is sufficient time, I will often open the problem to the other groups. This version of co-operative learning coupled with the Moore attitude of "you can find answers" has worked well in at least one nonmathematical setting. In the interdisciplinary freshman seminar that was once a basic part of the James Madison core curriculum, I used small groups to introduce the works under consideration to the seminar as a whole. Then each group assumed the responsibility of being the "expert" on what the content of the work they introduced was as the seminar grappled with the synthesis of the collective works under consideration.

7. Is there a list of features that define the method?

(Coppin) Normally, Moore Method courses have a set of rules that are made explicit at the beginning of the course. Mine are listed below.

1. Any result you want to present for credit must be your own work. Do not get help from anyone except your instructor. Do not go to any book for help.
2. You cannot receive credit for any result you have seen.

3. When someone is at the board do not give help or hints. You may ask questions but not in a way as to offer direct help to the one at the board. A student's proof must be his or her own.
4. You may step out of class from time to time under conditions, which I will state in class.

These rules are a paraphrase of those given by Moore at the beginning of his courses. The essence of the method may be something else but for a course to be classified as a Moore Method class, I believe rules comparable to the above would be followed. Rule 1 is the most important one.

(Mahavier) The features of the method that I list in the chapter "What is the Moore Method?" are: a set of rules governing the class, appropriate materials, time for discovery, and teacher attitude.

Lee Mahavier (Mahavier, L., 1999) elaborates on caring about the students, respect for learning, and enthusiasm and histrionics.

Creative Teaching by Reginald Traylor (Traylor, 1972) has the following list.

Criteria which characterize the Moore Method of teaching include:

1. *The fundamental purpose: that of causing a student to develop his power at rational thought.*
2. *Collecting the students in classes with common mathematical knowledge, striking from membership of a class any student whose knowledge is too advanced over others in the class.*
3. *Causing students to perform research at their level by confronting the class with impartially posed questions and conjectures which are at the limits of their capability.*
4. *Allowing no collective effort on the part of the students inside or outside of class, and allowing the use of no source material.*
5. *Calling on students for presentation of their efforts at settling questions raised, allowing a feeling of 'ownership' of a theorem to develop.*
6. *Fostering competition between students over the settling of questions raised.*
7. *Developing skills of critical analysis among the class by burdening students therein with the assignment of 'refereeing' an argument presented.*
8. *Pacing the class to best develop the talent among its membership.*
9. *Burdening the instructor with the obligation to not assist, yet respond to incorrect statements, or discussions arising from incorrect statements, with immediate examples or logically sound propositions to make clear the objection or understanding.*

In his biography on Moore, John Parker (Parker, 2005) lists features developed by Moore early in his career.

A number of key points which were to become a permanent feature of Moore's pedagogical style, and in turn the Moore Method, emerged in his point set theory course at Pennsylvania:

1. *There would be no textbooks linked to the course and none was to be consulted. Moore quickly developed his personal 'radar' for spotting those who had accidentally or otherwise become exposed to the work at hand.*

2. *Students were asked to prove theorems from given axioms and present their proofs in class without seeking help externally or discussing the problem with each other.*
3. *The rest of the class would then be encouraged to criticize weaknesses or inaccuracies in the presented proofs.*
4. *He encouraged a strong spirit of competitiveness among his students and devised various means to promote it.*

From the outset he placed a good deal of emphasis on logic.

The following excerpt from Joe Eyles' dissertation interview with John W. Neuberger (Eyles, 1998) sums up the features of the method as succinctly as possible. Neuberger simultaneously addresses the *culture in the classroom* discussed in an earlier chapter and defines the method in the following excerpt. Note that he implicitly reiterates the key ingredients of *time, teacher attitude, and materials.* He makes clear that there is material to work on varying widely in difficulty, that there is ample time to work on problems and discover solutions, and that there is no pressure to discover something by a certain date except the natural self-induced pressure that occurs as a result of the desire to solve a problem prior to a classmate.

> *As in his* [Moore's] *later courses and Wall's too, we were never lectured to. Neither of these people ever lectured to us. We were always given problems to work on and sometimes a large number of problems and then after a sufficient length of time we were questioned 'does anyone have this?' Often students were called on by name. 'Do you have it, Miss So and So? Mr. So and So, do you have this problem?' If you said yes you were given an opportunity to present an answer. If you said no you were simply passed over and asked again another day. So, we always had a great store of problems. Some of them were fairly easy and many were very difficult. Well, we knew, of course, that we would be called upon. It wasn't like law school where you're asked to recite. You could simply say no if you didn't have it and there were a lot of no's there, and there were a lot of mistakes when people presented. Perhaps, the most important part is that we worked very hard to prepare for these courses. There was at once some pressure, but in modern terms, it was a very supportive course. There wasn't a lot of fear there, it was mainly motivated by desire to get these problems. Above all, we worked hard. Now, it could be that almost any system that pulled that much work out of students would show a considerable amount of success. That makes it difficult to examine the benefits of the particular philosophy, because people just simply worked so hard and generally it was just completely joyful. The thought of cramming for a test or a final was just totally unknown. The finals were completely relaxed. People had done their work and they went in with no trepidation at all.*

(May) My version of the Moore Method is characterized by the following features:

1. A belief that it is my job to (re-)awaken in the students the ability to learn mathematics (or anything else that I am trying to teach), for the most part, on their own, and the recognition that it is their responsibility (and joy) to do so.
2. A classroom in which lecture is de-emphasized to the point of nonexistence and presentations and discussions by the students are the focus.
3. An environment, in my office as well as in class, that is as relaxed as possible.

Michael Henle (Henle, 1997) says, “It is best to read mathematics at a leisurely pace.” I could not agree more.

(Parker) As Mahavier has noted, this issue is discussed in depth in a full chapter. To my mind, the distinguishing feature of Moore Method that separates it from things such as “guided discovery” is that, in a Moore Method course, if the students don’t do it, then it doesn’t get done.

8. Does the Moore Method foster competition among students?

(Coppin) Any class or any endeavor where grades, ranks, certificates, ribbons, or medals are awarded will foster competition. The Moore Method is no different!

(Mahavier) Competition does not always occur in my courses. When it does occur, it is the natural consequence of two or more students trying to solve the same problem and tends to be both friendly and respectful. In the courses I took, whenever a problem was presented and I was unable to solve it, I either invoked the right to leave the room (a feature wherein any student working on a problem could prevent or postpone seeing a solution by stepping out during the presentation), or I watched humbly as my colleague got the problem and I saw where I went wrong or failed to see an opportunity in my own attempts. An illustrative example of this competition occurred when I was a student at the University of North Texas and is outlined in the boxed text on the following page.

(May) In my experience, the Moore Method has fostered only healthy competition. That is the competition that arises from each student’s wanting to solve as many problems as possible as soon as possible. It is like a race in which each runner is moving as fast as she can, not necessarily trying to beat everyone else. If someone pulls alongside her, she uses that occurrence to ask herself, “Can I run even faster than I have been up to this point?” She then proceeds to try to answer her question in the affirmative.

(Parker) This certainly can happen, and it can become an important part of the classroom culture. As a student, I was driven to try to get everything first for no real reason other than I wanted to be first. To me, mathematics was like a baseball game and getting there first was my version of winning. But I had racked up my share of L’s as a pitcher and not “winning” didn’t destroy me. Also I was enough of a sportsman to respect the work of others who “beat” me. I have had classes in which the students tried to outdo each other and I didn’t discourage it. But I also pay special attention to those who meet with success less often than others and make sure that the class knows that it is not the competition that I respect, but rather the mathematics that gets done. And I demand that students behave as if they are respectful of their peers. Students are free to ask questions and to point out possible errors. But when an error is made, it marks an opportunity to better nourish an idea, not a time to diminish the idea by suggesting that, somehow, someone has not measured up. In an education course I took the summer after I graduated from Guilford, my professor, Herb Appenzeller suggested, as a topic for one of our class discussions, that “competition is the lifeblood of achievement.” On my exam for that course I took this on and argued that the proper replacement was “achievement is the lifeblood of achievement.” Whatever it takes to get them working and keep them working is the issue. If that happens to be competition, ride it wherever it will take you.

Paul W. Lewis (a student of D. H. Tucker who took courses under Moore but received his degree from Ettlinger) used a book in his graduate analysis course. He used Royden's *Real Analysis*, supplemented by his own theorem sequence on vector valued measures. At least three of us in that class would later agree that it was one of the best classes we ever took in graduate school, one earning a doctorate under Lewis' direction. One evening during the first year of my doctoral studies, I was working late and was behind on my work for the class. Normally, I tried to stay ahead and always have a problem solved in advance. That way on the nights before classes, I was never working on something to present the next day, I was working ahead in the notes. Because I have always disliked pressure, this made working on the problems more enjoyable. On this night, I had nothing to present for the next day and the next problems from Royden were to provide proofs to Lusin's Theorem and Egorov's Theorem. Of course, we were not allowed other resources. As I worked on them, I thought, "I must solve Lusin's Theorem, because Mr. N. will have them and present them if I don't." Later in the night I proved Lusin's Theorem, relaxed momentarily, and then thought, "if Mr. N. were to solve and present Lusin's Theorem, then I must have Egorov's Theorem prepared as well." It was a long night. The next day, I went to class only to find Mr. N. missing. When Lewis called for solutions, only I had the first of these theorems. Upon presenting the theorem, I sat, only to find that I was the only student who had solved the other as well. This was a rare occurrence, as there were a large number of students in the class presenting on a regular basis. Upon defending my proof to the second theorem, the class ended. As I walked from the room I encountered Mr. N., dragging down the hall and looking like he hadn't slept a wink. There were no polite greetings—he wanted to know only one thing. "Did anybody present Lusin's Theorem?" I responded that I had. "Did anybody present Egorov's?" I responded that I had. Exasperated, he smiled and said, "I stayed up all night getting those problems, but I overslept." With a grin I said, "Sorry." Over the years, Mr. N. and I have attended conferences together, shared hotel rooms, camped, and rock climbed together. Yes, there was competition, but it was not a hostile competition, it was a natural occurrence of two people wanting the same results very badly because we believed that Dr. Lewis knew what we needed to know and was preparing us for successful careers in mathematics.[8]

Mahavier

9. Are there better ways for students to present than writing on the board?

(Coppin) There is no axiom which states that one mode of presentation is best. This is dependent on students. I allow students to present in writing, in my office at a board, on a pad of paper or just orally, as well as presentations on the board in class. One anecdote that is apropos is as follows. A student of mine was having difficulty creating proofs. Her habit was to use a pad of paper. I suggested that she use a whiteboard in the math lab. It worked well and consistently. I am not certain if it was the difference between using a horizontal medium (the pad of paper) or working on a vertical medium (whiteboard).

(Mahavier) I'm optimistic that the rapid advance of technology will continue to yield better solutions in the future. Some practitioners have required students to typeset materials via LaTeX prior to presentation, which allows uploading work to the web. If students review materials before class and spend less time in class note-taking, this can encourage more productive discussions in class. Others have required students to present arguments in class using a TabletPC which projects in real-time on a large screen. Such technologies have several potential advantages over the presentations at the board. Penmanship is often

[8] Mr. N. went on to become the highly productive Dr. J. M. Neuberger.

better than written work at the board and the presenter is facing the class rather than the board. The final work, including any modifications made during the presentation, can be saved and uploaded after class for students to review at their leisure. One caveat is that I have been told that the overhead associated with using the laptop is significant. My students present in color on a whiteboard and I upload photos of the work after each class.

(May) Salisbury University has equipped each of its mathematics classrooms with an impressive array of instructional technology. The best piece of this is a "document camera." An improvement on the "opaque projector" of my own school-days, this device allows a person to lay his or her work onto a flat surface from which it is projected onto a large screen at the front of the room. Because of the availability of the document-camera, a student in one of my courses can choose between writing his or her solution to a problem, or proof of a theorem, on the board, and slapping the finished product onto the viewing area of the document camera. This comes in very handy. For example, in a computational course such as freshman calculus, when a solution consists of knowing what buttons to punch on one's calculator, the student can walk his or her classmates through the solution, providing them with calculator instruction and calculus insights at the same time. In a proofs course, a presenter, especially one who likes to write out his or her entire argument before beginning to talk, can get right to the argument without making the rest of the class wait. In addition, if he or she drew pictures to accompany the argument, then the drawings are ready to go, perhaps in color, as soon as the presenter steps to the front of the room. The availability of the document camera lessens the tension of presenting, I believe.

(Parker) I can certainly imagine that the use by students of presentation software or those devices that project their written work onto screens could be effective time savers, particularly when a student presents an argument that holds up in the form in which the student had written it. In addition, such use likely encourages students to make careful write-ups before presenting. On the other hand, I have chosen not to switch since the board is a very convenient mode for making changes and adding comments and, for undergraduates, getting students to present is a much larger issue for me than getting them to present elegantly. I have also consciously not used the marvelous facility of the WEB to disseminate summaries of what has happened on class because placing individual responsibility on students to "sort out" other students' presentations is an important part of the experience that my Moore Method classes offer.

10. Does the Moore Method fail to equip those trained via the method, as students and later as professional mathematicians, with the ability to extract information from textbooks?

(Coppin) Of course not! Since students are taught to be self-sustaining, extracting information is trivial. They may completely rewrite the material to suit themselves and not use verbatim the language or style of the textbook. In my case, when I do use a textbook, I end up creating my own rendition. So many times, the textbook is confusing or full of useless jargon and falsehoods.

(Mahavier) I believe that training students through the Moore Method encourages students to read and consider the definitions and theorems very carefully and that this skill transfers to reading texts. To ease the transition to reading texts, at the conclusion of each Moore Method course that I teach, I tie any nonstandard notations back to the current mathematical norm and relay the names of the "famous" theorems.

On the other hand, teaching out of texts may handicap students' abilities to read from books since doing so implicitly reinforces the notion that to learn from a text the student first needs someone to condense, solidify, and explain the key points of the book.

(May) To some extent, it did for me. For the first ten or so years after I had earned my Ph. D., I experienced difficulty in picking up ideas and techniques from the authors of articles and textbooks. This was primarily due to my need to prove their results for myself. (Whether this need can be attributed solely to my Moore Method indoctrination is debatable.) Once I convinced myself that, if I continued this practice, I would never progress beyond the known results, or even become acquainted with the wide world of mathematics outside my own little sphere, and consequently gave myself permission to skim the work of others, I began to be able to use texts and articles more efficiently, and without guilt. (In this vein, see Mahavier's remarks in the chapter entitled, "Materials, Development and Selection.")

(Parker) If Moore Method does fail in this respect, then I don't know of any evidence of it. It may, however, influence its disciples to regard the literature as something other than the first source of ideas. The fact that a student or mathematician chooses not to read when first encountering a problem is not necessarily an indication that the person cannot read. From a personal perspective, I would claim that most of the successes I have had as a mathematician[9] are due to attacking a problem on its own before finding out what other people know, then using the wisdom of the literature to judge where any insights gained might apply. I am confident that when learning is done in this order, that other people's ideas complement my own rather than direct, overwhelm, or stifle them. In my last year of graduate school, after it was clear that I would be finishing that term, one of my mentors, W. S. Mahavier, advised me in a conversation in his office that the toughest call for a researcher is to recognize when it is time to go find out what other people know.

11. Are there professional risks associated with teaching a course by the Moore Method?

(Coppin) Yes, some administrators might have some difficulty. The more successful you are the less difficulty you will have. I would suggest that you be open with your Chair. Work with him or her. Explain what you would like to do; educate him or her. The atmosphere is so much better for different teaching philosophies than it was twenty years ago. Start with a "safe" course. Minimize student complaints. Administrators hate to hear complaints!

(Mahavier) Yes! When a mathematician chooses to teach by a nonstandard method, that mathematician takes risk. That's ok! The safest paths in life are rarely the most rewarding ones. But let's agree that while it's fun to take risks, it's also wise to control the risk associated with stepping outside the box. In the chapter "Grading" I advocated a modified

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[9] Moore Method is almost certainly the reason that I stayed with mathematics. The liberal arts curriculum in which I was educated offered lots of things that interested me, but seeing that, in mathematics, I could do things myself created a passion of the mind that just wasn't duplicated in the other disciplines.

Moore Method for testing the waters before attempting the pure Moore Method. If you are worried about administrative risks, start there. Before I was tenured, I consciously used two strategies when preparing to teach a Moore Method course to minimize such risks. They were, "don't ask, don't tell" and "full disclosure."

During much of the presidency of Bill Clinton, I followed his "Don't Ask, Don't Tell" policy, whereby I simply did my thing in the classroom without alerting my chair or other faculty members. Inevitably the word spread through the department, but in this case, the word was spread by (mostly) interested and satisfied students and the feedback was positive. In fact, other faculty decided to implement at least some aspects of the methods that the students seemed to find intriguing. One could hardly ask for a better outcome, but I am not unaware of what the results of this approach might be for the faculty member trying the method for the first time and having a more difficult batch of students or a less successful first attempt.

The second approach is to seek out both the necessary administrative support and create the environment whereby such a risk is most likely to be rewarded. As an example, I was able to develop an Honors Trigonometry course while at Nicholls State University. The administration of the Honors Program wanted a course in each department and the department wanted the course to be accessible to students unprepared for calculus. In this setting, I had the support of my chair, the Honors Committee, and the Provost. Furthermore, the enrollment was composed of a small number of Honors students, albeit students who either did not have the prerequisites for, or were not interested in taking calculus. This assured that, should there be difficulties, I had the time to give the needed individual attention to each student. Because I was developing notes from scratch for the first time, this was an ideal setting. Because the odds were very good that I would be able to teach such a course numerous times, the investment in the notes would likely be rewarded. In fact, the time has been rewarded many times over, as I taught the course four times at Nicholls and then the course was taught once at Stephen F. Austin by Dr. Pamela Roberson and three times at Lamar; once each by Dr. B. Dale Daniel, Dr. Kyehong Kang, and Mr. Jason Montgomery.

Here is an illustrative example that demonstrates just how real the risk can be. Having taught pure Moore Method courses for many years, I taught Real Analysis I in 2004 and received a set of less than favorable reviews. This caused me great pause for two reasons: it was my first set of evaluations with a significant number of concerned comments and it was the most successful analysis class I had conducted to date. There were nine students in the class and I could easily pinpoint each set of comments to a particular group of students. Two evaluations were outstanding. Three were neutral. Four were negative. Of the four, one claimed that I was unfair to working students who had no time to work on the material and no time to come to my office. In truth, he found time to come to my office ten minutes before numerous classes when he wanted a proof from me that he could present at the board. I was not forthcoming and he was frustrated. I take no credit for his failure, having accommodated numerous working students through such courses via phone, email, and off-campus meetings. The other three claimed that I spoke only to potential graduate students and they saw no value in what I made them do, as all were to be high school teachers where they would not need such material. A year-and-a-half later all three of these students came by my office together. It was the first time I had seen them since the annual departmental banquet. They said they wanted to confess that they did not like my class at the time, but now that they were in the school system as mathematics teachers, they thought about the class, and how much it helped them, every single day. Their evaluations were valuable to me and I had spent a lot of time thinking about that set of evaluations, giving them more time than I had ever spent on a set. Because of their evaluations, I have been careful not to allow the *perception* in my classes that I am speaking only to potential graduate students and not to future teachers. Clearly, the class was a much greater success than the evaluations demonstrated. Of the students in the class, five are successful high-school teachers, three went on to graduate school, and one (you can guess which one) was still looking for a job the last I heard. Hence the method will safely prepare your students for the world, but they may not realize it right away, and had those students never come back to me I would still carry their remarks with me today.

(May) I favor Mahavier's "Don't ask, don't tell" policy. For the most part, it has served me well. I have, however, experienced unfavorable consequences of being known as a practitioner of the method. The main one has been to be prohibited from teaching both semesters of my department's real-analysis course (the reason given was that it would be harmful to expose students to two consecutive semesters of Moore Method teaching in the same subject). Even so, I have never considered teaching by any other method.

(Parker) I would suggest that this depends on the administration of the university or college at which one works. I got my first job *because* it was an expectation that I would teach using Moore Method and I got my current job under circumstances in which the department *knew*[10] that I used Moore Method. Nevertheless my own career has not been without incident.[11] Fortunately for me, at each university at which I have worked, there has been a commitment to collegiality that included, at least for issues of teaching, peer review. And in each situation in which my methods have been openly challenged, I have been given the opportunity to give "my side of the story" or, as I would prefer to describe it, professional justification for my methods. I still have a job, and have been tenured and promoted at both institutions at which I have been a full-time faculty member.

On the positive side, the risks can have affirmative payoffs. When I first came to James Madison, I got opportunities within the Liberal Studies program *because* colleagues knew

[10] From Diane Spresser, Department Head at James Madison when I was hired: "Having… a faculty member… pedagogically adept with the Moore method is … a real asset in a mathematics department. Bringing such a faculty member into a department where the Moore method isn't currently being used, however, is not without some professional risk: risk with students who are unaccustomed to a Moore-method classroom environment, some risk perhaps with departmental faculty who have little or no experience themselves with the Moore method or who are concerned with its impact on the overall curriculum, and risk with administrators who may be responsive to complaints from students in Moore-method classes. … students, their first experiences with a Moore-method class may be a real departure from their prior experiences …students come to college thinking of mathematics as a body of facts to be learned and of the faculty as purveyors of those facts… departmental and college administrative support is critical in responding to student comments and complaints. The support of core faculty in the department is equally important…

Over time,…, student comments and complaints tend to recede and support from peer faculty increases. But, this takes time and patience and likely requires strategic discussions along the way with the Dean…

…important and positive outcomes from the Moore method will likely accrue… As a former department head in an institution that chose to hire and support a faculty member who was (and still is) adept in the Moore method, … there's no doubt in my mind that good decisions were made to hire and advance a Moore-method faculty member, decisions that I'd support again in a heartbeat."

[11] These have emanated from sources as high as the President's Office and as locally as student evaluations of teaching.

I taught a particular (peculiar?) way and, as a result, I got to explore the use of Moore Method in nonmathematical contexts. One of the veterans of Stan Yoshinobu's 2006 week-long Moore Method seminar, during the academic year in which she used Moore Method for the first time, created such a positive stir that she was contacted, from the dean's level, about giving colloquia/workshops on student-centered pedagogy.

Relative to risk, doctrinaire done poorly is probably more likely to go un-noticed than the Moore Method, but different done well has real possibilities for positive career payoffs.

12. Are the goals for a Moore Method course the same as the goals for a non-Moore Method course?

(Coppin) For me the goals are different. Usually, the main difference is that the depth of learning is greater in a Moore Method course. Moreover, I seek a transformational experience for my students. Privately, my main goal is give the students a first-hand experience with authentic mathematics, which is easier for me to do with a Moore Method course.

(Mahavier) Can one course be all things to all students? Let's take a specific example. Can we create a course in say numerical methods that:

1. addresses industry's desire for communication skills,
2. fosters cooperative learning as oft implemented in industry,
3. encourages discovery of the central topics of the subject,
4. covers a broad base of numerical techniques,
5. develops in students a deep enough intuitive feel for the central concepts of the topic to assure the student can apply the tools in a different setting,
6. addresses relevant hardware and software technologies,
7. prepares students bound for graduate school with the essential elements and proof techniques of the subject, and
8. demonstrates the myriad of applications and counterexamples of the subject?

All of these are aspects of the course that I would like to see. I am confident that there are more that the reader could add with which I would agree. Most importantly, I must develop the course (notes, syllabus, grading, etc.) within the context of a broader single goal that the course will have a long-term effect on my students. Information provides only a short-term benefit; either it is used immediately and becomes part of a larger, growing informational tool-kit or it is not used immediately and erodes with time. As B. F. Skinner wrote, "Education is what survives when what has been learned has been forgotten." Thus, process must be integral to my goals. I set my primary goal to actively involve my own students in the learning process. By "actively" I mean that we are regularly engaged in a discussion regarding the subject, as opposed to such passive involvement as graded homework. Hence, however I construct the class materials and syllabi, the following process-oriented goals are presupposed.

Goal 1: To be able to independently develop a position. Hard work and investigation of the topic are the prerequisites for developing this skill.

Goal 2: To communicate that position. Having spent the necessary time to solve a problem, the student must present to the class, in both written and oral form, his or her position.

Goal 3: To defend that position. During this process, either the presentation is adequate to convince peers of the validity of the position or the student must defend (or modify) his or her position. Perhaps the position is indefensible, in which case the student returns to Goal 1, repeating the cycle.

Assuming we have developed a set of notes that addresses the informational goals for the course and we teach in a way that respects the process oriented goals, then our course will likely be a success. The successful student in such a class has developed an understanding of the topics, defended some of these topics to peers, developed skill at independent problem solving, and developed communication skills. This student has transitioned from a passive recipient of knowledge to an active creator and conveyor of knowledge. These goals transcend the subject matter and the informational goals and will result in a long-term change in the student's approach to problems. By this approach in my own numerical methods course and by assigning weekly programming assignments requiring industry-style written reports, I manage to achieve 1,3,4,5 and 7 from the original list. My point here is that the process-oriented goal does not significantly reduce what I am able to accomplish in the course with respect to the original list of goals that I created.

(May) I teach every one of my courses – even those, such as calculus, that are taught by many other faculty and in which a textbook is mandated – by at least a modified version of the Moore Method. Nevertheless, in every course that I teach, the goals are established by a consensus of the faculty in my department. Not only do I strive to reach those goals; but also I believe that, with the help of the Moore Method, I achieve them as well as my colleagues. (One of my most-respected colleagues, primarily a lecturer, once stated that he could usually tell which students in his classes had taken a course from me. This was because, he said, of their ability to learn independently.)

(Parker) If the goals for a non-Moore Method course deal with mastery, at some level of sophistication, of a mathematics curriculum and/or the achievement of a greater level of mathematical maturity by the end of the course, then certainly the goals for a Moore Method course include these. The nature of the quality and scope of "coverage" may differ as well the nature of the mathematical maturation, but nevertheless the goals are common to both modes.

13. How does one prepare to teach a Moore Method course?

(Coppin) I prepare no differently for a Moore Method course than for any other course. Teaching is teaching. Any teacher will always have elements of a Moore Method class in any class he or she teaches. The only difference is to what degree these elements are emphasized.

Exception. In one instance, early in my career, I did use someone else's notes without modification; however, if I do use another's notes, I will make massive changes. I have my own ideas of what a good set of materials ought to look like. See my piece in the chapter entitled "Development and Selection of Materials." One thing you will realize is that Moore Method people are very independent and, if I may say so, a bit "bull headed." This is as it should be. As a Chair of a Department of Mathematics, I never impose this method or any other method on my faculty. I tell them to teach the way they want to. You will teach best when you teach out of your personality: be true to your pedagogical soul. Moreover, teachers will have the best results when they teach courageously and they develop the heroic student.

For any course.

Pre-preparation. So, what do I do when readying myself to teach a course? Here are some of the points I hit.

- What will be the maturity level of my students? Are they emotionally ready? How well will they handle academic pressure? Do they believe too highly in themselves but know too little?
- Will my students have holes in their background that will impede achieving my objectives? Do the students have deficiencies such as in algebra? Have they had any experiences doing proofs or have they had courses that just teach rote mechanics? Will the students be culturally ready for my class?
- What do I have to do to get the students ready for someone's course for which my course is the prerequisite? After all, I will not want to infringe on my colleague's ability to teach a viable course. I must cover the syllabus.
- Will the prescribed text support my teaching personality and my chosen teaching style? If not, should I use the text as a resource and use a set of notes as the "text?" Am I allowed to do this? (By the way, if you don't make an issue of this point, you can usually get away with supplementing the class with your own materials.) Am I free to pick my own text that fits my students and me?
- Will students remember what we spend time on? What will be remembered two years into the future?
- What will mathematically excite the students in a meaningful way? Will my course "tell a story?" Will they clearly discern the plot and the main characters?
- Remember that whatever the level and audience, you have a responsibility to any latently talented student that may be in your class.
- Will my evaluation scheme be well understood? Is it too complicated? Will it work to encourage learning? Will it be considered fair by the students?

Your answers to the above questions will guide you in the development of your course.

Prepare Syllabus. This should include contact information, evaluation scheme, objectives, issues, and policies on absences, cheating, and disabilities. You might include special dates such as for examinations, holidays, etc. The syllabus should be no more than two pages in length. Keep it simple!

Prepare Evaluation Scheme. Develop an evaluation scheme that supports your learning objectives and is easily understood. This is not the time to be nontraditional: students will not understand anything that deviates too far from expectations. It is acceptable to include presentations and even count presentations heavily. However, examinations and quizzes are compatible with a modified Moore Method class.

Prepare First Day. What you present to your class on the first day is very important. This will set the tone of the entire course. You are the coach: give your team a pep talk. Talk to them of "the agony and the ecstasy." Guarantee them "success" if they maintain a good attitude and work hard!

Prepare Day-to-Day Format. Do you have a good game plan? Do you have alternate game plans that are consistent with your syllabus, evaluation scheme and your first class pep talk? Students and you need to feel comfortable. Reinforce what has been covered. Have students to the board and/or interact with students. Attempt to have a focal point. Set up students for the next class: focus their activities away from class to ready them for the next class.

Again, most of what you need to do in preparation is essentially the same for the very reason you are not going to change your wiring as a teacher whether you teach a standard lecture course or a Moore Method course. Use your common sense, what you know about classroom psychology and what you know of the nature of mathematics. Treat each student as a precious individual and the discipline of mathematics as if it is a person as well. You love both of them and you want them to get to know each other: teach to that end!

(Mahavier) If preparing for a *first* attempt at a Moore Method course, I would suggest completing this check list:

1. Read this book. (Done!)
2. Secure a mentor through the Legacy of R. L. Moore website and begin a dialogue by email, phone, or visitation.
3. Decide if you want to go *pure* Moore Method or *modified* Moore Method. (See my essay on grading where the two methods are described.)
4. Secure or develop enough material to keep the class working for a full month. (Consider the *Journal of Inquiry-Based Learning in Mathematics* as a possible source.)
5. Create a syllabus that will support your strategy for the course. (See samples in the appendices.)
6. Put considerable time into deciding how to conduct the first day of class. (See my essay from "In the Classroom" for details on the first day.)
7. Go for it!

After the first successful attempt, skip steps 1 and 2.

(May) The best way to prepare to teach a Moore Method course is to take one from an experienced practitioner. If that is not possible, I suggest a four-step process. The steps are these:

Step 1. Gather materials from which to construct your Moore Method notes.

Step 2. Begin building a set of notes for your own course.

Step 3. Construct a syllabus for the course.

Step 4. If it is time for the course to begin, start teaching from the notes; otherwise, return to constructing the notes.

For me, Step 1 begins from one term to a full year ahead of the start of the course. For a description of the gathering technique, see my essay in the "Gathering Materials" chapter.

Once I have chosen one of more sets of materials from which I will build my development, I begin developing my own set of definitions, axioms, and theorems for the course. If the theorems are at all new to me, I prove as I go along. (I will not, of course, include my proofs in the version of the notes that I distribute to the students.) This gives me a feel for the level of difficulty of the material. During this step, I often take single theorems from my source, especially if it is a conventional textbook, and expand them into two or more theorems for my development.

After I have completed enough of Step 2 to be assured that I will be ahead of my students for at least the first two weeks of class, and seen where I would like the entire development to go, I construct a syllabus for the course. This will be published in hard copy in my department's display of syllabi of all courses to be offered during the upcoming term, and electronically on my department's and my web-sites. (I will also hand out a hard copy to each student of the course on the first day of class.)

Once I have completed the syllabus, I return to Step 2. I continue until I produce enough problems or theorems to fill out the syllabus, or the course begins.

I often continue preparing materials for the course while I am teaching it. At the very least, I find myself revising the notes throughout the term. Although this might seem irresponsible or lazy, it is unavoidable. I never know what sort of class I have until the students begin going to the board to present their results. Consequently, during some offerings of the course, I need to lower the level of rigor, whereas in other terms I need to raise it. (As an example of this phenomenon, in the first offering of my "Introduction to Abstract Mathematics" course, I seemed to have aimed the notes quite accurately for the level of mathematical and logical maturity of the students. During the second offering, I found myself lowering my expectations regarding rigor. During my current term, I am finding it advisable to aim somewhere between the two previous targets.)

I find it great fun to prepare a Moore Method course. Gathering the materials is enjoyable; for it allows me to see "what is out there," of both a conventional (traditional, lecture-format-oriented) and a Moore-friendly nature. Developing my own axioms, definitions, and theorems allows me to engage in my two chief loves, proving theorems for myself and giving students an opportunity to do the same thing. By the time I walk into the classroom on the first day, I find myself as excited as a brand-new college professor, eager for the chance to participate with another generation of students in a venture that, after more than thirty-five years, I continue to find uplifting, rewarding, and thrilling.

(Parker) The initial preparation for teaching a course using the Moore Method comes in choosing the materials for the course. The relative importance of this in the minds of the authors is shown by the fact that we have given the generation of materials for a Moore Method course its own chapter. In light of this, I begin my discussion with the premise that the materials for the course have been selected.[12]

Granted this premise, let me remind you of a couple of the features that I expect these materials to have. One is that there be some identifiable "goal" theorems in mind for the course, and a skeleton of problems that I anticipate will, in Mahavier's metaphor, provide the students with stepping-stones to the goal theorems. Second, there are some "starter" problems at the beginning of the materials that will gauge the preparedness of the students to address the main problems of the course. Major precourse preparation consists of readying oneself for the long- and short-term goals suggested by these two curricular vehicles.

[12] It should be noted that the materials with which I enter a course are always subject to change. As I observe how the students deal with the ideas that I have identified as being basic to attaining the goal theorems, what they do will tell me how to amend the materials as the course progresses.

For the long-term goals, we need to be able to settle the goal theorems for the course ourselves. Beyond having proofs in mind, it is also helpful to have in mind examples that, if worked through, have the capacity to illustrate the ideas behind the proofs we have conceived. For the proofs, this requires very little preparation that is not already done when generating or appropriating materials for the course since the stepping-stone problems are typically selected with proof ideas in mind. Having examples in mind, particularly the first time one teaches a course, is a little trickier. The intent is that the examples will not be shared with the class unless the class initiates ideas that make them appropriate. Unless your prescience greatly exceeds mine, you will likely be surprised at what your students actually do.[13] Nevertheless, even flawed arguments often contain nice ideas, and having anticipated how students *might* think will help prepare you to respond to the way they *do* think. If they do what you expect, you will have something available; if they do not, the contrast will sharpen your appraisal about which things that show in their thoughts are ripe to be nurtured. As we teach Moore Method courses, and particularly as we teach a single course multiple times, the experiences of the students we witness as they grapple with the problems will further inform our preparation. Regardless of the level of preparation we make, however, we must remember not to impose our proofs of the theorems on the students. If the students find the proofs we intended, that is fine. But we must be patient with the ideas they uncover and use our peripheral preparation to nurture those ideas rather than to coerce them back into some preconceived channel. Indeed, the "unexpected" ideas are typically the fuel for making our course materials dynamic; that is, these ideas may show us what problems that we did not *pre*pare might actually be especially pertinent to the way the students are thinking.

14. Are there upper or lower bounds on the size of a Moore Method class?

(Coppin) I have taught classes using the Moore Method with as few as one student and as many as thirty-five students; however, I have found that the best range is approximately six to approximately fourteen students. The major factors are the intellectual personality of the students and the goals of the course. The classes with student numbers in the twenty-seven to thirty-five range were geometry classes taught to liberal arts students at the University of Dallas in a course with a very sparse syllabus. These same courses at Lamar University have a cap of twenty-five students. When you have as few as five or less students, you run the risk of not getting a rich interaction among the students.

(Mahavier) I have used some form of the method with classes ranging in class size from one to fifty. In its purest form, the method ceases to be as effective with less than five students and I have needed to modify it considerably to make it work for classes with more than twenty students. With classes of one to five students, I use the method, but make myself an equal member of the class in terms of the number of presentations, being careful never to take a problem from a student that is working on it. This adds one student to the class in terms of sharing the presentation load. For classes of more than five and less than twenty, I implement the pure Moore Method, where almost all class time goes to individual students at the board. Probably twelve to seventeen students is optimal. For classes rang-

[13] One of the advantages of using Moore Method is that students make good mistakes; no one does the wrong thing on purpose so errors are typically the result of some fundamental misunderstanding.

ing in size from twenty to fifty students, I alternate between two modified versions of the method. I may use the modified Moore Method that I described in the "Grading" chapter where student presentations count as one test grade and we follow a book. Alternatively, I may use small group discovery where groups of students work and present as a team. This enables us to spend much more time on presentation and to use a set of notes rather than a book.

(May) The upper limit of optimality for a Moore Method class is twenty-five students. The lower limit is five. I have taught classes with fewer than five students, and ones with slightly more than twenty-five, which I would rate as successful. I believe, however, that every one of them would have been more effective for the students involved if the enrollment had been in the interval [5,25].

(Parker) Certainly, the lower bound is one. Just this past semester I gave topology to the single student who signed up for our topology course (even though the course was canceled so the time came out of my hide). I had never taught this student, and the student had never been taught with the Moore Method. The student turned out to have really good work habits, but was even slower than I am on my bad days. Nevertheless, the student proved some theorems and did enough mathematics that, in the context of a regular course, would have gotten the student a *C* if that were all the student did. The student got an *A* in the course since the student made, without the help of books or peers, all of the mathematics; it is highly likely that in a fully-populated course, the student would have made an *A* from the extras picked up from classmates' presentations. During the second semester, for graduation exigencies, I agreed to do a continuation with the student to include an open-ended component consisting of an investigation of an aspect of a problem that I had identified from one of the student's arguments during first semester. My resolution of the problem, which I had envisioned as a best-case outcome, was interesting enough and, if the student were to travel down the path I envisioned, the student would have learned some things that would have enhanced the student's education. My resolution of the problem pales in comparison to how the student solved it! In fact, an obvious (from my perspective, not the student's) generalization is fraught with mathematical possibilities that may, or may not, be in the literature somewhere. Unfortunately, because having worked with the student allows the student to graduate, continued work on the problem with its fascinating possibilities will have to wait until I get another topology student, that is, if I can maintain the discipline necessary to keep my own hands off of it.

An upper bound is certainly the number of seats that will fit in the classroom in which the course will be taught and that the fire marshal will allow. The largest class I have ever taught using Moore method exclusively is twenty-three, but I routinely do a Moore Method unit in my Math-for-people-who-don't-want-to-take-math course and enrollment there is usually capped at thirty-five and my sections typically have thirty to thirty-two students in them. Once, when in a faculty "teaching" seminar, I was challenged with the question "How can we best teach sections of 100 or more?". My response was "In such an academic year, while I am looking for another job, on the first day, I would take the twenty students who sat closest to the front and do Moore Method with them, holding the other students responsible, by test, for the results these students got. After two weeks, each day any one of the students among the current twenty would be allowed to join the throng if he or she so chose and a replacement from the throng could volunteer to fill the place." It would have been an interesting experiment. Would students have complained because they were excluded from the twenty or because they were part of the twenty?

I consider classes of between six and sixteen to be optimal for using Moore Method, but I am guessing. Fewer than seven may leave you without a critical mass for creating a culture and more than fifteen makes it difficult to be sure that there will be sufficient trips to the board to guarantee a good look into each student's mind and leaves you dangerously dependent on their taking the initiative to hand in their written work. For the student who struggles at the outset, such written work may not even exist. Nevertheless, my class of twenty-three (a geometry class) was one of the most productive I have ever taught and, since most of my classes with one student are supervised research projects with the student invited to the project because of what I have seen him or her do, classes of one tend to be successful on a routine basis. I cannot link those classes that have gone belly up with Moore Method (I have had three such classes in my thirty years of college teaching experience) to the number of students in them.

> "Dr. Parker, I am only taking education courses this coming semester and would like to take a math course. I now realize how much more I might have gained in 410 if I had been willing to try to be a maker of ideas and not just borrow those of my classmates. Would you be willing to let me do a 467 with you and teach it that way?"
>
> From a fifth-year student in the Mathematics Education master's program.

I
Coppin

I.A Syllabus for *Linear Point Set Theory*

Math 3321 Fall, 2001

SYLLABUS

Instructor: Charles Coppin
Office: SB 50 (lower level)
Office Hours: 2:00–3:00 MW; 2:00–3:30 F or by appointment
Text: Notes

Content: An axiomatic development of the numbers. A study of limit points, convergent sequences, compact sets, connected sets, dense sets, nowhere dense sets, separable sets

Objectives and Issues: The primary objective of Math 3321 is to begin to inculcate in you those mental processes that characterize mathematical thought. This course will develop your powers of deduction and imagination. You will learn to use language precisely and concisely. Settling conjectures, creating counterexamples and conceptualizing an abstract definition will stretch your imagination.

I have taught this course twenty-seven years at the University of Dallas. We will attempt to run this course in the spirit of R. L. Moore, the father of the American school of point set topology. In his prime, he was considered to be one of the top ten mathematicians in the world. He was considered to be one of the most effective teachers in collegiate mathematics in the first half of this century.

The format is not lecture. There are no surprises about the way people learn. Vince Lombardi, the coach of the Green Bay Packers in their heyday, said that football is nothing more than blocking, tackling, and running—the fundamentals. We will also stress the fundamentals of mathematics—logic and imagination. Class members present all material, not the teacher. Most class periods will be a sort of laboratory where student presentations will be critiqued by the class. You are going to learn mathematics the best way I know—by doing rather than by watching. My experience is that students who do well in this course develop an enthusiasm and a sense of accomplishment quite analogous to that of climbing a mountain. There is no substitute for doing proofs or solving problems on your own which you know to be challenging, especially if you know you received little or no help.

Evaluation: The course grade is based on:

Item	Weight
Presentations	90%
Final	10%

1. The student presentations are, by far, the most important component of the course. I will keep track of which results you say you have, which ones you attempt and which ones you successfully present. Grades here will be based on the overall quality of the presentations, as well as the number and difficulty of your presentations. It has been my experience that the top students and those students with positive, enthusiastic attitudes toward proving theorems, settling conjectures, or answering questions are identical groups of people.
2. The final examination will be comprehensive covering all topics. The examination will be split between material you will have seen before and new results.

Finally, because the student presentations will be done in a way that may not be familiar to you, I will state some rules that must be followed if the course is to succeed and if you are going to succeed. They are as follows:

a. Any result you want to present for credit must be your own work. Do not get help from anyone except your instructor and do not go to any book for help.
b. You cannot receive credit for any result you have seen.
c. When someone is at the board do not give help or hints. You may ask questions but not in a way as to offer direct help to the one at the board. Their proof must be their own.
d. You may step out of class from time to time under conditions which I will state in class.

If, at any time during the course, you have questions concerning objectives, methodology, etc., please feel free to discuss them with me. My office is Science Building 50 and my office phone is 721-5361. I will be available to help you during my office hours or you may make an appointment to see me.

I.B Notes for *Neutral Geometry*

Neutral Geometry (sometimes called Absolute Geometry) is that geometry which is common to both Euclidean and Non-Euclidean geometry. We start with some familiar axioms.

State:

"Point" and **"line"** are undefined. Lines will consist of points.

Axiom 1. If L is a line, then there exist at least two points belonging to L.

Axiom 2. If L is a line, then there exists at least one point that is not on L.

Axiom 3. There exists at least one line.

In-Class 1. What is the minimum number of points possible based on the Axioms 1, 2 and 3 above? Prove your answer.

State the following theorem only after In-Class 1 is settled.

Theorem 1. There are three different points. (Must they be collinear?)

In-Class 2. Find a model that satisfies Axioms 1 and 3 but not Axiom 2.

Settle:

Problem 1. Create a model for Axioms 1, 2, and 3 in which there are at least two points x and y with the property that no line contains both x and y.

Problem 2. Is the following a theorem of the axiom system?

Each point belongs to at least one line.

I think you would agree that any geometry that contains two points not belonging to any line is not very interesting. We have an unacceptable situation as represented by Problem 1. **Therefore, we <u>need</u> the next axiom only after Problems 1 and 2. State:**

Axiom 4. If A and B are different points, then there exists at least one line which contains both A and B.

In-Class 3. Explain what is wrong with the following "proof" that each point belongs to some line. Find the invalid line of the proof.

By Axiom 3, there is some line L. By Axiom 1, L contains two points A and B. Thus, for any point A, there is a line containing A.

Prove the following theorem.

Theorem 2. If P is a point, then there is a line containing P.

Settle:

Problem 3. Which of the following is true of the axiom system consisting of axioms 1, 2, 3, and 4?

A. There is a model of the axiom system in which at least one point belongs to only one line.
B. It is a theorem that each point belongs to at least two different lines.

Prove your response.

We state the theorem proved in Problem 3 but only after it has been settled.

Theorem 3. If P is a point, then there exist two different lines containing P.

In-Class 4. Is it a theorem of the axiom system that each point belongs to at least three different lines?

In-Class 5. Find a model which satisfies the axiom system in which there are two different points A and B such that at least two different lines contain both A and B.

State the following axiom only after In-Class 5 is settled.

Axiom 4′. If A and B are different points, then there exists one and only one line which contains both A and B.

In-Class 6. Give a definition of what we would mean by the statement that a set of points is collinear.

Prove the following Theorem.

Theorem 4. If P is a point, then there are two other points Q and R such that P, Q, and R are non-collinear.

Settle:

Problem 4. Axiom 2 says that each line excludes some point. Can we turn that statement around and say that each point is not contained on some line? Prove your answer.

State the following theorem only when Problem 4 is settled.

Theorem 5. If P is a point, then there exists at least one line that does not contain P.

In-Class 7. Suppose L and H are two different lines. What can you say about the maximum number of points, which can belong to both L and H? Prove your answer.

We have proved the following theorem.

State only after In-Class 7 is discussed.

Theorem 6. If L and H are two different lines, then there is at most one point belonging to both L and H.

In-Class 8. Find a model with the smallest number of points, which satisfies Axioms, 1, 2, 3, 4′. Prove your response.

State:

Notation. By the symbols $\overleftrightarrow{AB}$, we mean that A and B are two different points and $\overleftrightarrow{AB}$ denotes <u>the</u> line containing A and B.

Definition 1. The statement that a set of points M is collinear means that M is contained in some line.

We now have an additional undefined term to add to "**point**" and "**line**."

We now state that the word "between" is undefined but satisfies the axioms to come:

The word "**between**" is undefined.

State:

Notation. *ABC* means that *A*, *B* and *C* are points such that *B* is between *A* and *C*.

Before proceeding, we develop a motivation for the axioms that follow. Suppose we say nothing about "between" other than it is an undefined term. Consider the following model.

In-Class 9. Consider the following model for axioms 1, 2, 3, 4′.

Model. Suppose by "point" we mean point in the plane, by "line" we mean the normal "straight" line of the plane and by "*B* is between *A* and *C*" we mean that the point *A* is to the left of *B* and *C* is to the right of *B*.

Using the above model as a guide, is it a consequence of Axioms 1, 2, 3, 4′ that if *XYZ*, then *X*, *Y*, and *Z* are collinear? Explain.

Note that it is possible for a point to be between two other points but not be collinear with them. This we do not want. Thus, we have to have the following axiom but only state when In-Class 9 is settled.

State:

Axiom 5. If *ABC*, then *A*, *B* and *C* are three different points of some line and *CBA*.

In-Class 10. Based on Axioms 1–5, if *XYZ*, then can we say *X*, *Y*, *Z* are collinear? Explain.

Settle:

Problem 5. Create a model for Axioms 1–5 that violates the following statement.
If *A*, *B*, and *C* are three different points of a line, then one and only one of the following hold: *ABC*, *BCA*, *CAB*.

Problem 6. Suppose *ABC* and *XYZ*. How can we change the hypothesis *XYZ* in order for it to be said that the set $\{A, B, C, X, Y, Z\}$ is collinear? Prove your answer.

State:

Axiom 6. If *A*, *B*, and *C* are three different points of a line, then one and only one of the following hold: *ABC*, *BCA*, *CAB*.

In-Class 11. Suppose *ABC* and *XYZ*. How can we change the hypothesis *XYZ* in order for it to be said that *A*, *B*, *C*, *X*, *Y*, and *Z* are four different points? Prove your answer.

State the following only after In-Class 11 is settled.

Theorem 7. If *ABC* and *ACD*, then *A*, *B*, *C* and *D* are four different points and all belong to the same line.

Definition 2. *ABCD* means that *ABC*, *ABD*, *ACD*, and *BCD*.

Axiom 7. If *A*, *B*, *C* and *D* are four different collinear points and *ABC*, then one and only one of the following is true: *ABCD*, *ABDC*, *ADBC*, *DABC*.

Prove the following theorems.

Theorem 8. If *ABC* and *ACD*, then *ABD* and *BCD*.

Theorem 9. If *ABD* and *BCD*, then *ABC* and *ACD*.

Settle:

Problem 7. One of the following two conjectures is true (a theorem) and one is false. Determine which is which. Prove the theorem and find a counter example for the other.

Conjecture A. If ABC and BCD, then $ABCD$.

Conjecture B. If ABD and ACD, then ABC or ACB.

Don't assign the next two theorems until Problem 7 is settled.

Prove the following:

Theorem 10. If ABC and BCD, then $ABCD$.

Theorem 11. If ABD, ACD and B is not C, then ABC or ACB.

Prove the following:

Theorem 12. If ABC and ABD and $C \neq D$, then BCD or BDC.

Theorem 13. If ABC and ABD and $C \neq D$, then ACD or ADC.

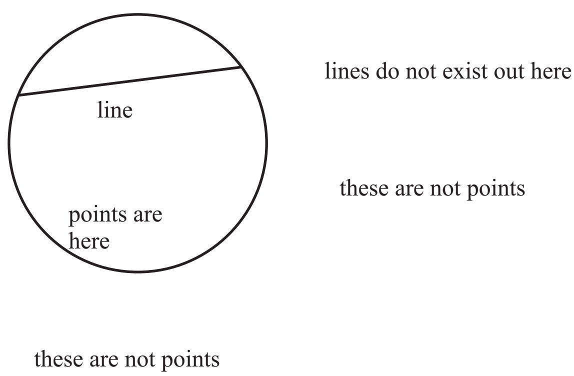
Settle:

Problem 8. (Group work allowed) Suppose L is a line and A is a point of L. Using the notion of "between," define two subsets, S and T of L such that

a. each point of L is in S or in T,
b. A is the only point in both S and T,
c. A is between any other point of S and any other point of T, and
d. A is not between any two points of S and T.
(Two credits for a proper definition of S and T.)

Problem 9. Prove a, b, c and d.
(One credit per each proof.)

In-Class 12. Remember the model where "point" meant a point on the circle or interior to the circle and by "line," we meant a chord of the circle. "Between" can be understood in normal sense of the word. See below.



This is a model for Axioms 1–7; however, the following axiom is not satisfied. In our geometry, we most definitely want a point to precede a given point and we want a point to follow a point as in the following axiom.

State:

Axiom 8. If A and B are different points, then
(a) there is a point C such that ABC,
(b) there is a point D such that ADB,
(c) there is a point E such that EAB.

State:

Problem 10. Modify the model of the above example so that Axiom 8 is true.

Definition 3. Suppose A and B are two different points. Then AB (called "the interval AB") is the set to which X belongs iff $X = A$, AXB, or $X = B$; that is, AB is the set of all points between A and B inclusive. Moreover, A and B are called the end points of AB.

Problem 11. Suppose AB is an interval. Must there be a third point in AB? Prove your response.

Problem 12. Suppose A and B are different points. Prove $AB = BA$.

Problem 13. Suppose A and B are different points. Prove every point in the interval AB is a member of $\overleftrightarrow{AB}$, that is, AB is a subset of $\overleftrightarrow{AB}$.

Problem 14. Suppose AB is an interval and APB. Prove
(a) AP is a subset of AB and PB is a subset of AB,
(b) $AP \cup PB$ is a subset of AB,
(One credit for (a) and (b).)
(c) if x is in AB, then x is in AP or x is in PB,
(d) AB is a subset of $AP \cup PB$,
and
(e) $AB = AP \cup PB$.
(One credit each.)

Problem 15. Suppose AB is an interval and APB. Prove $AP \cap PB = \{P\}$.

Problem 16. Suppose AB and CD are intervals where $AB = CD$. Prove $A = C$ and $B = D$ or $A = D$ and $B = C$. First, prove ACB is false.

Problem 17. Find a model satisfying all axioms where there are three different points A, B and C, a line L not containing A, B, and C, contains a point of AB but does not contain a point of AC or BC.

State

Axiom 9.(Pasch) If A, B, C are three distinct non-collinear points and L is a line which does not contain A, B, or C but does contain a point of AB, then L contains a point of AC or BC but not both.

In-Class 13. Suppose L is a line. Find a set U containing at least two points such that no point of L is between any two points of U.

State

Definition 4. Suppose L is a line. Then the statement that M is a side of L means that there is a point O not on L such that M is the set to which the point X belongs if and only if X is O or the interval XO does not contain a point of L. M is called "O-side of L."

Problem 18. Suppose L is a line and Q is a point not on L. Complete: The statement that X does not belong to the Q-side of L means ....

Problem 19. Suppose L is a line and A and A' are distinct points not on L where no point of L is between A and A'. Prove that the A-side of L is the A'-side of L. (Remember how to prove two sets are equal)

Problem 20. Suppose L is a line. Prove that L has at least two sides.

Problem 21. Suppose L is a line. Prove that no two different sides of L have a point in common.

Problem 22. Suppose L is a line. Prove that L does not have three or more different sides.

Definition 5. The statement that the point set M is convex means for each pair of distinct points P and Q of M the interval PQ is contained in M.

Problem 23. Prove that the following are convex:

a. Any line.
b. Any interval.
c. Any side of a line.

Problem 24. The common part of a convex set with a convex set is a convex set.

Problem 25. Suppose A, B, C are three distinct non-collinear points. Prove there exists a point which belongs to the C-side of $\overleftrightarrow{AB}$, the A-side of $\overleftrightarrow{AB}$, and the B-side of $\overleftrightarrow{AB}$.

I.C Worksheets for *Neutral Geometry*

NG Worksheet 1

Neutral Geometry (sometimes called Absolute Geometry) is that geometry which is common to both Euclidean and Non-Euclidean geometry. We start with some familiar axioms. Respond to the problems below.

"Point" and "line" are undefined. Lines will consist of points.

Axiom 1. If L is a line, then there exist at least two points belonging to L.

Axiom 2. If L is a line, then there exists at least one point that is not on L.

Axiom 3. There exists at least one line.

Problem 1. What is the minimum number of points possible based on the Axioms 1, 2 and 3 above? Prove your answer.

Problem 2. State the theorem you discovered and proved in Problem 1.

Problem 3. Find a model which satisfies Axioms 1 and 3 but not Axiom 2.

a. "point" means ....
b. "line" means ....
Explanation.

NG Worksheet 2

The results of NG Worksheet can be applied to this one. That is, the "truths" of that worksheet can be used in arguments here.

Problem 1. Create a model for Axioms 1, 2, and 3 in which there are at least two points x and y with the property that no line contains both x and y.

"point" means ....

"line" means ....

Explanation.

Problem 2. Is the following a theorem of the axiom system?

Each point belongs to some line. Yes ___ No ___ Prove your response.

I.D Diary Entries for *Euclidean and Non-Euclidean Geometry*

As you read this diary, I hope that you will see the psychological themes that run through the various "days" of the diary. To me this is the most important thing that I can communicate to you. The bottom line is that students learned deeply and that they have claimed the class for themselves; that is, they have invested in their learning at such a deep level that they have learned the material in a way that "fits" their personality. In most cases, this means they will have proved theorems and solved problems on their own without recourse to extra help. The teacher is the "coach," the facilitator. The so called mechanics of running a Moore Method class is something you must discover for yourself. Study other teachers of the Moore Method. In true Moore Method style, you must discover your own way to this style of teaching: you must claim this method for yourself, so much so, that your style of teaching the Moore Method could be called "your name method."

Class 1. (Introduction to the Course, Part I) Notable Observations

Introduction. After taking roll, I gave the class a short introduction to the course giving out an information sheet, syllabus and the first set of notes, *Language and Logic*.

Evaluation. Concerning evaluation, I made certain that students understood the two major components, presentations and examinations, each of which I give a grade. If a student's presentation grade and examination grade do not differ by more than one letter grade, I will give them as a grade for the course, the maximum of the two grades. For example, if the presentation grade of a student is A and the Examination grade is B, then this student earns a grade of A. However, if the student's presentation grade and examination grade differ by more than a letter grade, I will give the students as a grade for the course, the average of the two grades. For example, if the presentation grade is C and the examination grade is A, then the final grade in this case is B. If the student does nothing in the way of presentations and earns an A for the examination grade, then the student fails the class. The last case has never occurred and the second case very rarely ever happens. The first case occurs almost every time.

This method allows each student to work to his or her strength in evaluation. However, he or she must make presentations if the student is to pass the class.

I strove to develop student understanding that while the grading scheme was very forgiving, they would need to make presentations. Most presentations would be done at the board and that they could not obtain help from any individual or outside source. In spite of my emphasis on solitary work, I had one thirty-seven-year-old speak with me concerning getting help from a tutor/grader. With large classes (more than 25), I sometimes allow a math major to listen to proofs much as a grader scores papers for the teacher. In those situations, I will give rules to the math major as a designated listener. These rules must be followed. This has worked moderately well in the past.

Presentation Grade. Each presentation (the number is unspecified and not known by the students) earns one of the following marks:

√–: Some engagement, but substantial questions remain
√: A well reasoned argument, but some questions remain
√+: A complete and thorough explanation, no further questions

Presentation grade will be computed as follows:

- D: eventually earn at least a √ on 50% of the presentations.[1]
- C: eventually earn at least a √ on all of the presentations.
- B: eventually earn at least a √ on all the projects, and a √+ on 40% of the presentations.
- A: eventually earn at least a √ on all the projects, and a √+ on 80% of the presentations.

Note: A √– is cancelled by a √+ (so one √– and one √+ is the same as two √s. Each in-class presentation is worth an extra +.

Synopsis. As is my usual practice, in an attempt to allay fears, I told a story about a previous student who had made a *D* in the same course the previous semester and needed an *A* in my course (but, not taught by the Moore Method) to graduate. Talk about Pressure with a capital P! To add to the challenge, later in the course, he informed me he had never made an *A* in a math course at any level. Generally, I attempt to communicate to the students that I strive to run a humane course where their concerns are addressed but that the integrity of mathematics is maintained. I emphasize that if they work hard and have a positive, can-do attitude they have a high probability of "success." The student did receive an *A*.

I gave some background on the notes. Among other things, the notes were more like Euclid's *Elements* than a modern day geometry course such as they had in high school. David Hilbert while maintaining the spirit of the *Elements* attempted to fix some of the "flaws." From *Monographs on Topics of Modern Mathematics*,[2] it appears that Moore and Oswald Veblen had a part in developing *Synthetic Geometry*. I told the class that after the first three sets of notes, we would start geometry in earnest. Then, we would begin applying the Moore Method exclusively.

Class 2. (Introduction to Course, Part II) Notable Observations[3]

Because I had trouble with my voice, I continued with motivation into a second class period. Usually, one day is sufficient.

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[1] A presentation is a proof of a theorem, a solution to a problem or an answer to a question, all original to the student.

[2] *The Foundations of Geometry* by Oswald Veblen found in *Monographs on Topics of Modern Mathematics*, J.W.A. Young, editor, Dover Publications, 1955.

[3] Class 2 Bullets

- **Roll/Time**
- **Perspective and motivation**
    - o Have FUN and RELAX!!!!!
    - o The course is different!
    - o You can be successful here where you haven't before. Explain.
    - o Moore Method (used for 70 years, film, R. L. Moore, etc)
    - o Swim analogy and Coppin's story and anything else that strikes me to change their perspective.
    - o Speak to their fears
    - o Hard work and positive attitude will guarantee success.
- **Initiate Discussion**: *Language*
- **Next Time:** Read and study *Language*

Again, I stressed that they should relax and have fun. This course was different because of its design. Many of them will be successful in this course in ways they would not have believed. The course will invert the emphasis on information and techniques over process that is standard fare in most mathematics classes.

In the spirit of tradition, we discussed Moore, one of my teachers in graduate school. He is the "Moore" of the Moore Method. During his prime, he was one of the top ten mathematicians in the world.[4] What is seemingly a paradox, Moore was considered to be one of the top mathematics teachers in the twentieth century. Moreover, he was a member of the prestigious National Academy of Sciences. It is rare indeed that one mathematician be a superior researcher *and* a superior teacher, a true teacher-scholar, worthy of emulation in the "publish or perish" atmosphere found in much of academia. We would view *Challenge in the Classroom,* a film about Moore and his teaching method. I believe that if students understand the origins and culture of the Moore Method, they will have respect for the method and, therefore, be more successful.

Attempting to speak to the pervasive feeling of mathematical inadequacy that exists among liberal arts students and the fact that most students do not know how to study hard and smart, I related the following three stories:

1. On entering as a freshman at the University of Texas in the fifties, Mr. John Worrell was told by a counselor that he should not major in mathematics. Apparently, Mr. Worrell did take that counselor's advice and went on to become a physician. While practicing medicine in Denver, Colorado, he decided that he wanted to return to the University of Texas to study mathematics under Moore. The story goes that Mr. Worrell, after only eighteen credit hours of graduate mathematics; finished his doctorate under Moore. Now, Moore's doctoral students were quite exceptional with two named members of the National Academy of Sciences. After, Mr. Worrell's oral defense of his dissertation, Moore was reported to have said, "I don't know but I think he (Mr. Worrell) is the best student I have had."[5] John Worrell went on to have two successful careers, one in medicine and one as a mathematician.

2. When I entered graduate school, I did not know how to prove theorems, or so I thought. In my first year, my wife and I and another couple lived in the two apartments on the first floor of a two-story house converted into a four-unit apartment house. The man, who I will call Ed, wanted to get me started doing proofs. He gave me a theorem to work on. I did not do so immediately. You see, I was not receptive to this near stranger giving me extra work; I was already having a rough time keeping up with my studies. However, it seemed like every day, he would poke his head through the open top half of the Dutch door to our apartment, asking if I had worked on the theorem. Just to get him off of my back, I started work on a proof of the theorem. Eventually, I had what I thought was a proof. He did not like it! I created another proof but with identical results. What a shortsighted man! Now, you have to know that I am a very stubborn person myself. I thought my style of proof was acceptable and that my proofs were actually correct. I thought that he was very rigid and inflexible. My thinking was

[4] This was communicated to me by Ralph Whitmore, a mathematics professor at Southwestern University circa the decade of the 1960s.

[5] The moral of this story is that one should not judge mathematical talent by some standardized test or even performance in previous mathematics courses.

that I was open to his way of using language but he was closed minded to my style. However, I wanted to get him off of my back. I had a plan. I had noticed how he used language. It was simpler and more stilted than my own or so it seemed. Therefore, I created a proof using Ed's language style. He accepted this proof and I was very happy. However, I learned a surprising thing. The proof I created was actually much clearer than I thought a proof could be. This experience reminded me of the time I got my first pair of eyeglasses when I was in the third grade. The elementary school I attended had been attempting to get my family to get me glasses since first and second grades. But, I did not carry the message home until the third grade. I just did not see the need. However, I humored my mother and the optometrist and submitted to the indignity of becoming "four eyes." But, even as the doctor put the new pair of glasses on my head, I still believed this act to be fruitless. The doctor asked me to look out the window at a tree situated diagonally from the second floor of the doctor's office up on the street corner. Boy, was I a surprised little boy! I did not know you were supposed to actually see the leaves on a tree at that distance, just as I did not know that one could create a proof or any explanation, for that matter, that would be so clear! This is what I want for my students. I want them to have the power to see and create proofs on their own. As a result they will see mathematics for the first time.

3. To communicate the importance of a student doing proofs or solving problems on his or her own, I use the metaphor of the Monarch butterfly that appears in my essay in "What is the Moore Method?" The butterfly becomes strong by struggling to emerge from its pupal home. Similarly, the student of mathematics must encounter great resistance in the form of challenges. The resistance that learning to do proofs offers does wonders for the education of the mathematics student. And, at the risk of stretching this metaphor too far, under this method of teaching, students have the opportunity to become transformed mathematically.

Class 3 Notable Observations[6]

To explain logical implication, I used an analogy of a washing machine guarantee.[7] Interestingly, students understood that this analogy was like the liar's analogy[8] we had

[6] **Class 3 Bullets**

- **Roll/Time**
- **Implication:** Go over two handouts quickly.
- **Explain:** Go over the three example proofs on page three of *Language Notes*.
- **Next Time**: Bare denials

[7] "Suppose you buy a washing machine, and it has a guarantee. Essentially, the guarantee says: "If the machine breaks, then a repairman will come to fix it." (or, "The machine breaks implies that a repairman will come and fix it.") Let p be "the machine breaks" and let q be "the repairman will come and fix it." Then, to say that a guarantee is not valid is to say that the statement $p \Rightarrow q$ is false. The only way that the guarantee can be invalid is when the machine breaks (p is true) and no repairman comes to fix it (q is false)." From *Introduction to Mathematics* by W. Barnier and N. Feldman. 1990, Prentice-Hall.

[8] Under what circumstances is the statement, "if P, then Q" true? Or, a better way is to ask when is "if P, then Q" not true? When does someone lie using implication? It occurs when he or she says "I will do such and such" and then does not do "such and such." That is, the implication "if P, then Q" is false only when P is true but Q is false. Thus, "if P, then Q" is true only in case one of the following holds:

P and Q are both true,
P an Q are both false
P is false and Q is true.

discussed previously. However, the washing machine analogy was more convincing.

We moved on to discuss three proofs on page 3 of *Language and Logic*. We discussed the use of symbols in language. I even restated the first theorem where I used the symbol "$\forall$" for the phrase "for each." I explained how I prefer the English to the shorthand. At some deep level, the words, "for each" carry more meaning than the sterile symbol "$\forall$." To exaggerate the point, I stated the definition of limit as follows:

$$\exists L \ni \forall \varepsilon > 0 \exists \delta > 0 \ni \left(0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$$

In order to emphasize the power of geometric language, I translated this statement into English followed by the horizontal line-vertical line definition. Then, we did the proofs.

First Proof. The class collectively discovered the proof, as we would not begin the pure Moore Method for a few class periods. Universal quantifiers were emphasized. For example, when we would say $a + y = y$ for *each* number y that since b *was* a number, we could conclude that $a + b = b$. Similarly, $b + a = a$. This discussion was deep and long in both classes.

Counter examples were covered but students find these easy to understand.

Second Proof. The class was expected to parse the statement of the theorem to determine which part was hypothesis and which part was conclusion. The following resulted from class collaboration:

Given:

1. $w \geq 0$
2. $w < c$ for each positive number c.

Prove:

$w = 0$.

We stated $w \geq 0 \equiv (w > 0) \vee (w = 0)$. We reviewed the conditions under which disjunctions are true and the fact that we do not know that $w = 0$. If we could prove $w > 0$ were false then we would be done. This discussion should not be scripted but should evolve from class discussion. My goal was that almost everyone should understand some proof at the end of the day. Students should leave the room generally encouraged.

Class 4. Notable Observations[9]

We discussed bare denials (negatives), emphasizing that a bare denial is minimally a negative. Students make the mistake of "over-negating" a statement. A good rule of thumb is to move the negative, "no" or "not", as far to the right in the sentence as possible. It is easy to say that if S is a statement, then its negative is "it is not true that S is true." Although correct, it is not very useful. This becomes clear as the course develops.

I asked the class to give bare denials of the following:

[9] **Class 4 Bullets:**

- **Roll/Time**
- **Read:** Poincaire article for next time, to discuss the part the subconscious plays in the creative process.
- **Bare Denials:** First, give a general idea of negation; start simple; apply to conjunction, disjunction, existential and universal quantifiers.
- **Next Time**: Initiate *Naïve Set Theory*.

Statement	Class Response
Grass is green.	Easily done.
Grass is green and roses are red.	Students would make the mistake of saying "Grass is not green and roses are not red. "Most understood that "and" should be "or" but many did not.
Grant is angry or Sally is happy.	Ditto here.
All fish have fins.	Easily done but some wanted to say, "All fish do not have fins." One student proposed that we consider the statement that "All fish have fins and all fish can swim."
Each horse has a head.	Done well—some students wanted to offer up alternatives—good discussion ensued.
Some cows have horns.	Done well.
If a student is enrolled at the University, then that student attends some class.	This one took lots of discussion. Many students wanted to say, "If a student is enrolled at the University, then that student does not attend class." Eventually, someone got this one.
If P is a point, then there is a line containing that point.	Done well.
There is a line containing a given point.	Ran out of time.

We attempted to have a thorough/unhurried discussion of each situation sometimes using diagrams or visualizing physical situations.

Class 5. Notable Observations[10]

We discussed *Mathematical Creation* by Henri Poincaré. The class gave their own impressions first. I then amplified or added to their remarks. I focused my part on the following:

- Poincaré's opinion that not all can do mathematics.
- The work of the subconscious in the intellectual lives of the students. Many recognized the phenomena in themselves.

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[10] **Class 5 Bullets:**

- **Roll/Time:** Have students indicate on the roll sheet if they had read Poincaire's article.
- **Discuss:** Poincaire's article. Starter question is: What did you learn?
- **Set Theory:** idea of set, notations($\cup$, $\cap$, $\in$), and equality of sets (do a proof by me or student)
- **Next Time**: *Naïve Set Theory*, proofs

- We discussed discernment and emotion.

Students asked many questions but the following were most noteworthy. Hopefully, the essentials for proving theorems are covered. Philosophical issues were discussed with the aim of moving students from preconceived, erroneous notions of mathematics. Opinions and philosophical stances should be nurtured and respected.

- What is the empty set? It would seem contrary to what one would mean by a set. I usually will pick a student and say, if I believe in the empty set, Mr. *X* has a coin collection. I can say that even if he does not have a coin collection in common terms. I wanted the students to feel free to challenge the conventional practices.
- We discuss that the number 0 and negative numbers were heretical at one time. I used Isaac Newton's "ghosts of departed quantities." I referred to John Leslie (Olsen, 1971), a Scottish mathematician who did not believe that one would actually have to have negative numbers. He advocated the use of Synthetic Geometry.

Class 7. Notable Observations[11]

My 10:00 A.M. class did not do as well as the 11:00 A.M. class.

10:00 A.M. Two students put up Theorem 1 in the first class but only one even wanted to try Theorem 2a. In the latter case, the student had a semblance of a proof. We talked about syntax errors in her proof that stalled her whole thinking process. She had an idea that could work. I gave some direction and asked her to work on it some more.

11:00 A.M. The class did Theorem 1 without fanfare and did Theorem 2a well. One student could understand that $p \Rightarrow (p \vee q)$ is a true statement. I solicited some help from the class to nudge him toward an understanding of his problem. When he finally realized what was going on he saw what he needed to do and finished the proof. He then proceeded to tell me that next time I should tell him he had an error and not take so much class time. My judgment was that we needed to discuss this point as we did, that situations like this will arise in the future and the student needs to adapt to the class format. The situation was sufficiently subtle that I could not determine whether he really understood the issue.

Essentially, I had two objectives:

- Get students used to going to the board.
- Discuss proper syntax and different styles of proofs.

It is imperative that students at the board learn the format of board work and how to take criticism. Students in the class must learn how to give criticism without directly aiding the presenter.

[11] **Class 7 Bullets**

- **Roll/Time:** Ask students to sign up for Theorems 1, 2a,b,c on the roll sheet.
- **Handout**: Axiom Systems
- **Student Presentations:** As a minimum, Theorem 1(two students) and Theorem 2a.
- **Axiom Systems**: Set up for student presentations.
- **Next Time**: Finish giving example proofs and start *Axiom Systems*.

I sometimes find it useful to get two or more people at the board at a time. It takes some pressure off when two are presenting. It also increases the probability that one of the students will have a proof. Student mistakes provide fertile teaching moments. As much as possible, I attempt to affirm the student's style of proof but, also, I discuss a mathematician's style. I read both proofs as they are written on the board and allow the one who finishes first to present first. If one presenter's proof appears worse than the other's proof but the student is emotionally strong, then I allow that student to present first and then finish with the proof that is essentially correct. If one student is really sensitive to criticism and his or her proof is shaky, I may spend a lot of time on the other, if I feel I can have a robust conversation with the person. When I go to the sensitive one, we need to get well into the presentation but "run out of class time." Then, that student can reflect on what she has learned at the board. Next time, she will hopefully have a passable proof.

> Professor William S. Mahavier tells the following story. One day, when Mahavier was in one of R. L. Moore's classes, he was at the board proving a theorem. Moore started talking about the squirrels outside the classroom. He continued this way until the end of the period. I believe that Mahavier was upset at that point. What had happened is that Moore had discovered a hole in Mahavier's proof but did not point out the error himself. He just let the class end. Mahavier went home, started working on the proof again but this time he discovered the flaw. He patched up the proof and finished it the next class. Moore must have sensed that it would not be beneficial to point out the error but let Mahavier discover the flaw himself.
>
> Coppin

I attempt to get the class to expose errors or flaws in proofs at the board. This day, the class made a good start, getting a total of seven students up to the board in both classes. Five were successful. I decided to spend more time on the Theorem1, 2abc. The fact that the first class seemed clueless on Theorem 2a told me we needed some discussion in both classes concerning the nature of proofs.

Class 8. Notable Observations[12]

JB [11:00] proved Theorem 2c using a method much like a proof using statement calculus. However, to my displeasure, her approach was very mechanical. I yearned to make the point that when a proof is presented, we, the listeners, would like to hear its "music" and not only the "lyrics." A proof should be like a beautiful song: great lyrics *and* a great tune. However, I chose not to risk embarrassing her but decided that next class period, we would put two other proofs on the board along with hers. It was my hope that the budding mathematical discernment of the students would perceive the differences between these proofs without my overt intervention.

[12] **Class 8. Bullets**

- **Roll/Time:** Students may indicate they have Theorems 2a [10:00AM only], b, c on the roll sheet.
- **Explain**: iff
- **[10:00AM] Student Presentations:** Students will present Theorem 2a,b.
- **Set up**: *Axiom Systems*
- **[11:00AM] Student Presentations:** Students will present Theorem 2b,c.
- **Questions**: Set up last class period. Do students have questions concerning *Axiom Systems*.
- **Next Time**: Students will present 1 and 2, page 8, of *Axiom Systems*.

Class 9. Notable Observations[13]

I reminded the students of two very important policies of the course:

- The Absence Policy. If the students do not attend every class diligently, then there is a tendency for the teacher to re-teach materials others have already seen. If there is too much unwarranted re-teaching, then students will become bored and skip classes. Moreover, I have observed that, too many times, when an important teaching moment is about to occur, the student who needs it is absent. Therefore, to address absenteeism, I do one of the following: (1) If a student has incurred four unexcused absences, I drop them from the course. (2) For each three unexcused absences I drop the student's grade by one-third of a letter grade. For example, *B*+ drops to *B* or *C* drops to *C*–.
- Solitary Proofs. One of our rules is that proofs must be solitary. It is important that in doing proofs, all students should do proofs on their own without the aid of a person or a book. This is a matter of pride. When you get a proof in a group, did you get the main idea or did someone else? Just as it is important that you learn to walk on your own, you must learn to do proofs on your own. Only then, would they lead rather than just follow someone else's proof.

My plan was to have JB, ST and Coppin present proofs of Theorem 2c and the students compare the three different proofs. Moreover, I planned to give the class the opportunity to ask questions concerning the notes, *Axiom Systems*.

However, nothing eventful occurred. Students did very well with problems 1, 2 of *Axiom Systems*. I wondered if these problems were too easy. I should have had other models ready.

Unfortunately, ST was absent. Her presentation of Theorem 2c would have been very interesting compared with JB's more mechanical proof. However, I detected that JB might be aware that her proof is superficial. I say this because she asked if she should present a proof that is more "English."

Class 10. Notable Observations[14]

I presented to the class the authentic Euclid's first five postulates. I especially focused on the following:

[13] **Class 9 Bullets**

- **Roll/Time:** Ask students to sign up for Theorems 2c [10:00AM only] and Problems 1, 2 on the roll sheet.
- **Reminders:** Reiterate absence policy and that attendance is essential to learning. When they are absent, they have lost an opportunity to make a break through. Moreover, I suspect some students are working together and that they must "Do proofs on your own."
- **[10:00AM Class] Student Presentations:** Students present Theorem 2c. *Axiom Systems*: 1 and 2 page 8 of *Axiom Systems*
- **[11:00AM Class] Student Presentations:** As a departure from the routine, JB, ST and I will present Theorem 2c up on board. Compare the different styles of the same proof.
- **Field Questions**: Deal with questions about concerning the handout, *Axiom Systems*.
- **Initiate**: Neutral Geometry and the film, *Challenge in the Classroom.*
- **Next Time:** History, Film

[14] **Class 10 Bullets**

- **Roll/Time**
- **Introduce Neutral Geometry**: Explain the fifth postulate, negation of parallel postulate, neutral geometry, and the three geometries.

- The implicitly undefined terms in *Euclid's Elements*.
- Euclid's fifth postulate and the fact that it appears to be a theorem rather than an axiom. It took over 1500 years before there was any resolution.
- How the negation of Euclid's fifth postulate led to two new geometries in addition to Euclidean Geometry.
- The Greek philosophy of mathematics circa 500 B.C. versus the formalist school of the early twentieth century.

We viewed *Challenge in the Classroom.*

Class 11. Notable Observations[15]

Discussed *Challenge in the Classroom.* My main reason for doing this is to continue to acculturate the students to the Method. I warned the students that Moore states that competition is "the main feature" of his method. He stated that he wanted students to want to get Theorem X and to want no one else to. I want to let students know that is not true of this class, quite the contrary. I do not think the "cut-throat" type of competition will be well received by students of today's culture.

Class 12. Notable Observations[16]

- In-Class 1.[17] This was quickly done. Several knew the correct answer. Many seemed to have the proof. The only glitch was a tendency for some students not to apply Axiom 3 in the beginning of their proofs.
- In-Class 2. Several knew how to do this one. Students did this one at the board.
- Problems 1 and 2. Several had each of these.
- In-Class 3. We skipped this one for the lack of time.
- Generally, in creating models, I discovered that I had to reinforce the notion that students must supply specific, concrete meaning for the undefined terms, "point" and "line." They must carefully articulate which axioms are satisfied and explain why. Some students were bothered that two points could belong to two different lines and that it was possible for some point not to belong to any line. It was good they expressed their concerns because those concerns were well founded. Their observa-

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- **Initiate Viewing Film:** *Challenge in the Classroom* (20–25 minutes)
- **Next Time: Continue student** presentations and finish viewing *Challenge in the Classroom.*

[15] **Class 11 Bullets**

- **Roll/Time**
- **View Moore Film:** Allow 30 minutes to finish *Challenge in the Classroom.* Start at the top of the hour.
- **Discussion of Film:** Just get students talking but guide in the direction of the "learning culture."
- **Next Time**: Continue presentations.

[16] **Class 12 Bullets**

- **Roll/Time**: History
- **State and discuss:** In-Class 1. After discussion, state Theorem 1 just proved. In-Class 2.
- **Student Presentations**: Present Problems 1 and 2.
- **State and discuss:** Axiom 4. In-Class 3.
- **State and explain:** Theorem 2, Problem 3.
- **Next Time:** Student presentations of a proof of Theorem 2 and students settle Problem 3.

[17] Problems with the prefix, In-Class, are designed to be taken up in class and not assigned. These problems were designed for in class discussion and collaboration only.

tions set up the need for additional axioms. If Axioms 1, 2, 3 were all sufficient, then we would have a very boring and useless axiom system. We would add additional axioms as we progressed through the course allowing the material to become increasingly enriched. Ultimately, we will have a geometry that is familiar to them.

- Students remained confused about what they could and could not use. One student even called the statement of a particular problem an axiom. I must reinforce what can be legally used and what is the nature of an axiom system. Students use the term "straight line." I explained that use of the word really is misleading. Euclid used it but he did not define the term. I mentioned the idea of curved space where what appears to be "straight" to us may be curved to an outside observer of our physical world.
- Some students attempted to create "lines" that did not exist. They did not how know to "play the game"—not yet.
- It was a very good day—lots of good work was done and we covered a good deal of ground. Realistically, I have to assume many are lost or on the verge of being lost. However, many students are really getting the hang of the material and its processes.

Class 13. Notable Observations[18]

The entire class period was spent on Theorem 2. The students made typical errors but it still surprised me that after all this time they had so much trouble getting started. The proof should start with "Suppose *P* is a point." This day I had no indication that anyone knew how to start the proof even though we had had a similar situation with proofs on set theory. There were two basic false proofs:

- Apply Axiom 3. Obtain a line. Then, apply Axiom 1 to get a point on *L*. That point becomes the arbitrary point.
- Apply Theorem 1 to obtain three points. Call one of them the arbitrary point *P*. Then, use another of those points along with Axiom 4 to obtain a line through *P*.

I would use metaphors or practical situations to demonstrate errors in logic.

- Worse case scenarios and drawings were used.
- Models developed earlier in the course, which did not have the property they desired but still fit their logic.
- Use "randomness" as a metaphor for "universal." "You have qualified your arbitrary point *P*."

Much time was taken with students at the board. Two or three students had to be asked to be seated, which I always find hard to do. One young man seemed almost arrogant—wanting to be told exactly where he was wrong. When he found out his errors, he thought he almost had the proof and what was missing in his proof was minor. This last sentiment is typical.

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[18] **Class 13 Bullets**

- **Roll/Time**
- **Student Presentation:** Theorem 2
- **Next Time:** Settle Problem 3.

We ended up with three good but very different proofs. JB's was especially nice.

I repeated what was expected in the course and that students were not to help each other directly. Only critical questions were acceptable. Suggestive or leading questions were unacceptable.

Class 14. Notable Observations[19]

Next class was supposed to have been done today but students had difficulty with Problem 3. Many went to the board but no one was successful. I forced some to the board but I am not sure I will do that again for a while. I almost always give students one of the following choices:

- Work with me at the board.
- Sit down.
- Go in the hall and work more on the problem.

The issues were the same as before. Although I had many correct proofs after class—we just ran out of time. I must *be patient*—"*IT"* will come. Keep encouraging the class and reiterate what we are doing and why.

Class 15. Notable Observations[20]

We finally had a correct presentation of Problem 3. AW did a great job! She was one I had worried about last time—not wanting to be at the board. She had said, "If I have to, I will present next time." I had gone to her to give her the opportunity not to go to the board—she was much happier today. I offered to look at her proof ahead of time—told her that it was correct but there were a couple of points I would question her on. In fact, one of the students beat me to the punch—I liked that. It was good I had warned her—it was good a classmate caught her errors—even though some students may have thought her classmate was just being picky. This was a great day!

Class 20. Notable Observations[21]

I learned a lot from my class today. I apologized for botching Part 2 of the examination. I encouraged them by telling them that they had done better than any other class to

[19] **Class 14 Bullets**

- **Roll/Time**
- **Tell Story:** Bill Mahavier squirrel story.
- Student Presentations: Problem 3

[20] **Class 15 Bullet**

- **Roll/Time**: state on the roll sheet if you have started your portfolio.
- **Student Presentation (second try):** Problem 3 (these credits were given last time but I want to get/give second try)
- 10:00 JR(up/not/hall)(office credit), AW(up/not)
- 11:00 SJ(up/not/hall/in), KM (after class credit), ST(after class credit), PH(after class credit)
- **State and discuss:** In-Class 4.
- **Skip:** In-Class 5.
- **State and discuss:** Axiom 4'.
- **State and Discuss:** In-Class 6 (QUICKLY).
- **State:**. Theorem 4, Problem 4.
- **Next Time:** Prove Theorem 4 and settle Problem 4.

[21] **Class 20 Bullets**

- Go over Examination 1

which I had given that exam. I thought that we should have taken the exam on Wednesday rather than Friday. They were fatigued from taking midterm exams. The following were my reflections:

1. They are having a "double predicate" problem. The definition of collinear defines "a set is collinear" but the students got confused and thought "a set is collinear with another set." This is where the problem lies with Part 2- I-1 -2.
2. They are still having trouble with bare denials. Therefore, before starting Part 2, I had them write down the definition of collinear and its negation. I talked about what components of the statement to focus on especially "if P" can be thought of as "for each P" even though the two are technically different. I warned them off "formulaic" negations.
3. They had conflicting/contradictory argumentation. For example, on Part 2 I-1, they applied Theorem 2 but went on to prove there is a point that is not collinear by proving there is a line not containing the point.
4. I should have put easier problems on the exam, i.e., I should have had them negate Axiom 4 or something even easier than Axiom 4'.
5. I should not have introduced "finite" for the first time on the exam.
6. I modified the last problem by replacing "finite" with "six." They could do it for presentation requirement.
7. I think the class will focus a little more. As long as I do not overdo it I think an apology from the teacher helps. I said that the poor performance was a collaborative effort between teacher and students.

Class 23. Notable Observations[22]

Students in both classes knew to apply Theorem 7 to prove Theorem 8 and reversed Theorem 8 for a proof of Theorem 9. One student knew to do a substitution in Theorem 8 to get Theorem 9. One of my best students in the 11:00 a.m. class was a little reluctant to go to the board because she had a proof of a conjecture that was not true. She discovered her error while at the board. I gave her credit for proof of c. A discussion of an error and a proof patched up conjecture B. Both classes did very well!

Class 24. Notable Observations[23]

10:00 A.M. Class

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[22] **Class 23 Bullets**

- **Roll/Time**
- **10:00AM**
  - **Students Settle:** Theorem 8, Theorem 9
  - **State**: Problem 7
  - **Next Time**: Students present Problem 7
- **11:00AM**
  - **Students Settle:** Theorem 9, Problem 7
  - **State**: Theorem 10, 11 (if Problem 7 done), Theorems 12, 13
- **Next Time:** Theorems 12, 13

[23] **Class 24 Bullets**

- **Roll/Time — No Class Friday as I will be giving a talk in Houston**
- **10:00AM Class**

- SM finished Theorem 9 with difficulty but she did finish after much questioning by the students and me. Her spirits were very good—I had spoken with her after class, just to be sure she was okay. She even thanked me for my questions. One interesting fact is that she had made one of the high grades on Examination 1 but she had trouble at the board. Contrast that with low scoring students (SL and CN) who did very well at the board one class period after receiving their grades.
- I left Theorems 11 and 13 to be turned in for credit.

11:00 A.M. Class

- SR had informed me during the preceding class that she did not want to go to the board. I worked with her in my office later on Theorem 12. I was surprised today when she wanted to go to the board. Her proof was rough and too formulaic. She had problems with the application of Axiom 7 and disjunction; however, she finished with some good work. I must remember to encourage her. I asked her to write up the proof of collinearity and proper application of Axiom 7.
- I spent a great deal of time setting up Problem 8. I told the students that if they have felt shaky or they were confused up to this point in the course, maybe they should work on the less open-ended theorems. However, at the risk of sending a confused message, some may want to attempt problem 8 even if they are having difficulty getting proofs. I told them that a determining factor might be that if my talk has scared them that they ought not try Problem 8; however, if my talk almost "dared" them to attempt the problem then they ought to try it. I discussed the problem by purposely not drawing any pictures. I told them I wanted them to create their own pictures mentally; that would help them create the formal definition. What I did was to discuss by class interaction set theoretic phrasing of the four properties. I was very frank about why I did not draw a picture. At the end of class after SR's presentation, AB (undecided and a freshman) and ER (theology and a freshman), came up to me after class with an approach that almost worked. I was thrilled although I probably did not show it. I encouraged them—told them that they almost had it and gave some direction as to why. I told them to work on it and improve their definition. They certainly did have the correct idea!

Class 30. Notable Observations[24]

I announced "listeners" and what counts as presentation when a "listener" gives credit. I explained the rules that "listeners" must abide by. "Listeners" can be students in the class

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  - **Students Settle:** Theorem 9 (SM finish), Problem 7
  - **State**: Theorem 10, 11 (if Problem 7 done), Theorems 12, 13, Problems 8, 9
  - **Next Time**: Theorems 12, 13, Problems 8, 9 (talk about)
- **11:00AM Class**
  - **Students Settle:** Theorems 12, 13
  - **State**: Problem 8, 9 (talk about)
  - **Next Time:** Problem 8, 9

[24] **Class 30 Bullets**

- **Roll/Time** — Student Notes
- **Announcement**: Examination 2 Monday April 22, 2001 (in Class/Take Home)
- **Review**: Worksheet Template
- **Counsel**: Talk with students one-on-one concerning their standing in class and what they need to do.

who have made so much progress that I can use them to listen to a classmate's proof of a theorem or a solution to a problem as long as they as they follow the following rules:

1. Before listening to a proof, discern the key to the proof. As you listen, "protect" the key.
2. You may tell the student whether proof is correct or incorrect but do not devulge more than that as far as the correctness of a proof.
3. Always look for something positive to say or some way to encourage.
4. Students who are having a very difficult time of developing a proof should see the instructor.

A "listener" can be someone who has a great deal of mathematical maturity who may have a role analogous to a grader. Thus, in addition to the above rules, this higher level of listener would have a bit more latitude and would follow the additional rules below:

1. When asking students critical questions don't ask the questions in such a way as "to hound" the students to discover a proof. This is demeaning. Craft your questions very carefully. A few structured well placed questions in a dialogue can do more good than many "micro-managed" questions. The latter tend to lead the student to a correct proof; however, the student may not have learned deeply and could feel demeaned. Manage the environment with just the right questions so that the student will at least make some progress. End the session on a high note.
2. You may ask critical questions of the sort you would ask in a Moore-method class you have previously taken.
3. Document the extent to which you may have helped the student and the nature of the help.
4. Fear is the most significant barrier to doing mathematics especially proofs or problem solving. Another impediment is that the student may have had previous experiences in mathematics that worked against proofs. Intellectual personality is a factor. The way students use their natural language is another factor.

II
Mahavier

II.A Syllabus for *Analysis*

This is the syllabus that meets the University and Southern Association of Colleges requirements and I pass it out after the first few class meetings.

Course: MATH 3380, Advanced Calculus, Fall 2007, MWF 11:15–12:05, Lucas 114

Requirements: Introduction to Advanced Mathematics or permission of instructor

Instructor: W. Ted Mahavier, Lucas 200E, 409.880.2290 (office), 409.347.1809 (home), 985.381.0292 (cell), wtm@mathnerds.com

Office Hours: MTWRF: 9:00–11:00, 1:30–3:00 or stop by my office anytime.

Course Materials: All materials provided by instructor. See also Honesty Policy.

Course Subject Outline: limit points, sequences, continuity, differentiability, and integration theory for functions of one real variable, culminating in the Fundamental Theorem of Calculus

Student Learning Outcomes: The two primary objectives for students in this course are (1) to master the material listed in the Course Subject Outline and (2) to develop their ability to create mathematics, to make conjectures, to test hypotheses, and to prove theorems. Additionally students will hone their mathematical writing skills, their ability to follow the proofs of other students, their ability to question the work of other students, and their ability to communicate mathematics orally.

Attendance Policy: You are responsible for everything that goes on in class regardless of the reason for an absence. I reserve the right to drop any student for three unexcused absences.

Course Goals and Objectives: To actively involve the students in the process of learning the mathematics provided while developing their appreciation of the subject. Building on Math 3322, students will continue to learn proof techniques and principles of analysis while making proper connections with the study of continuity, derivative, and integral that were begun in Math 2413 and Math 2414. Develop an understanding of the significance of the Completeness Axiom and its connections to the Heine Borel Theorem and the Bolzano-Weierstrass Theorem. Study the theory of continuity, derivative and integral in more general settings.

Grading: Your grade will be no less than the average of three grades: (1) your presentation grade, (2) your written work grade, and (3) the average of your midterm and final.

Honesty Policy: Please see Student Conduct Code, Lamar University Student Handbook pages 67–80. You are not allowed to look on the web, to other texts, to other faculty, or to other students for help without my explicit permission. The sources of aid should be your intuition, hard work, the materials provided, and me.

Make-ups: Make-ups are given during the last week of classes. Contact me and set up a date for make-ups.

Services for Students With Disabilities: I'll help you in any way that I can according to university policy. For additional information see pages 64 and 91 of the Student Handbook.

Note: I reserve the right to vary from this syllabus and probably will.

I pass this, or a modification of it, out to most of my lower-level modified Moore Method classes. The goal is to set their minds at ease with respect to the presentation aspect. It is a compilation of my own writing, the writing of Robert Kaufman and the writing of Parker.

How This Class Works. You have just entered a class that will be taught in a way that is (most likely) quite different from the way the mathematics classes you have encountered in the past were taught. Much of the class will be devoted to students working problems on the board and much of your grade will be determined by the amount of mathematics that you produce in this class. I use the word produce because it is my belief that the best way to *learn* mathematics is by *doing* mathematics. Therefore, just as I learned to ride a bike by getting on and falling off, I expect that you will learn mathematics by attempting it and (occasionally) falling off! You will have a set of notes (provided by me) that you will turn into a book by working through the problems with the help of me, your classmates, and a lot of hard work on your part. If you are interested in watching someone else put mathematics on the board, working ten problems like it for homework, and then regurgitating this material on tests, then you are not in the correct class. Still, I urge you to seriously consider the value of becoming an independent thinker who tackles doing mathematics (and everything else in life) on your own, rather than waiting for someone else to show you how to do things.

A Common Pitfall. There are two ways in which students often approach my classes. The first is to jump right in and start trying the material. The second is to say, "I'll wait and see how this works and then see if I like it and put some problems up later in the semester after I catch on." These students often have great difficulty. If you *try* every night to do the problems then either you will get a problem (YEA!) and be able to put it on the board with pride and satisfaction or you will struggle with the problem, learn a lot in your struggle, and then watch someone else put it on the board. When this person puts it up you will be able to ask questions that help you and others understand it, as you say to yourself, "Ahhhh, now I see where I went wrong and now I can do this one and a few more for next class." If you do not try problems each night, then you will watch the student put the problem on the board, but perhaps will not quite catch all the details and then when you study for the tests or try the next problems you will have only a loose idea of how to tackle such problems. Basically, you have seen it only once in this case. The first student saw it once when he or she tackled it on his or her own, again when either he or she put it on the board or another student presented it, and a third time when he or she studied for the next test or quiz. Hence the difference between these two approaches is the difference between participating and watching a movie. I hope that each of you will tackle this course with an attitude that you *will* learn this material and thus will both enjoy and benefit from the class.

Boardwork. Because the board work constitutes a reasonable percentage of your grade, let's put your mind at ease regarding this part of the class. First, by coming to class today you have a sixty percent on board work. If you come to class every day, your cell phone never rings, and you participate in the class then your board work grade will not go down. Every problem you present pushes that grade higher. You may come see me any time for an indication of what I believe your current level of participation will earn you at the end of the semester for this portion of the grade. Here are some rules and guidelines associated with the board work. I will call for volunteers every day and will pick the person with the least presentations to present a given problem. You may inform me that you have a problem in advance (which I appreciate), but the problem still goes to the person with the least presentations on the day I call for a solution. Ties are broken either randomly (at the beginning), or by test grades (lower test grades taking priority), or by my own top-secret algorithm. A student who has not gone to the board on a given day will be given precedence over a student who has gone to the board that day. To "present" a problem at the board means to have written the problem statement up, to have written a correct solution using complete mathematical sentences, and to have answered all students' questions regarding the problem. Since you will be communicating with other students on a regular basis, here are several guidelines that will help you. First, the whole class is on your side and wants to see you understand and put the problem up correctly.

Don't use the words "obvious," "stupid," or "trivial." Don't attack anyone personally or try to intimidate anyone. Don't get mad or upset at anyone (and if you do, try to get over it quickly). Don't be upset when you make a mistake—brush it off and learn from it. Don't let anything go on the board that you don't fully understand. Don't say to yourself, "I'll figure this out at home." Don't use concepts we have not defined. Don't use or get examples or solutions from other books. Don't work together without acknowledging it at the board. Don't try to put up a problem you have not written up.

Do prepare arguments in advance. Do be polite and respectful. Do learn from your mistakes. Do ask questions such as, "Can you tell me how you got the third line?" Do let people answer when they are asked a question. Do refer to earlier results and definitions by number when possible. "By Definition 29…"

I pass this out to my analysis classes to help clarify my policies on written work, board work, and grading.

RULES FOR THE COURSE

All work presented or submitted is to be your own. You are *not* to discuss any problems with any one other than myself, nor are you to look into any other reference such as books or the internet for further guidance.

Grading for the course will be the average of three grades: your presentation grade, your submission grade, and the average of your midterm and final exam grades. Anyone who is regularly presenting material at the board will certainly have adequate work for good submission grades and thus will likely do well on the midterm and final. The midterm and final grades are opportunities for those who do not regularly make it to the board. However, it is my experience that those who do not work toward successful presentations rarely do well on the midterm and final. *Thus, I emphasize that the goal of the course for each student should be well prepared, well presented problems.* You will know your grades on submissions and tests. My policy for your presentation grade is:

D = this student made it to class every day, was attentive and alert, and their cell phone never rang

C = requirements for *D* plus made a few successful presentations

B = requirements for *C* plus made numerous successful presentations

A = requirements for *B* plus tackled some of the difficult problems successfully

Turn-ins. You must turn in exactly one "new" problem each week. A "new" problem means one which you have *not* turned in before. This problem should be neatly written and double spaced. You should label this problem with TURN IN at the top of the page along with the problem number and statement.

Grading for turn in assignments and board work will be based on the following scale.

A = This is a correct proof.

B = You *know* how to prove the theorem but some of what you have written is incorrect.

C = You have a mistake in your work or I do not understand what you have written.

D = There is at least one *major* flaw in your argument.

Please understand that the purpose of these exercises is to *teach* you to prove theorems. It is not expected that you started the class with this skill. Hence, some low grades are to be expected. Do not be upset—just come see me.

Resubmissions. If you earn a grade of less than *B*, you may resubmit a TURN IN problem on the following week. You only get one resubmit chance! Please write RESUBMIT at the top. Feel free to come see me anytime if you do not understand my comments. It is expected that a certain amount of time in my office will be required to help students. I prefer to give guidance in my office rather than in class because this allows me to tailor the guidance to the student who needs help. Still, I will always answer questions in class as well.

Boardwork. If you have a problem which is about to be presented at the board or you feel you have made significant progress on a problem which is about to be presented, then you may opt to leave the room for the presentation. In this case, you may turn in a write up of this problem for credit as BOARD WORK. You must write BOARD WORK at the top of the page. There is no limit on the number of BOARD WORK problems you may submit.

Last Comments. Be sure that everything you turn in has your name, problem number, and problem statement on it. Be sure to write either TURN IN, RESUBMIT, or BOARD WORK at the top of each problem turned in. I reserve the right to vary from this syllabus and probably will.

II.B First and Last Pages of *Analysis* Notes, by Mahavier < Mahavier

These notes can be found on-line at The Journal of Inquiry-Based Learning in Mathematics or by contacting the author for a more recent two-semester version. The reason for their inclusion is to illustrate the very low level of the first day of class as we attempt to develop the students' mathematical ability and the depth of the theorems near the end of the course that the students are creating independently.

First page of notes.

Definition 1: By a **point** is meant an element of the real numbers, $\Re$.[1]

Definition 2: By a **point set** is meant a collection of one or more points.

Definition 3: The statement that the point set M is **linearly ordered** means that there is a meaning for the words "less than (<)", "less than or equal to ($\leq$)", "greater than (>)", and "greater than or equal to ($\geq$)" and that if each of a, b, and c is in M, then

- if $a \leq b$ and $b \leq c$ then $a \leq c$ and
- one and only one of the following is true:
  - i. $a < b$,
  - ii. $b < a$, or
  - iii. $a = b$.

Axiom 4: $\Re$ is linearly ordered.

Axiom 5: If p is a point, then there is a point less than p and a point greater than p.

Axiom 6: If p and q are two points then there is a point between them, for example, $(p+q)/2$.

Axiom 7: If $a < b$ and c is any point, then $a + c < b + c$.

Axiom 8: If $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$. If $c < 0$, then $a \cdot c > b \cdot c$.

Axiom 9: If x is a point, then x is an integer or there is an integer n such that $n < x < n + 1$.

Definition 10: The statement that the point set O is an **open interval** means that there are two points a and b such that O is the set of all points between a and b.

Definition 11: The statement that I is a **closed interval** means that there are two points a and b such that p is in I if and only if $p = a$, $p = b$, or p is between a and b.

Notation: We use the notation (a,b) to denote the open interval consisting of all points p such that $a < p < b$. Similarly, we use the notation $[a,b]$ to denote the closed interval determined by the two points a and b where $a < b$. We do not use (a,b) or $[a,b]$ in case $a = b$, although many mathematicians do.

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[1] *I tell them to assume the algebra they know about the real line, that the set of axioms are not complete, and that they may need properties of the real numbers that are not stated. Twice over the years a student has recognized the need for the multiplicative inverse of a real number and we have added the existence of inverses as an axiom, named after the student. Starting without developing the real numbers allows us to go farther and it is almost always the case that a discussion arises later in the course that leads me to pass out a few pages on the development of the real numbers.*

Definition 12: If M is a point set and p is a point, the statement that p is a **limit point** of the point set M means that *every* open interval containing p contains a point of M different from p.

Problem 13: Show that if M is the open interval (a, b), and p is in M, then p is a limit point of M.

Problem 14: Show that if M is the closed interval $[a, b]$, and p is not in M, then p is not a limit point of M.

Problem 15: Show that if M is a point set having a limit point, then M contains (at least) two points.

Problem 16: Show that if M is the set of all positive integers, then no point is a limit point of M.

Definition 17: The statement that p is the **first point to the right of the point set *M*** means that p is to the right of every point of M and if q is a point to the left of p, then q is not to the right of every point of M.

Question 18: Create a definition for "q is the first point to the left of p" by completing the following sentence. "If M is a point set and p is a point in M and q is a point in M then q is the first point to the left of p in M means..." Assume M is a point set such that if p is a point of M, there is a first point to the left of p in M and a first point to the right of p in M. Is it true that M cannot have a limit point?

Definition 19: If each of A and B is a set, then the set defined by ***A* union *B*** is the set consisting of all elements that are in A or are in B. This is typically denoted by $A \cup B$ and written in set notation as: $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$.

Definition 20: If each of A and B is a set and there is at least one point common to A and B, then the set defined by ***A* intersection *B*** is the set consisting of all elements that are in A and are in B. This is typically denoted by $A \cap B$ and written in set notation as: $A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$.

Problem 21: Show that if H is a point set and K is a point set and p is a limit point of $H \cap K$, then p is a limit point of H and p is a limit point of K.

Problem 22: Show that if H is a point set and K is a point set and every point of K is a limit point of H and p is a limit point of K, then p is a limit point of H.

Problem 23: If H is a point set and K is a point set and p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.

Problem 24: Show that if M is the set of all reciprocals of positive integers, then 0 is a limit point of M.

Last page of notes.

Theorem 136: If f is a continuous function with domain the closed interval $[a, b]$, and ε is a positive number, then there is a partition $\{x_0, x_1, x_2, \ldots, x_n\}$ of the closed interval $[a, b]$ such that for each positive integer i not larger than n, if u and v are two numbers in the closed interval $[x_{i-1}, x_i]$, then $|f(u) - f(v)| \leq \varepsilon$.

Theorem 137: If f is a continuous function with domain a closed interval, then the range of f contains only one value or it is a closed interval.

Theorem 138: If f is a bounded function with domain the closed interval $[a, b]$ and for each positive number ε, there is a partition P of $[a, b]$ such that $U_P(f) - L_P(f) < \varepsilon$, then f is Riemann integrable on $[a, b]$.

Theorem 139: If f is a continuous function with domain the closed interval $[a, b]$, then f is Riemann integrable on $[a, b]$.

Theorem 140: Every nondecreasing bounded function on $[a, b]$ is Riemann integrable on $[a, b]$.

Theorem 141: If $[a, b]$ is a closed interval and $c \in (a, b)$ and f is integrable on $[a, b]$, then $\int_a^c f + \int_c^b f = \int_a^b f$.

Theorem 142: If f is a continuous function with domain the closed interval $[a, b]$, then there is a number c in $[a, b]$ such that $\int_a^b f = f(c)(b - a)$.

Theorem 143: If f is a continuous function with domain the closed interval $[a, b]$ and F is the function such that for each number x in $[a, b]$, $F(x) = \int_a^x f$, then for each number c in $[a, b]$, F has a derivative at c and $F'(c) = f(c)$.

Theorem 144: If f is a function with domain the closed interval $[a, b]$ and f has a derivative at each point of $[a, b]$ and f' is continuous at each point in $[a, b]$, then $\int_a^b f' = f(b) - f(a)$.

Theorem 145: If f is an integrable function, then $|\int_a^b f| \leq \int_a^b |f|$.

Theorem 146: If f is a function with domain the closed interval $[a,b]$ and there is a sequence $x_1, x_2, \ldots$ of points in $[a, b]$ such that for each positive integer n, $f(x_n) = n$, then there is a point in $[a, b]$ at which f is not continuous.

Lemma 147: Suppose f is a function whose domain includes $[a,b]$, $f(a) = 0 = f(b)$, and f has a derivative at each of its points. Then there is a number c in (a, b) such that $f'(0) = 0$.

Theorem 148: Suppose f is a nondecreasing function whose domain includes $[a,b]$ and f has a derivative at each of its points, then there is a number c in (a, b) such that $f'(c)$ is the same as the slope of the line joining the two points $(a, f(a))$ and $(b, f(b))$.

Problem 149: Show that there exists a function f that is discontinuous at each point of $[0, 1]$.

Theorem 150: If f is continuous on $[0,1]$ and $g(x) = \int_a^x f$ then g is continuous at each point of $[0,1]$.

II.C Questions and Presentation Guidelines

Only once in 20 years of teaching have I needed to pass out these guidelines to calm a class that did not respond to my early and gentle suggestions regarding respect in the classroom. Two students were socially immature and mathematically talented. They were very direct in pointing out errors at the board and the presenters interpreted the directness of the questions as personal attacks. Another student had failed the course previously and attempted to present material she had seen, but did not understand. She would try to use the "proof by intimidation" method at the board to discourage students from asking questions. This did not work with the students who knew the mathematics was wrong and recognized that she could not explain, defend, or correct mistakes. Several students simply knew and disliked one another from previous classes taught by other faculty. During the first weeks, at the beginning of a class or after an incident, I would make comments about asking questions politely. I would point out how challenging it was to present, defend, and answer questions at the board and that we needed to have respect, collegiality, and patience while students presented. When after three weeks this had not produced the desired culture in the classroom, I put my foot down, passed out the following guidelines, and stated that I would remove from class any student who did not follow these rules. The class proceeded flawlessly thereafter. Around mid-term when I referred to the progress we had made considering the "challenging environment in which we had started," one student responded, "you mean, when we were acting like sharks in a tank?" I laughed, the class laughed, and we went on to complete a successful semester.

Question and Presentation Guidelines

We seem to be struggling with the art of asking questions. That's O.K., there are faculty on campus who haven't mastered the art, yet. Questions should satisfy two criteria. The first criterion is mathematical. Questions should be well posed, clearly stated, and address one specific point. The second criterion is social. Questions should be politely presented and the speaker should be allowed time to respond before the poser or other members speak. Here are some guidelines that I hope will resolve this issue.

Question Guidelines

DO's:

1. First stop the speaker politely by either raising your hand to be recognized or by a phrase such as "excuse me," or "pardon me…"
2. Ask a specific question such as "on the second line, you wrote, $p > x$. Why is $p > x$?" If you can't be that specific perhaps ask "can you say more about the second line?"
3. Before you ask a question, ask *yourself,* "Do I have a specific question?" and "Can I ask it politely?" If the answer is no, **don't speak.**

DON'Ts:

1. Make statements such as "I don't think that is right because…" or "I don't think that is a proof because…" Statements are not questions.

2. Suggest solutions or alternatives like "couldn't we just define p so that…" This is not your proof and demonstrating to me and the class that you can solve it, once someone has put an idea into your head is not relevant to the presentation.

Presentation Guidelines

DO's:

1. Before you offer to present an argument, go over every line of the proof and ask yourself, "Why do I think this line is correct?", "What are they going to ask me about this line?", and "Is everything in this line defined?"
2. Before you offer to present an argument, read the argument aloud—you'll be surprised at what you catch when you slow down to read it to yourself.
3. As you present, offer the class the opportunity to address each statement or line by saying, "is it clear that I've defined a, b, and p at this point?" You'll feel more confident with each correct line and you'll get a feel for any confusion from the audience early in the presentation.

DON'Ts:

1. Ask vague questions like "Is everybody OK to here?" or threatening questions like "Does anybody have a problem with this?"
2. Use words like "obvious" or "trivial."
3. Use concepts or notation at the board that we have not defined *unless* you are prepared to define them.
4. Be upset when you make a mistake; brush it off and learn from it.

III
May

III.A Syllabus for *Introduction to Abstract Mathematics*

SALISBURY UNIVERSITY
DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE
SYLLABUS *(Tentative)*
MATH 300 *Introduction to Abstract Mathematics*

Intended Audience: Students minoring in mathematics, particularly prospective teachers, will find this a good capstone to their undergraduate mathematical experience. Students majoring in mathematics who have not already completed a 400-level mathematics course will find this a valuable course to help them develop a better understanding of the connection between computational and theoretical mathematics.

Objective: To provide students with an opportunity to develop the foundations of abstract mathematics in a manner similar to that employed by professional mathematicians.

Prerequisite: Discrete Mathematics, completed with a grade of *C* or better.

Text: Notes distributed by the instructor.

Foundations, I: The Theory of Sets (2 weeks)
Points and sets. Axioms on sets. Subsets. Operations on sets.

Foundations, II: Functions (3–4 weeks)
Ordered pairs. Relations, functions, and operations. Injections, surjections, and bijections. Binary operations and their properties.

A Deeper Look at the Real Numbers (3–4 weeks)
The integers. Algebraic and order properties. Completeness. Supremum Property. Intervals, decimals, and rational and irrational numbers. Mathematical induction. The Archimedean Property. Recursion. Primes. Finiteness and infinity. Countability and uncountability.

Equivalence Relations (2 weeks)
Reflexivity, symmetry, and transitivity. The Equivalence-Relation Theorem. Elementary functions.

Logic (2–3 weeks)

Conjunction, disjunction, negation, and implication. The rules of premises, conjunction, fantasy, reiteration, implication, disjunction, 0 and 1, and negation. Logical equivalence. Contradiction.

A Peek into Abstract Algebra (0–2 weeks)

Definition of group. First theorems on groups. Cyclic and dihedral groups. Examples of groups: the symmetries of an equilateral triangle; cyclic groups generated by modular addition of integers; dihedral groups.

EVALUATION

Portfolio	10–30%
Presentations at the Board	30–70%
Midterm	0–25%
Final	0–25%

Overview of *Introduction to Abstract Mathematics*

The Subject

This course consists of a study of the concepts fundamental to all of abstract mathematics. Those concepts are sets, functions, the properties of the real-number system, and logic. (For more information about the objectives and content of the course, see the syllabus.)

In this course, we shall learn mathematics as professional mathematicians do. That is to say, we shall prove theorems and raise and test conjectures. The notes that I have handed out (more will follow) form the backbone of our study. Your assignment for the next class is to prove, solely on your own (note: you are never to use a book in this course without first receiving permission from me), Theorem 1.1, and to come to class prepared to explain your proof, at the board, to the rest of the class. If, between today and the next meeting of the class, you prove Theorem 1.1 and have more time that you would like to devote to the course, then try to prove Theorem 1.2; and so on. When we next get together, at least one of the people who have proved Theorem 1.1 will be asked to present his or her proof of it. If we finish with Theorem 1.1. and there is time left in the period, someone who has a proof of Theorem 1.2 will be asked to present that proof; and so on. If there is time and it seems beneficial for me to say something, I will. This is how almost every one of the meetings of the class will proceed.

Administrative Matters

Evaluation of Your Work. Your performance in the course falls into four categories: written work, presentations and other work in class, and midterm and final examinations. The portfolio of your written work must contain your own writeups of proofs (not necessarily your own) of five theorems from the notes. It may contain also exercises from the notes that we have not covered in class. It may contain one or more conjectures and work that you have done – examples, proofs, or commentary, for example – to try to decide whether the conjectures are true or false. The portfolio may also contain journal- or diary-like comments by you on your work, or on what goes on in class.

There is no maximum length to the portfolio. The minimum length is five (not-necessarily-full) pages, one for each theorem that you include. With the exception of mathematical symbols and illustrative diagrams, the portfolio should be typed in at least 12-point font with double spacing. The spelling and grammar should be error-free. Mathematical symbols and any diagram that you choose to include may be hand-written or drawn in the document. (For examples of a style in which to compose your portfolio, consult *Mathematics Magazine*, *The College Mathematics Journal*, *The American Mathematical Monthly*, or some other reputable journal of mathematics.) The portfolio will count as p percent of your final average, where p is a number chosen by you so as to satisfy the conditions to be presented below.

My evaluation of your presentations at the board or in my office, and of other comments that you make in class or in my office, will count as b percent of your final average, where b, like p, satisfies the conditions to be presented below.

The midterm and final examinations will count as m percent and f percent, respectively, of your final average. The numbers p, b, m, and f must satisfy the following conditions:

each of p, b, m, and f is an integer;

p is in [10,30]; b is in [30,70], and each of m and f is in [0,25];

$$p + b + m + f = 100.$$

Let me know your choices for p, b, m, and f not later than the end of the second week of the term. You may change your choices until one week after the middle of the term.

The grading scale for the course is as follows:

90–100,	A;
80–89,	B;
70–79,	C;
60–69,	D;
< 60,	F.

The Integrity of Your Work. I expect you to conduct yourself with honor, integrity, and respect for every member of the class, including yourself. By presenting or turning in a piece of work, you will be pledging that you have neither given nor received any unauthorized help on the work. My response to discovering a violation of this pledge might include, but will not necessarily be limited to, the following:

1. assigning a score of *0* on any offending work;
2. assigning a grade of *F* for the course;
3. reporting each cheater to an appropriate authority, such as the Provost.

Attendance. Regular attendance of, and participation in, class is a very important part of this course. Nevertheless, because I believe that university students should make their own decisions, I hereby declare that attendance of the class meetings of this course is optional, subject to the conditions below.

1. The student and not the professor is responsible for the consequences of an absence. This means, for example, that I will not be obligated to repeat to an absentee material that has already been covered. An exception to this rule is the following: if a

student voluntarily leaves class because he or she wishes to work further on some of the assigned work without seeing others' results, I will go over with him or her anything that was missed thereby.

2. A test from which a student will be or was absent may be made up only when the absentee can convince me, preferably in advance, of the necessity and worthiness of the absence.
3. Assigned homework that is late will not be accepted.

Regarding Learning Styles and Difficulties. There are many styles of learning. Some people learn better with their eyes, some with their ears. There are many effective ways to acquire knowledge. If you have a learning style that does not seem to accommodate well to my method of teaching—in particular, if you have a learning disability—please let me know. If, for example, taking notes in class is difficult for you or hampers your learning, arrangements can almost certainly be made to help you work around this problem.

Some Thoughts at the Beginning of a Semester

I want to help you learn. I will help you with any legitimate need. I will not help you with anything that you need to do for yourself. I want this course to be an enjoyable experience for all of us, and I will do all I can to make it so.

I am making certain assumptions about you. You enjoy mathematics. You are here because you want to be. You want to learn the material of this course, at least to the point of earning a passing grade. You are willing to work and study. You are prepared to do, throughout the semester, at least two hours of diligent work *on your own* outside class for each hour spent in class. (If you do not possess all of these characteristics and you are unable or unwilling to develop them, then you should probably drop this course.)

If at any time you would like to discuss this course, this university, or any other aspect of your life, I would be happy to do so with you.

III.B Midterm Examination for *Introduction to Abstract Mathematics*

General Directions.

Use a separate sheet of paper for your solution to each problem stated below. Make your work neat and legible. Circle your answer whenever possible. You may use, in any proof that you provide, the statement of any of the theorems of Chapters 1 and 2.

You may use only your notes, calculator, graph paper and a straight edge, and a computer (but not the Web) as resources for the exam.

DO NOT DISCUSS ANY ASPECT OF THIS EXAM WITH ANYONE EXCEPT ME BEFORE 1:00 P.M. ON FRIDAY, OCTOBER 14, THE DEADLINE FOR TURNING IT IN.

A. Computational Problems (50 points). You are required to do each of Problems 1 through 5.

1. Denote by $\mathcal{U}$ the set $\{0,1,2,3,4,5,6,7,8,9\}$, by A the set such that $\sim A = \{1,3,5,7,9\}$, and by B the set $\{1,2,3\}$.
   Display each of (a) $A \cap B$, (b) $A \cup B$, and (c) $A \sim B$.
2. Twenty-five people went to dinner. There were two vegetables on the menu, peas and carrots. Three people ordered both, ten had only carrots, and eight ordered neither. How many people ordered only peas?
3. (a) Create a function on the set $\{2,4,6,8,10\}$. (That is to say, display a function the domain of which is $\{2,4,6,8,10\}$.)
   (b) Create a relation on $\{2,4,6,8,10\}$ that is not a function.
4. Suppose that $f(x) = x$ for each number x and $g(x) = |x|$ for each number x. Does $f = g$? Back up your answer.
5. Suppose that A is the set $\{1,2,3,4,5,6,7,8,9\}$, and f is the relation defined by the table below.

x	1	2	3	4	5	6	7	8	9
$f(x)$	8	7	9	3	1	2	6	4	8

   Is f a function? Is f one-to-one? Is f onto A? Back up your answers.

B. Proofs (30 points). You are required to do only two of Problems 6 through 8, each of which calls for a proof.

6. **Definition.** If A is a set, then **the power-set of** A, which is denoted as $\mathbf{P}(A)$, is the set to which S belongs only in case S is a subset of A. For example,
   $\mathbf{P}(\{0,1,2\}) = \{\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\}\}$.
   Theorem i. If each of A and B is a set and A intersects B, then $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cap \mathbf{P}(B)$.
7. **Definition.** If each of x, y, and z is a point, then the **ordered triple** (x,y,z) is the ordered pair $(x,(y,z))$.

Theorem ii. If each of x_1, x_2, and x_3 and each of y_1, y_2, and y_3 is a point, and if $x_i = y_i$ for each integer i in [1, 3], then the ordered triple $(x_1, x_2, x_3) = (y_1, y_2, y_3)$.

8. **Theorem iii.** If each of A and B is a set, then the following statements are equivalent:
   (a) $A \cup B = B$;
   (b) A is a subset of B;
   (c) $A \cap B = A$.

C. Conjectures (20 points). Each of Problems 9 through 12, only two of which you are required to do, is the statement of a conjecture. If the conjecture is true, prove it. If the conjecture is false, prove this by means of a counterexample or a proof by contradiction. (Indicate clearly on which two problems you wish to be graded.)

9. If each of A and B is a set, then $A \cap B = B \cap A$.
10. If each of A, B, and C is a set such that $B \cup C$ does not contain A, then $A \sim (B \cup C) = (A \sim B) \sim C$.
11. If each of A, B, and C is a set, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
12. There is a set $\mathcal{U}$ with the property that, if A is a set, then A is an element of $\mathcal{U}$.

IV
Parker

IV.A Syllabus for *Real Analysis*

Following is an excerpt from the syllabus and introduction to the notes from a course in real analysis.

COURSE: Mathematics 410 Advanced Calculus I (An Introduction to Analysis)

INSTRUCTOR: Ed Parker

OFFICE: Burruss 001, JMU6938, parkerge@jmu.edu

OFFICE HOURS: MF 8–9, W 2:30–3:30, Tu 9–10, Th 10–11 (student priority, MF 2:30–3:30, W 8–9, Tu 10–11, Th 9–10 (advisee priority), or by appointment

TEXTBOOK: There will be no formal textbook. Classes will be conducted from problem sets distributed by the instructor; solutions to those problems should effectively allow the student to write his/her own text.

COURSE CONTENT: Number functions and continuity, intermediate and extreme values, derivatives, and (time permitting) integrals.

POLICIES: This course is concerned with the creation and application of the theory that supports elementary calculus. Its conduct will be studentoriented in the sense that you will be asked to create solutions to problems, present your solutions for the scrutiny of the class, and to "criticize" the work of others when presented. You may earn credit during the semester by presenting your solutions in written or oral form. Oral presentation carries twice as much credit as written presentations. A final examination covering the entire course will be given at the end of the semester. The final grade will be determined from the problems solved during the semester (both quantity and quality are factors) and from the final examination. An important objective of the course is for you to develop your creative and critical mathematical skills; the dynamics of classroom interaction are thus crucial. You are expected to be in class.

AN INTRODUCTION TO DOING MATHEMATICS: In this course, not only will you be responsible for understanding why the mathematics we cover is correct, but the responsibility for discovery will also be assigned to the class. One of the immediate results of this responsibility for doing mathematics yourself rather than just learning how someone else did it will likely be an acute awareness of the difference between the challenge associated with understanding why something is correct and discovering for yourself whether or not a conjecture is a theorem.

Doing mathematics can be extremely exhilarating when one succeeds in the discovery process; failing to do mathematics when one is putting in the time trying to do mathematics can be extremely frustrating. This introduction is designed to alert you to some tips that are designed to enhance your chances for success.

First, you must put in the time necessary to give your creative intelligence a chance to work. Flashes of insight typically occur after information is organized and mulled over. Commitment to solving problems often leads to help from the subconscious. Students often tell me that they got "the big idea" while walking across campus or after turning in for the night.

Second, solutions to problems need not come all at once. You may need to solve many small problems on the way to proving a theorem or disproving an incorrect conjecture. Some of the most important work in mathematics is the creation of technique. Take pride in progress toward a goal as well as reaching the goal. Any information you uncover is more than you knew before, and solving a problem is usually just a matter of putting together enough small solutions to allow you to see why the big problem is correct.

Students often tell me that they would be glad to put in the time if they just knew where to start. The following scheme is offered toward that end.

(THE AWARENESS STAGE)

1. Identify all the words in the problem and make sure that you KNOW the definition of each of them. Try to recall examples that have dealt with these notions before. If a definition is new, make some examples for the definition.
2. Identify any theorems that may have dealt with ideas present in the problem already. Put techniques that gave rise to proofs in those contexts firmly in mind.

(THE DIRECT APPROACH)

3. Make an example that models the hypothesis to the problem and try to show that the example exhibits the properties of the conclusion. (If you can prove that your example fails to have the properties of the conclusion, you will have shown that the problem is not a theorem!)
4. See if what allowed you to establish the conclusion in the example is a property of all examples covered by the hypothesis. If it is, write a proof. If not,
5. … make an example which models the hypothesis but fails to have whatever special properties you used to get the conclusion in the previous example. GO TO 3

(THE INDIRECT, OR CONTRAPOSITIVE, APPROACH)

3. a. Suppose that the conclusion is false and try to show that the hypothesis must be false as well. If the problem is not a theorem, any conclusions you get must be qualities an example that disproves the conjecture must have.

6. Try to be aware of properties that, if they were added to the hypothesis, would guarantee the conclusion. Alternatively, you might also try to find conclusions that follow from the hypothesis, even if they do not include the one you seek. Even if you are not able to solve the problem as stated, you may be able to create a substitute theorem.

The main mindset is to be aware that even when arguments do not come quickly or easily, the hunt itself may be an important learning experience. Working on problems yourself is the central ingredient. Not only will it provide you with theorems that are "your own", but even when someone beats you to a solution, it will put you in a much stronger position to analyze the argument given.

IV.B Handout on logic

I often use this hand out in majors courses.

A BRIEF REVIEW OF THE LOGIC OF QUANTIFICATION

1. There are two types of quantifications, universal and existential.
2. The most common syntax for an existential quantification is

   There is… so that… .
3. The most common syntax for a universal quantification is

   If... then... .

   (THIS IS ALSO THE SYNTAX FOR CONDITIONALS!)
4. Quantifications are statements, but the sentences they connect are not.
5. The sentences connected in a quantification describe properties.
6. The sentence following “There is” or “If” in a quantification is called the sentence of inclusion. The sentence following “so that” or “then” is called the sentence of property.
7. An existential quantification asserts the existence of at least one object about which the properties of the sentences of inclusion and property are true.
8. A universal quantification asserts that any object about which the properties from the sentence of inclusion are true must also have the properties from the sentence of property true about it.
9. To prove directly that an existential quantification is true, you must exhibit an object about which the properties from the sentences of inclusion and property are true.
10. To prove directly that a universal quantification is true, on the assumption that an object has the properties from the sentence of inclusion, you must show that it must also have the properties from the sentence of property.
11. The negation of a universal quantification is logically equivalent to the existential quantification with the same sentence of inclusion and whose sentence of property is the negation of that of the universal.
12. The negation of an existential quantification is logically equivalent to the universal quantification with the same sentence of inclusion and whose sentence of property is the negation of that of the existential.

I often use this hand out in Math-for-people-who-don't-want-to-take-Math.

A statement is a sentence that is true or false and not both.

A **steap** is a sentence containing a pronoun so that if the pronoun is replaced by a noun, then the result is a statement.

In a **steap**, when a pronoun is replaced by a noun to form a statement, the process is called instantiation.

And, **or**, and **it is not the case that** connect statements to make statements.

If…,then… and **there is…so that…** connect steaps to make statements.

Standards of truth:

A statement formed by connecting two statements with **and** is true provided that both of the statements connected are true; otherwise the statement formed is false.

A statement formed by connecting two statements with **or** is true provided that at least one of the statements connected is true; otherwise the statement formed is false.

A statement formed by preceding a statement with **it is not the case that** is true provided that the statement preceded is false; otherwise the statement formed is false.

A statement formed by connecting two steaps[1] with **if…,then…** is true provided that every instantiation of the steap after **if** that makes it true ALSO instantiates the steap after **then** and makes it true; otherwise the statement formed is false.

A statement formed by connecting two steaps with **there is…so that…** is true provided that at least one instantiation of the steap after **there is** that makes it true ALSO instantiates the steap after **so that** and makes it true; otherwise the statement formed is false.

[1] "Steap" is an acronym for *sentence that expresses a property* (what the logicians call open sentences) which was suggested by one of my 1991 103 classes and which I have been using ever since.

IV.C Handout on Sets

I use these handouts on sets in both major and non-major classes.

A THEORY OF SETS AND ORDERED PAIRS

We will not create an axiomatic set theory. Following, however, is an idiomatic presentation of the conventions that axiomatic set theory implies and presupposes the existence of formal English as a language for expressing properties.

The primitive words are set, element, ordered pair, first co-ordinate, and second co-ordinate.

i. A set consists of an element or elements.
ii. An element of a set and the set consisting of that element are different objects.
iii. A set is defined by stating the properties its elements have.
iv. Given a definition for a set, any object having the properties specified is an element of the set; and any element of the set has the properties specified in the definition.
v. An ordered pair consists of a first co-ordinate and a second co-ordinate.
vi. The first co-ordinate of an ordered pair may be the same set theoretic object as the second co-ordinate, but as a part of the ordered pair, being the first co-ordinate is distinguishable from being the second co-ordinate.

We reserve a notation for the creation of definitions of sets and for defining ordered pairs.

Reserved symbols for definitions of sets are { : }. A symbol is created to follow the open brace and precede the colon and then properties that an element must have are stated in terms of that symbol after the colon and before the closed brace. Thus

$$\{x : x \text{ is a number and } x > 5\}$$

stands for "the set to which an element belongs provided that it is a number and it is greater than 5."

Reserved symbols for definitions of ordered pairs are (,). The first co-ordinate of the ordered pair is written after the open parenthesis and before the comma; the second co-ordinate of the ordered pair is written after the comma and before the closed parenthesis. Thus

$$(p,5)$$

stands for "the ordered pair whose first co-ordinate is p and whose second co-ordinate is 5."

IV.D Final Examinations for *Real Analysis* and *Numbers*

Following are the final examinations I gave in my most recent offerings of the first semester of analysis (Mathematics 410) and a course on the numbers (Mathematics 315) that, at the time, was being used as a bridge course.

In the analysis course, there were multiple students who still needed to make their grade. Furthermore, the class had fulfilled only minimal expectations in coverage, so in order to meet the assessment condition on testing removable discontinuities and the intermediate value theorem, I could not provide choices in items I.5 and I.6.

Mathematics 410 : Final Examination

This exam is due in Dr. Parker's office, Burruss 1, by 12:30pm on Wednesday, December 14, 2005(revised Gregorian calendar). You may use your notes, but contact about the exam with anyone other than Dr. Parker will be considered a breach of the honor code. You will not be there to explain what you meant by what you wrote, so make sure that what you write is convincing as written.

I. Answer one question from each of 1., 2., 3., 4., 5., and 6..

1. a. Suppose that X is a set and P is an order on X. Show that P is not a function from X into X.
   b. Suppose that each of X, Y, and Z is a set and that h is a function from X into Y and that k is a function from Y into Z. Show that the composition of k with h is a function from X into Z.
2. a. Suppose that each of m and b is a number and that $g = \{(x, y) : x$ is a number and $y = (m * x) + b\}$. Show that g is continuous on $\Re$.
   b. Suppose that $Q = \{(x, y) : x$ is a number and $y = x * x\}$. Show that Q is continuous on $\{x : x > 0\}$.
   c. Suppose that $s = \{(x, y) :$ there is a number, call such a number w, so that $x = w * w$, and $y = \sqrt{x}\}$. Show that s is continuous on $\{x :$ there is a number, call such a number w, so that $x = w * w\}$.
   d. Suppose that $\gamma = \{(x, y) : x$ is a number different than 0 and $y = \frac{1}{x}\}$. Show that γ is continuous on $\{x : x$ is a number different than $0\}$
3. a. Suppose that a, b, c, and d are numbers and w is an element of $(a,b) \cap (c,d)$. Show that there are numbers, call them p and q, so that $(a,b) \cap (c,d) = (p,q)$.
   b. Suppose that w is a number and $w > 0$, and $L = \{x : x \geq 0$ and $x * x < w\}$, and $M = \{x : x \geq 0$ and x is not an element of $L\}$. Show that if u is an element of L and v is an element of M, then $u < v$.
4. a. Suppose that each of X, Y, and Z is a subset of $\Re$, and that p is an element of X, and that f is a function from X into Y that is continuous at p, and that g is a function from Y into Z that is continuous at $f(p)$. Show that the composition of g with f is continuous at p.
   b. Suppose that X is a subset of $\Re$, p is an element of X, and each of f and g is a function from X into $\Re$ and each of f and g is continuous at p. Show that $f + g$ is continuous at p.

5. Suppose that p is a number. Show that $2*p$ is the slope of Q at $(p, p*p)$.
6. Suppose that f is a function from $[a,b]$ into $\Re$, and f is continuous on $[a,b]$, and $f(a) < f(b)$, and w is an element of $(f(a), f(b))$. Show that there is a number in $[a,b]$, call such a number p, so that $f(p) = w$.

In II. and III., you may submit solutions or any progress you are able to make on any of the problems. A complete proof of a problem, or an example that shows that a problem is not a theorem, is worth more than several parts of different problems.

II.

1. a. Suppose that X is a subset of $\Re$, and each of f and g is a function from X into $\Re$, and p is an element of X. Show that

$$f(p)*g(p) = \left(\frac{1}{2}*\big(Q\circ(f+g)\big)(p)\right) + \left(-\frac{1}{2}*(Q\circ f)(p)\right) + \left(-\frac{1}{2}*(Q\circ g)(p)\right).$$

   b. Suppose that X is a subset of $\Re$, and each of f and g is a function from X into $\Re$, and p is an element of X, and each of f and g is continuous at p. Show that $f*g$ is continuous at p.
2. Problem 18.
3. Problem 13.
4. a. Suppose that F is a sequence in $\{m$: there are numbers, call them a and b, so that $m = [a,b]\}$ and if n is a natural number, then $F(n+1)$ is a subset of $F(n)$, and
   $L = \{x : x$ is a number and there is an element of the range of F, call such an interval $[p,q]$, so that $x \le p\}$, and
   $M = \{x : x$ is a number and x is not an element of $L\}$.
   Show that $L \cup M = \Re$, and if p is an element of L and q is an element of M, then $p < q$.

   b. Suppose that F is a sequence in $\{m$: there are numbers, call them a and b, so that $m = [a,b]\}$ and if n is a natural number, then $F(n+1)$ is a subset of $F(n)$. Show that there is a number, call such a number w, so that if n is a natural number, then w is an element of $F(n)$.
5. a. Suppose that f is a function from $[a,b]$ into **R**, f is continuous on $[a,b]$, and p is an element of $[a,b]$, and w is a number so that $f(p) > w$. Show that there is an open interval, call such an open interval (u,v), so that p is an element of (u,v), and if q is an element of $(u,v) \cap [a,b]$, then $f(q) > w$.

   b. Suppose that p is a number and p is an element of (u,v). Show that there is an interval, call such an interval $[a,b]$, so that p is an element of $[a,b]$, and $[a,b]$ is a subset of (u,v).

   c. Suppose that f is a function from $[a,b]$ into $\Re$, and n is a natural number, and the range of f is not bounded. Show that $\{x : x$ is an element of $[a,b]$ and $f(x)$ is not an element of $[-(n+1), n+1]\}$ is a subset of $\{x : x$ is an element of $[a,b]$ and $f(x)$ is not an element of $[-n,n]\}$.

d. Suppose that f is a function from $[a,b]$ into $\Re$, and f is continuous on $[a,b]$, and w is a number, and p is a number in $[a,b]$, and $f(p) > w$. Show that there is an interval, call such an interval $[c,d]$, so that if x is an element of $[c,d]$, then $f(x) > w$.

e. Problem 16.

6. a. Suppose that f is a function from $[a,b]$ into $\Re$, and f is continuous on $[a,b]$, and S is the least upper bound of the range of f, and $f(a)$ is not S, and
   $L = \{x : x$ is an element of $[a,b]$ and there is an element of $[a,b]$, call it y, so that if $x < y$, then $f(x) < f(y)\}$, and
   $M = \{x : x$ is an element of $[a,b]$ and x is not an element of $L\}$.
   Show that if p is an element of L and q is an element of M, then $p < q$.

   b. Problem 28.

III.

1. a. Suppose that X is a subset of $\Re$, and f is a function from X into $\Re$, and p is an element of X; and, if $E > 0$, then there is a number, call it D, so that if q is an element of X and $|q + -p| < D$, then $|f(q) + -f(p)| < E$. Show that f is continuous at p.

   b. Suppose that X is a subset of $\Re$, and f is a function from X into $\Re$, and p is an element of X, and f is continuous at p. Show that if $E > 0$, then there is a positive number, call such a number d, so that if q is an element of X and $|q + -p| < d$, then $|f(q) + -f(p)| < E$.

2. a. Suppose that p is a number and p is not 0. Show that $SA\gamma p = \{(x,y) : x \textit{ is not } 0 \textit{ and } y = -1\} * (Q \circ \gamma)$.

   b. Problem 21.

In the course on the numbers (Parker, G.E., 2008)*, the role as a bridge course is paramount. Even though the class had not made the progress I would have hoped for with cuts, there had been broad-based participation and the class had made it to the point where some issues on the algebraic aspects of the model had been addressed. Given the many choices offered, I demanded near-perfect write-ups in I. as the standard for C.*

Mathematics 315 : Final Examination

This examination is due by 10am on Wednesday December 8, 2004 (revised Gregorian calendar). You may turn it in at Dr. Parker's office, Burruss 1. You may use your notes for the exam and the university's research facilities; however, any interchange about the exam with any person other than Dr. Parker once the exam is handed out and before it is turned in is a violation of the honor code. Each student is expected to answer all of the questions in I. In questions II and III, you may submit partial answers or parts of questions; however, the answer to the last part of each question is where the big credit lies. Remember that you will not be present to explain what you meant by what you wrote; be clear and be correct.

Definitions are from the course notes. On I, you may cite theorems proven during the class so long as their proofs do not depend on the theorem on which you are working. All of our theorems are available for II and III.

I. Places we have been before. Solve one problem from each of 1, 2, 3, 4, and 5.

1. a. Show that G' is an order on U'.
   b. Suppose that x and y are elements of U' and (x,y) is an element of G'. Show that there is an element of U', call such an element p, so that (x,p) is an element of G' and (p,y) is an element of G'.
   c. Show that it is not the case that U' has a max by G' and it is not the case that U' has a min by G'.
2. a. Suppose that X is a set and Y is a subset of X. Show that X commands Y.
   b. Show that U' commands the natural numbers.
   c. Show that ρ is a function from
   $\{(x,y) : x$ is an element of U' and y is an element of $U'\}$ into U'.
   d. Suppose that each of P, Q, and S is a set, and P commands Q, and Q commands S. Show that P commands S.
3. a. Show that G is not an order on U.
   b. Show that there is a set, call it X, and a subset of X different than X that commands X.
   c. Show that if each of x and y is an element of U', then it is not the case that $\rho((x,y)) = x$.
4. a. Suppose that x and y are elements of U'. Show that there are open intervals of U' by G', call them A and B, so that x is an element of A and y is an element of B, and if p is an element of A, then p is not an element of B.
   b. Suppose that x and y are elements of U' and (x,y) is an element of G'. Show that there is a cut of U' by G', call such a cut (J,K), so that x is an element of J and y is an element of K.
   c. Suppose that (H,K) is a cut of U' by G'. Show that it is not the case that both H has a max by G' and K has a min by G'.
5. a. Suppose that x and y are elements of U'. Show that $\rho((x,y)) = \rho((y,x))$.
   b. Show that there is an element of U', call such an element v, so that if x is an element of U', then $\tau((x,v)) = x$.

II. Unfinished business.

1. a. Suppose that n is a natural number greater than 1.
   Show that $\{k : k$ is a natural number and $k \leq n - 1\}$ commands
   $\{(p,q) : (p,q)$ is an element of U and $p + q = n\}$.
   b. Suppose that n is a natural number greater than 2.
   Show that $\{j : j$ is a natural number and $j \leq \sum_{k=1}^{n-1} k\}$ commands
   $\{(p,q) : (p,q)$ is an element of U and $p + q \leq n\}$.
   c. Show that the natural numbers commands U.

2. $P = \{(a,b) : (a,b)$ is an element of U' and $a*a < 2*(b*b)\}$ and
   $Q = \{(a,b) : (a,b)$ is an element of U' and $a*a > 2*(b*b)\}$
   a. Suppose that x is an element of P. Show that there is an element of P, call it y, so that (x,y) is an element of G'.
   b. Suppose that x is an element of Q. Show that there is an element of Q, call it y, so that (y,x) is an element of G'.
   c. Show that G' does not have the Dedekind Cut Property.
3. Suppose that each of x, y, and w is an element of U'.
   Show that $\tau((x,\rho((y,w)))) = \rho((\tau((x,y)),\tau((x,w)))$.
4. Suppose that (x,y) is an element of G'. Show that there is an element of U', call such an element w, so that $\rho((x,w)) = y$.

III. Uncharted waters

1. Suppose that s is a function from the natural numbers into U' so that if n is a natural number, then $(s(n),s(n+1))$ is an element of G', and there is an element of U', call it y, so that if x is an element of the range of s, then (x,y) is an element of G'.
   Show that
   $(\{x$: there is a natural number, call it n, so that $(x,s(n))$ is an element of $G'\}$,
   $\{x$: if n is a natural number then $(s(n),x)$ is an element of $G'\})$
   is a cut of U' by G'.
2. Suppose that (H,K) is a cut of U' by G' and x is an element of U'. Show that
   $(\{p : p$ is an element of H or there is an element of H, call it y, so that $p = \rho((x,y))\}$,
   $\{p : p$ is an element of K and if y is an element of H, then p is not $\rho((x,y))\})$
   is a cut of U' by G'.
3. Suppose that $U^* = \{x$: there is a cut of U' by G', call it (A,B), so that $x = A\}$ and
   $G^* = \{(x,y) : x$ and y are elements of U^* and x is a subset of $y\}$.
   Show that G^* is an order on U^*.

About the Authors

Charles A. Coppin

I received my baccalaureate from Southwestern University, Georgetown, Texas, and my doctorate in Mathematics from the University of Texas, Austin, Texas. I was a student of H.S. Wall, a colleague of R.L. Moore. Having studied Point Set Topology under Moore, I had significant exposure to the Moore method. I was a member of the mathematics faculty at the University of Dallas where I was Chair for several years. I was a Distinguished Visiting Professor of Mathematics at the United States Air Force and Visiting scholar at the Visualization Lab at Texas A&M. In 2002, I became Chair of the Department of Mathematics at Lamar University.

Nothing I have accomplished would have been possible without my wife, Alaine Fay. We have been married forty-five very good years. Alaine Fay has been my professional partner. My teaching has been greatly influenced by many years of conversation with her and watching our children grow up. Our family now includes three children, Stephen, Peter, Sarah, and son-in-law, Michael.

W. Ted Mahavier

I am a grandson of the Texas Method[1] in two senses. My father, William S. Mahavier, was a student of Moore, and my advisor, John W. Neuberger, was a student of Wall. Thus, two influential people in my life had considerable exposure to Moore and his colleagues. As a child, I was constantly surrounded by mathematicians and graduate students at departmental social functions at Auburn, Emory, and North Texas. What struck me most as a child growing up around this group of devoted mathematicians was how in one sentence they would be debating the properties of indecomposable continua while in the next they would be bragging proudly about how an elementary education major had proved that the square root of two was an irrational number in some particularly interesting way. Their devotion to their research and teaching at every level has stayed with me to this day and my teaching methods are strongly influenced by the courses I took over the years.

In total, I took twenty-two courses under nine descendants of either Moore or Wall. Michel Smith's calculus course at Auburn University caused my change in major from physics to mathematics. My father's examples, as I sat in on his classes at Emory, stayed

[1] I prefer Texas Method because it designates the school at which it gained fame and because Wall played a very large role in producing descendants who have practiced the method.

with me and eventually led me to teach via Moore's method. At the University of North Texas, Paul Lewis taught me more about mathematics and teaching mathematics than anyone, and John W. Neuberger, a man whose infinite optimism is an inspiration to all who know him, directed my dissertation. While I am a firm believer in the method and my education was largely guided by mathematical descendants of Moore, H. S. Wall, and H. J. Ettlinger, I remain a strong supporter of other methods, perhaps because the non-Texan Dean Hoffman set the stage for my return to graduate school in a lecture-style class by convincing me that I could prove theorems. I feel deeply indebted to my teachers and I repay them in the only way I can, by carrying their examples into my classroom and passing the time they offered me onto my own students.

Two side notes on Dean Hoffman and his class are in order. First, Dean Hoffman may have been influenced by his close friend and colleague at Auburn, Kurt Lintner, who studied at Emory under the Moore Method. Second, Hoffman's teaching technique was completely unique in my experience, and worth elaborating on. The senior-level course was Graph Theory and he assigned several problems every day as he lectured, some written on the chalk board and some from the text. We were expected to turn in, on average, a specified number of solutions (proofs, examples, counter-examples) per week for each of the ten weeks in the quarter. While we could turn them in for grading early, the only deadline was the last day of class. I recall working late into the night on many nights, completely hooked on the idea of actually choosing which problems to work on and not feeling any pressure to complete them by the next day or even the next week. I recall colored pencils and graphs and papers spread all over my small kitchen table. I remember liking very much the freedom to choose which problems I worked on and the ability to come back to a problem week after week. And, I recall that on the last week of the semester, I resolved one of the problems posed during the first week. At the end, I still remember that I had solved only forty-two problems, but received an *A*. I was surprised and went to see him to discuss my grade. When he told me that I had solved more problems than any member of the class, I felt a real pride and sense of mathematical accomplishment. This was my first major break-through experience in a theory class and I immediately recognized that this must be what mathematical research was like—working on problems for an extended time with no pressing deadlines. This class strongly influenced my decision to attend graduate school.

I like to think that the method has produced a productive, if perhaps overly diverse, academician. My vita includes two years in defense contracting, consulting on an America's Cup team, educational consulting associated with the Moore Method and mathematics education, and two-million dollars in grant support related to education projects. I have taken an active service role in each of the departments in which I have served and currently serve as a Board Member for MathNerds, Inc., a nonprofit corporation I co-founded in 1996.[2] My research has addressed algorithms associated with LASIK surgery, papers on numerical solutions to differential equations, and pedagogical papers. My research results have been extended and applied to the study of pressing problems in elasticity at Los Alamos National Laboratories and to the furtherance of the theory of differential algebraic equations. In addition to the usual teaching and service duties aligned with an associate professor, I have served for seven years as the faculty advisor to a martial arts club that has produced five black belts in Kuk Sool Won. I have achieved an eighteen-year marriage resulting in two children. I list these accomplishments because *they are all products of the Moore Method*

[2] MathNerds, Inc. was originally MathNerds, LLC and was later converted to a nonprofit.

training. The method has been constantly influential as a force in all my efforts, because it convinced me that I could do anything and taught me how to rely on my own planning and strategies to address problems.

The training changed my philosophy toward tackling problems. Before I tackle a problem now, I ask what procedure I want to take. Perhaps I want to take my time and study and learn from the problem. Perhaps I want the quickest and simplest fix. In mathematics, I ask the same question. Do I want to learn this field and study it or do I want the answer quickly? But fundamental to the process is that before I start, I assume that *I* am the individual who will both define and solve the problem and that process has benefited me in all my endeavors.

E. Lee May

I have been practicing the Moore Method throughout thirty-seven years of teaching. I have taught at Emory University; Kennesaw State University; Wake Forest University; and, for most of my career, at Salisbury (Maryland) University. The courses that I have designed bear titles such as "Discovering Affine Transformations," "Introduction to Abstract Mathematics," "A History of Mathematics from $-\infty$ through 1600," and "Statistics through Baseball." I have received a Maryland Council of Teachers of Mathematics award as an Outstanding College Teacher of Mathematics, and the John M. Smith Award of the Maryland-District of Columbia-Virginia (MD-DC-VA) Section of the Mathematical Association of America (MAA) for Distinguished College or University Teaching. My publications include "Localizing the Spectrum," "The Local Resolvent Set of a Locally Lipschitzian Transformation Is Open," "Real-Linear (Including Semilinear) Operators," "An Experiment with Mathematical Statistics," "Are Seven-Game Playoff Series Fairer?", and "Is the Integral Test Wrong? A Research Adventure in Calculus." (My latest two manuscripts are "0 for April, or, Are Batting Slumps Inevitable?", and "The Spectra of Affine Operators.") I have served as chairman of my department and of my MAA section. I have participated in seven of the eleven Legacy of R. L. Moore Conferences, and chaired or co-chaired three. For ten years I was the founding director of Salisbury University's Center for Applied Mathematical Sciences, an organization that paired faculty-directed teams of undergraduates with local businesses for the purpose of solving problems for those organizations. I have also spent two years as a fulltime, independent computer-consultant to small businesses.

John Neuberger, my beloved Ph. D. advisor at Emory University, likes to say something to the effect that he stumbled out of the hills of South Texas and onto the campus of the University of Texas, fell under the spell of the Moore Method, and began a life of joy and satisfaction as a student and teacher of mathematics. My story is similar. I chose both my undergraduate and graduate schools without any awareness of the existence of the Moore Method or any of its practitioners. Nevertheless, at Wake Forest I was prepared for the method, and at Emory I was immersed in and converted to it. My life has never been the same.

It was not a case of love at first sight, however, for my graduate career began abysmally. I do not remember, in either Moore-style courses or ones taught by the traditional lecture method, proving a single theorem during my first two quarters. I do remember, however, going to the board numerous times. A typical trip consisted of my placing a number of mathematical terms and symbols on the board, permuting them several times, and then stepping back from my final permutation and looking beseechingly at my classmates and professor. When it was inevitably explained to me that my argument was not a proof, I nodded in assent but not understanding and sat down, clueless as to why my sequence of permutations was not a proof. I was able sometimes to understand others' proofs and wonder why I had not thought of them, but for the most part I spent the fall of 1966 and winter of 1967 practicing "creative hypocrisy," acting like a graduate student in mathematics in the hope that, somehow, through some miracle, I would someday become one.

The miracle occurred during the spring of 1967. My third-quarter Topology I class, like those of the first two quarters, was taught by Dr. W. S. Mahavier, the father of the lead author of this book. The class consisted of eight students. Three of them, two men of my age and a fifteen-year-old high-school student, clearly had "cracked the code" of doing mathematics. For the first two or three weeks of the quarter, they took turns in going to the board and presenting correct and, for the most part, clear and even elegant proofs to the theorems. Meanwhile I sat among the other five students, dutifully copying down what was being put on the board and fuming that, if only I had had one more day, I could be up there at the board strutting my stuff. At the beginning of class during the third or fourth week of the quarter, Dr. Mahavier announced that he intended to split the class. He would, he said, continue to meet five of us at the regular class-time, but he would see the other three one hour later. The three later-meeting ones were, of course, the three hot-shots. I immediately concluded that Dr. Mahavier had pulled the classic "sheep and goats" maneuver (see the Bible, the book of Matthew, Chapter 25, Verses 31 through 46); and it was clear to me who were the goats. I was furious.

Nevertheless, I decided to play the game. So, apparently, did the rest of the Goats. We began to meet without the Sheep, and I actually enjoyed the latter's absence. I wondered, however, would we Goats now rise to the challenge? Would we really, given more time and less Sheep-pressure, prove theorems?

We would and we did. The five of us began to take fairly regular turns at the board, presenting our arguments to one another, and being correct. (The most eminent Goat was the late Dr. Etta Falconer of Spellman College in Atlanta.) Our confidence grew with our successes, to the point where we could absorb the disappointment of having a flaw in our reasoning pointed out on Wednesday and return on Friday with a correction and the rest of the proof. We were rolling! But this exulting was quickly followed by The Question: What would happen at the beginning of the fall quarter and Topology II? Would Dr. Mahavier reunite the Sheep and the Goats, and if so would the old pattern of Sheep dominating the Goats return?

The answers were yes and no. Dr. Mahavier did indeed reunite the class. The old pattern, however, did not return. While we Goats had been honing our skills at proving theorems and presenting our proofs, Dr. Mahavier had kept the Sheep from getting ahead of us in the sequence of theorems for the course by assigning them "diversionary" problems of an enrichment nature. As a result, as the fall of 1967 began, the Goats found themselves at the same point as the Sheep in the sequence of theorems. Even better, it became readily apparent that the remaining Goats (one or two had dropped out of the graduate-math program) had become Sheep. The usual pattern now was for all of the members of the class to share uniformly in the presentation of arguments and in the satisfaction of having arguments certified as correct by the class. The miracle was complete.

The Sheep-and-Goats Maneuver is the most dramatic demonstration I have ever experienced of the effectiveness of the Moore Method, and of the dedication and generosity of its practitioners. I seriously doubt that, without it or them, I would ever, at Emory or anyplace else, have learned to enjoy mathematics as I do, much less earn a Ph. D. in it. This is why I am the committed "Moore-on" that I am.

G. Edgar Parker

I have taught mathematics since 1969 (or since 1963 if you count the tutoring I did as a high school student and as an undergraduate college student), settling for this second choice in careers because of an insufficient fastball (or at least that is the reservation the scouts had). My teaching experience includes four years in a public high school; courses at Atlanta Junior College; four years as a teaching assistant during my graduate education at Emory University, by most measures a highly selective university; seven years at what is now University of Texas-Pan American and was, at that time, an open-admission university; and the last twenty-three years at James Madison University. At James Madison, I have taught a full range of the service and pure curriculum, including precalculus, the calculus sequence, math-for-students-who-don't-want-to-take-math, the mathematics for prospective elementary/middle-school teachers sequence, linear algebra, differential equations, algebraic structures, point-set topology, geometry, analysis, and a course on the structure of the numbers. At both Pan American and James Madison, I have directed undergraduate research projects. The topics for these investigations have ranged through topics as diverse as applications of mathematics to football and basketball, square roots by composition for nonlinear functions, models for the numbers in which numbers are represented by sets of natural numbers, and algebraic comparisons within finite groups and fields different from homomorphism and isomorphism. I have taken baby steps into the applied curriculum by teaching the modeling course and developing a version of math-for-students-who-don't-want-to-take-math that I call The Nature of Applied Mathematics. For the five years that we had it at James Madison, I taught the interdisciplinary freshman seminar. This broad range of curricular content and student audiences has informed my teaching as it has evolved, and continues to evolve, over the course of a career in education.

My mathematical interests are eclectic; indeed, for pragmatic reasons as a young mathematician, I had to learn to resist following curiosities that promised no professional return. I no longer feel such constraints. Most of my early published mathematical work concerns nonlinear semigroups in their role of describing the structure of the solutions of differential equations and centers around pathology within the structures as I tried to hone a vision of the edges of tractability. A collaboration begun in 1988 with Jim Sochacki, also of James Madison, has been the focus of much of my recent research. We have written a sequence of four papers, the last two in collaboration with several other colleagues in the department, on what we call "polynomial projection." I have also presented many talks over the past 25 years on the application of mathematics to tennis, baseball, and basketball.

I am a career-long practitioner of the Moore Method and have used discovery-based pedagogy at all of the levels at which I have taught. I take particular pride in producing problem sequences that make it more convenient for students to solve the problems themselves than to go find them in books and translate back to the context of the course. Although my greatest pleasure in teaching comes from nurturing majors' use of the tools I love, I take pride in involving the students in my (numerous) sections of math-for-students-who-don't-want-to-take-math in the examination of mathematics as they have experienced it and the creation of mathematics themselves. I am also hopeful that my extensive involvement in teaching prospective teachers will have long-term societal benefits.

My baccalaureate degree is from Guilford College where I took courses in the Moore style from J. R. Boyd, Ken Walker, and my older brother, Elwood. I did my graduate work at Emory University, where John W. Neuberger directed my thesis. While at Emory, I took Moore Method courses with David A. Ford, Phillip C. Tonne, William S. Mahavier, John W. Neuberger, and Mary Frances Neff. I owe them all big time, mostly for things to emulate, but also, on occasion, for things to avoid. For the past decade, I have been active in the Legacy of R. L. Moore movement through which I began professionally collaborating with Mahavier, co-founder of MathNerds. I have been a career-long colleague and friend of May, whom I met initially at my first AMS meeting in 1977. My published works include nine refereed papers in mathematics journals, three refereed papers in mathematics education journals and two sets of course notes in the *Journal of Inquiry-based Learning in Mathematics*.

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