notes

ENGAGING STUDENTS IN INTRODUCTORY MATHEMATICS COURSES THROUGH INTERDISCIPLINARY PARTNERSHIPS: **THE SUMMIT-P MODEL**

Susan L. Ganter
Debra Bourdeau
Victor Piercey
Afroditi V. Filippas
Editors

MAA PRESS

Engaging Students in Introductory Mathematics Courses through Interdisciplinary Partnerships: The SUMMIT-P Model

Library of Congress Control Number: 2022941675

Print ISBN 978-0-88385-211-8
Electronic ISBN 978-1-61444-329-2

Printed in the United States of America

Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1

Engaging Students in Introductory Mathematics Courses through Interdisciplinary Partnerships: The SUMMIT-P Model

Edited by

Susan L. Ganter
The University of Texas Permian Basin

Debra Bourdeau
Embry-Riddle Aeronautical University

Victor Piercey
Ferris State University

and

Afroditi V. Filippas
Virginia Commonwealth University

Published and Distributed by
The Mathematical Association of America

The MAA Notes Series, started in 1982, addresses a broad range of topics and themes of interest to all who are involved with undergraduate mathematics. The volumes in this series are readable, informative, and useful, and help the mathematical community keep up with developments of importance to mathematics.

MAA Notes

14. Mathematical Writing, by *Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts.*
16. Using Writing to Teach Mathematics, *Andrew Sterrett,* Editor.
17. Priming the Calculus Pump: Innovations and Resources, Committee on Calculus Reform and the First Two Years, a subcommittee of the Committee on the Undergraduate Program in Mathematics, *Thomas W. Tucker,* Editor.
18. Models for Undergraduate Research in Mathematics, *Lester Senechal,* Editor.
19. Visualization in Teaching and Learning Mathematics, Committee on Computers in Mathematics Education, *Steve Cunningham and Walter S. Zimmermann,* Editors.
20. The Laboratory Approach to Teaching Calculus, *L. Carl Leinbach et al.,* Editors.
21. Perspectives on Contemporary Statistics, *David C. Hoaglin and David S. Moore,* Editors.
22. Heeding the Call for Change: Suggestions for Curricular Action, *Lynn A. Steen,* Editor.
24. Symbolic Computation in Undergraduate Mathematics Education, *Zaven A. Karian,* Editor.
25. The Concept of Function: Aspects of Epistemology and Pedagogy, *Guershon Harel and Ed Dubinsky,* Editors.
26. Statistics for the Twenty-First Century, *Florence and Sheldon Gordon,* Editors.
27. Resources for Calculus Collection, Volume 1: Learning by Discovery: A Lab Manual for Calculus, *Anita E. Solow,* Editor.
28. Resources for Calculus Collection, Volume 2: Calculus Problems for a New Century, *Robert Fraga,* Editor.
29. Resources for Calculus Collection, Volume 3: Applications of Calculus, *Philip Straffin,* Editor.
30. Resources for Calculus Collection, Volume 4: Problems for Student Investigation, *Michael B. Jackson and John R. Ramsay,* Editors.
31. Resources for Calculus Collection, Volume 5: Readings for Calculus, *Underwood Dudley,* Editor.
32. Essays in Humanistic Mathematics, *Alvin White,* Editor.
33. Research Issues in Undergraduate Mathematics Learning: Preliminary Analyses and Results, *James J. Kaput and Ed Dubinsky,* Editors.
34. In Eves' Circles, *Joby Milo Anthony,* Editor.
35. You're the Professor, What Next? Ideas and Resources for Preparing College Teachers, The Committee on Preparation for College Teaching, *Bettye Anne Case,* Editor.
36. Preparing for a New Calculus: Conference Proceedings, *Anita E. Solow,* Editor.

37. A Practical Guide to Cooperative Learning in Collegiate Mathematics, *Nancy L. Hagelgans, Barbara E. Reynolds, SDS, Keith Schwingendorf, Draga Vidakovic, Ed Dubinsky, Mazen Shahin, G. Joseph Wimbish, Jr.*
38. Models That Work: Case Studies in Effective Undergraduate Mathematics Programs, *Alan C. Tucker,* Editor.
39. Calculus: The Dynamics of Change, CUPM Subcommittee on Calculus Reform and the First Two Years, *A. Wayne Roberts,* Editor.
40. Vita Mathematica: Historical Research and Integration with Teaching, *Ronald Calinger,* Editor.
41. Geometry Turned On: Dynamic Software in Learning, Teaching, and Research, *James R. King and Doris Schattschneider,* Editors.
42. Resources for Teaching Linear Algebra, *David Carlson, Charles R. Johnson, David C. Lay, A. Duane Porter, Ann E. Watkins, William Watkins,* Editors.
43. Student Assessment in Calculus: A Report of the NSF Working Group on Assessment in Calculus, *Alan Schoenfeld,* Editor.
44. Readings in Cooperative Learning for Undergraduate Mathematics, *Ed Dubinsky, David Mathews, and Barbara E. Reynolds,* Editors.
45. Confronting the Core Curriculum: Considering Change in the Undergraduate Mathematics Major, *John A. Dossey,* Editor.
46. Women in Mathematics: Scaling the Heights, *Deborah Nolan,* Editor.
47. Exemplary Programs in Introductory College Mathematics: Innovative Programs Using Technology, *Susan Lenker,* Editor.
48. Writing in the Teaching and Learning of Mathematics, *John Meier and Thomas Rishel.*
49. Assessment Practices in Undergraduate Mathematics, *Bonnie Gold,* Editor.
50. Revolutions in Differential Equations: Exploring ODEs with Modern Technology, *Michael J. Kallaher,* Editor.
51. Using History to Teach Mathematics: An International Perspective, *Victor J. Katz,* Editor.
52. Teaching Statistics: Resources for Undergraduate Instructors, *Thomas L. Moore,* Editor.
53. Geometry at Work: Papers in Applied Geometry, *Catherine A. Gorini,* Editor.
54. Teaching First: A Guide for New Mathematicians, *Thomas W. Rishel*.
55. Cooperative Learning in Undergraduate Mathematics: Issues That Matter and Strategies That Work, *Elizabeth C. Rogers, Barbara E. Reynolds, Neil A. Davidson, and Anthony D. Thomas,* Editors.
56. Changing Calculus: A Report on Evaluation Efforts and National Impact from 1988 to 1998, *Susan L. Ganter*.
57. Learning to Teach and Teaching to Learn Mathematics: Resources for Professional Development, *Matthew Delong and Dale Winter.*
58. Fractals, Graphics, and Mathematics Education, Benoit Mandelbrot and Michael Frame, Editors.
59. Linear Algebra Gems: Assets for Undergraduate Mathematics, *David Carlson, Charles R. Johnson, David C. Lay, and A. Duane Porter,* Editors.
60. Innovations in Teaching Abstract Algebra, *Allen C. Hibbard and Ellen J. Maycock,* Editors.
61. Changing Core Mathematics, *Chris Arney and Donald Small,* Editors.
62. Achieving Quantitative Literacy: An Urgent Challenge for Higher Education, *Lynn Arthur Steen.*
64. Leading the Mathematical Sciences Department: A Resource for Chairs, *Tina H. Straley, Marcia P. Sward, and Jon W. Scott,* Editors.
65. Innovations in Teaching Statistics, *Joan B. Garfield,* Editor.
66. Mathematics in Service to the Community: Concepts and models for service-learning in the mathematical sciences, *Charles R. Hadlock,* Editor.
67. Innovative Approaches to Undergraduate Mathematics Courses Beyond Calculus, *Richard J. Maher,* Editor.
68. From Calculus to Computers: Using the last 200 years of mathematics history in the classroom, Amy Shell-Gellasch and Dick Jardine, Editors.
69. A Fresh Start for Collegiate Mathematics: Rethinking the Courses below Calculus, *Nancy Baxter Hastings,* Editor.
70. Current Practices in Quantitative Literacy, *Rick Gillman*, Editor.
71. War Stories from Applied Math: Undergraduate Consultancy Projects, *Robert Fraga*, Editor.
72. Hands On History: A Resource for Teaching Mathematics, *Amy Shell-Gellasch*, Editor.
73. Making the Connection: Research and Teaching in Undergraduate Mathematics Education, *Marilyn P. Carlson and Chris Rasmussen*, Editors.

74. Resources for Teaching Discrete Mathematics: Classroom Projects, History Modules, and Articles, *Brian Hopkins*, Editor.
75. The Moore Method: A Pathway to Learner-Centered Instruction, *Charles A. Coppin, W. Ted Mahavier, E. Lee May, and G. Edgar Parker*.
76. The Beauty of Fractals: Six Different Views, *Denny Gulick and Jon Scott*, Editors.
77. Mathematical Time Capsules: Historical Modules for the Mathematics Classroom, *Dick Jardine and Amy Shell-Gellasch*, Editors.
78. Recent Developments on Introducing a Historical Dimension in Mathematics Education, *Victor J. Katz and Costas Tzanakis*, Editors.
79. Teaching Mathematics with Classroom Voting: With and Without Clickers, *Kelly Cline and Holly Zullo*, Editors.
80. Resources for Preparing Middle School Mathematics Teachers, *Cheryl Beaver, Laurie Burton, Maria Fung, and Klay Kruczek*, Editors.
81. Undergraduate Mathematics for the Life Sciences: Models, Processes, and Directions, *Glenn Ledder, Jenna P. Carpenter, and Timothy D. Comar*, Editors.
82. Applications of Mathematics in Economics, *Warren Page*, Editor.
83. Doing the Scholarship of Teaching and Learning in Mathematics, *Jacqueline M. Dewar and Curtis D. Bennett*, Editors.
84. Insights and Recommendations from the MAA National Study of College Calculus, *David Bressoud, Vilma Mesa, and Chris Rasmussen,* Editors.
85. Beyond Lecture: Resources and Pedagogical Techniques for Enhancing the Teaching of Proof-Writing Across the Curriculum, *Rachel Schwell, Aliza Steurer and Jennifer F. Vasquez,* Editors.
86. Using the Philosophy of Mathematics in Teaching Undergraduate Mathematics, *Bonnie Gold, Carl E. Behrens, and Roger A. Simons,* Editors.
87. The Courses of History: Ideas for Developing a History of Mathematics Course, *Amy Shell-Gellasch and Dick Jardine,* Editors.
88. Shifting Contexts, Stable Core: Advancing Quantitative Literacy in Higher Education, *Luke Tunstall, Gizem Karaali, and Victor Piercey,* Editors.
89. MAA Instructional Practices Guide, *Martha L. Abell, Linda Braddy, Doug Ensley, Lewis Ludwig, Hortensia Soto,* Project Leadership Team.
90. What Could They Possibly Be Thinking!?! Understanding your college math students, *Dave Kung and Natasha Speer*.
91. Mathematical Themes in a First-Year Seminar, *Jennifer Schaefer, Jennifer Bowen, Mark Kozek, and Pamela Pierce,* Editors.
92. Addressing Challenges to the Precalculus to Calculus II Sequence through Case Studies: Report based on the National Science Foundation Funded Project Precalculus through Calculus II, *Estrella Johnson, Naneh Apkarian, Kristen Vroom, Antonio Martinez, Chris Rasmussen, and David Bressoud*, Editors.
93. Engaging Students in Introductory Mathematics Courses through Interdisciplinary Partnerships: The SUMMIT-P Model *Susan L. Ganter, Debra Bourdeau, Victor Piercey, and Afroditi V. Filippas,* Editors.

MAA Service Center
P.O. Box 91112
Washington, DC 20090-1112
1-800-331-1MAA FAX: 1-301-206-9789

Contents

Table of SUMMIT-P Implementation Sites

Institution	Institution Type	Partner Disciplines	Courses	Module(s) Included in This Volume
Augsburg University	Small, Private	Economics and Chemistry	Calculus	3.13: New Product Adaptation 3.14: Titration
California Polytechnic State University, Humboldt	Medium, Public, Hispanic-Serving	Biology, Mechanical Engineering, and Physics	Calculus and Trigonometry	3.18: Solar Energy
Embry-Riddle Aeronautical University	Medium, Private	Humanities and Communication	Quantitative Reasoning	3.22: Media Journal
Ferris State University	Medium, Public	Business, Social Work, and Nursing	Quantitative Reasoning	3.3: Refugee Camp Administration 3.6: Human Trafficking
LaGuardia Community College	Medium, Public, Minority-Serving	Economics	College Algebra and Trigonometry	3.7: Demand
Lee University	Small, Private	Education, Natural Science, and Social Science	Algebra for Calculus, Concepts of Mathematics, and Introduction to Statistics	3.8: Medicine Dosage
Norfolk State University	Small, Public HBCU	Electrical Engineering	Calculus and Differential Equations	3.16: ECG Signals 3.19: Second Order Circuits 3.20: Wireless Power Transfer
Oregon State University	Large, Public	Biology, Chemistry, and Zoology	Calculus	3.15: Glucose Absorption
Saint Louis University	Medium, Private	Business and Economics	College Algebra and Business Calculus	3.11: Fitting Exponential Curves 3.12: "What-If" Analysis 3.17: Riemann Sums in Excel
San Diego State University	Large, Public, Hispanic-Serving and Asian American/Pacific Islander-Serving	Biology and Physics	Precalculus and Calculus for Life Sciences	3.10: pH
University of Tennessee – Knoxville	Large, Public	Anthropology and Geography	Data Science and Mathematical Reasoning	3.4: Marshmallow Shooters 3.5: Health Insurance
Virginia Commonwealth University	Large, Public	Engineering and Physics	Calculus and Differential Equations	3.21: Nuclear Engineering

Preface

William Haver[1], *Virginia Commonwealth University*

Gay Stewart, *West Virginia University*

Talk to a colleague in another discipline before you explore this volume.

Or even better. . .

Gather a team of colleagues from multiple disciplines and read this volume together.

As educators, our natural tendency when given a book like this is to immediately jump to the section with classroom resources and figure out how to use them in class as soon as possible. We hope these resources will help us to improve the courses we teach so that students will be more successful. Perhaps, then, the question to ask when considering new classroom resources is:

How are these classroom resources different from those in other publications?

The answer for the current volume is the process that resulted in the materials in Chapter 3. This process includes two decades of targeted exploration with colleagues across 22 disciplines about the mathematical needs of students pursuing careers in those disciplines. This work led to the conclusion that interdisciplinary faculty partnerships are key in the development and implementation of a mathematics curriculum that allows students to really learn mathematical concepts while moving seamlessly to concepts in other disciplines.

What does this mean for next steps at my institution?

Begin by contacting some colleagues at your institution from another discipline. Together, explore questions and issues related to courses across those disciplines that prevent student transfer of knowledge. You may want to select a module or two from Chapter 3 to add to the discussion by providing illustrations of those issues. Simultaneously, your group can explore the tested tools described in Chapter 2 to help broaden the circle of faculty involved in the interdisciplinary conversation. As you build these partnerships, draw upon the research in Chapter 1 that provides historical context and verifies the importance of interdisciplinary faculty collaborations.

While we hope you will want to use the classroom resources presented in Chapter 3, the real power of this volume is the ideas presented in the first two chapters that set the stage for the nature of effective interdisciplinary faculty partnerships (Chapter 1) and present a model for creating and sustaining effective partnerships across disciplines (Chapter 2). These ideas, supported by extensive research and utilized across a consortium of 15 unique institutions, will help you and your colleagues to not only implement the classroom resources (Chapter 3) but also to form a strong faculty team that can continue to develop and offer engaging courses through the creation of additional resources long into the future.

As an illustration, when one of the authors (a mathematician) was invited to read this volume and co-author a preface, he knew nothing about the co-author except that she is a faculty member in a partner discipline and that we both are committed to extensive collaboration across disciplines. When we met to discuss this volume, we found that we each had the same reaction: the volume provides exactly the information needed to initiate or deepen curriculum

[1]Contact: whaver@vcu.edu

development collaboration involving mathematics faculty and faculty from partner disciplines. It includes research results and recommendations from respected sources to support conversations with colleagues, administrators, and possible funding sources. It includes a number of proven processes to foster on-going collaboration. And it includes a collection of classroom resources developed by interdisciplinary teams that bring concepts from partner disciplines into mathematics courses. These resources provide excellent starting points to enrich courses and provide prototypes for further collaboration.

Navigating this Volume

This volume was developed as a part of the consortium *Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships* (SUMMIT-P). SUMMIT-P was formed in 2016 in partnership with MAA's Curriculum Renewal Across the First Two Years committee (CRAFTY). It now includes 15 institutions and is committed to improving the undergraduate mathematics curriculum in response to the recommendations of CRAFTY's Curriculum Foundation (CF) project.

Chapter 1, *Leveraging Interdisciplinary Partnerships to Create an Impactful STEM Curriculum*, highlights research and recommendations from many groups that call for students to have more meaningful experiences within their mathematics coursework. These experiences must enable students to see how mathematics is used to understand real situations. The chapter makes the case that developing and offering courses that provide these experiences requires ongoing collaboration among faculty from mathematics and from other disciplines.

Notable among the recommendations are those from participants in CRAFTY's CF project, a collection of workshops that brought together approximately 15 faculty members from a specific partner discipline and a smaller number of mathematicians. At the conclusion of each workshop, the partner discipline participants prepared a report describing their views concerning the introductory collegiate mathematics that students majoring in their area should study. One unanimous message from the 22 different disciplines was that courses should focus on providing opportunities for students to develop an understanding of fundamental mathematics topics while grounding the discussions in context.

Chapter 1 also emphasizes that the faculty from partner disciplines mostly do not care about computational skills and symbol manipulation; rather, they present an overriding need to develop in students a conceptual understanding of basic mathematical tools. Meaningful mathematical experiences provide students the opportunity to appreciate the connections between mathematical concepts and their many applications in other disciplines. And these experiences can be developed and implemented through interdisciplinary faculty collaboration.

Chapter 2, *SUMMIT-P Processes for Interdisciplinary Faculty Collaboration: Transforming the Undergraduate Experience*, describes how collaborations have been initiated, nurtured, and sustained within and across the SUMMIT-P institutions. The goal of the collaboration is to develop meaningful experiences in mathematics courses, increasing student understanding of the relevance of the mathematics studied through real disciplinary examples and in turn engage partner discipline faculty in the appropriate application of the mathematics in their own courses.

The chapter presents tested processes for developing and carrying out interdisciplinary collaborations. Because of the diversity of SUMMIT-P institutions that have implemented these processes (e.g., liberal arts colleges, community colleges, minority-serving institutions, comprehensive research universities), they can be adapted and adopted at any institution. The processes discussed include 1) the formation of an ongoing multidisciplinary team of faculty, 2) utilization of the "fishbowl" to increase understanding across disciplines, 3) participation in protocols designed to foster structural engagement, and 4) organization of comprehensive site visits.

The formation of an ongoing multi-disciplinary team of faculty is key. It can be organized in various ways, depending on the culture of a particular institution. For example, it could be launched as a learning community or as a standing committee. The leadership should consist of both mathematics and partner discipline faculty with a long-term commitment to a) make suggestions concerning select mathematics course(s), b) respond to these suggestions, c) follow-up by reviewing developed materials, d) visit classes and to talk to students to determine how the suggestions have been implemented, and then e) make appropriate adjustments in partner courses that make use of these mathematical ideas.

A "fishbowl" activity can be used to increase understanding among faculty from different disciplines. Typically, mathematics faculty develop questions for consideration by a group of partner discipline faculty. These faculty discuss the questions among themselves while being observed by the mathematics faculty, who remain silent until the discussion is complete. The questions delve into the mathematical needs of the partner discipline in their courses.

A particularly useful approach is to ask the participating faculty to read the CF report written by faculty from their discipline and to ask if the report accurately describes the current needs of their undergraduate academic programs.

SUMMIT-P has incorporated two protocols to foster structured engagement. In one protocol, a "presenter" is assisted in thinking more extensively about a specific dilemma with the implementation of new classroom resources and to obtain advice from colleagues on how to resolve it. The chapter describes a tiered eight-step process that allows a problem to be framed to enable the presenter to move toward a focused solution. The second protocol allows for best practices to be shared. Each protocol can be modified and used by a team of faculty to help them move toward their goals.

Site visits have been an effective means of strengthening the work at the host site while providing new ideas to the visiting team. Chapter 2 describes in detail how site visits have been organized within SUMMIT-P and how they can be adapted to a wide variety of institutional settings. Teams of faculty can initiate visits within or across institutions by inviting colleagues who represent key aspects of their specific goals to observe and discuss the local work.

Chapter 3, *Interdisciplinary Classroom Modules from the SUMMIT-P Consortium*, provides a collection of resources developed by interdisciplinary teams working within SUMMIT-P, implementing the CF project recommendations. While the primary efforts of SUMMIT-P to date have focused on the modification of mathematics courses, there increasingly is greater focus on complementary modifications in partner discipline courses. Such modifications, explicitly bringing the use of mathematics into focus in these contexts, can deepen the connectedness of student learning and further promote the ability for students to effectively apply their mathematical knowledge and skills more broadly.

Twenty classroom modules are presented, with information that will allow for easy implementation by interdisciplinary faculty teams. Each module includes information about the course and institutional context for which they were designed; mathematical content, purpose, and inspiration; solutions when appropriate; and any necessary prerequisites. Additional information needed to use the modules is available from a companion website.

While all modules require conceptual problem solving, exploring mathematical relationships between variables, and modeling in context, some include complex societal issues while others require students to communicate findings from the application of mathematical processes to a broader audience. While most of the activities bring the partner discipline into mathematics, one is focused on bringing mathematics into the humanities.

Even seemingly small connections across disciplines can help students experience the power of mathematics, greatly increasing their chances for success in targeted courses. And the connections might not be ones a mathematics instructor would even realize are missing. For example, one author observing a calculus-based physics class noticed that not all students realized that the $f(x)$ they saw in one class was the same as the y they saw in another. When the concepts and representations in mathematics and physics were explicitly connected, students were then able to recognize the significance of such expressions in a range of courses. The ability of students to integrate dx/x, but NOT dt/t is another example for which contextualizing information can help students understand why they need to learn something. A physicist may find mathematics beautiful and powerful, but not all students will see those connections. To help these students become successful participants in a STEM-based economy, and to achieve the goal of providing the U.S. with the workforce it needs, it is necessary to empower students to make these connections. This means finding ways out of our artificial silos! This volume provides a roadmap to help all of us make strides in the right direction.

1

Leveraging Interdisciplinary Partnerships to Create an Impactful STEM Curriculum

Susan L. Ganter[1], *The University of Texas Permian Basin*

Debra Bourdeau, *Embry-Riddle Aeronautical University*

Some readers may be familiar with—or even have participated in—the long-standing Curriculum Foundations project of the Mathematical Association of America (MAA). This project contributed to an understanding of the relationship between mathematics and partner disciplines via two decades of targeted exploration with colleagues across 22 disciplines about the mathematical needs of students pursuing careers in those disciplines. This work led to the conclusion that interdisciplinary faculty partnerships are key in the development and implementation of a mathematics curriculum that allows students to really learn mathematical concepts while moving seamlessly to concepts in other disciplines.

In order to fully appreciate the classroom resources presented in Chapter 3, it is important that the reader begin by exploring Chapter 1 with a team of interdisciplinary colleagues. This essential background material will set the stage for the importance of leveraging faculty partnerships across disciplines in the creation of an impactful STEM curriculum. The chapter discusses the Curriculum Foundations process and outcomes, as well as other important research and projects that motivate the need for interdisciplinary conversations.

1.1 Introduction

Mathematics plays a critical role in undergraduate education. In fact, global employers have emphasized for years that they seek individuals who can think mathematically, reason through problems, and work effectively on interdisciplinary teams (Singapore Ministry of Education, 2018; Steen, 2001; Finley, 2021). As such, graduates who have meaningful mathematical experiences are better able to face the challenges of careers in both mathematics and other disciplines—including those in non-scientific areas. Add to these skills the appropriate use of technology, the ability to model complex situations, and an understanding and appreciation of the specific mathematics appropriate to their chosen fields, and students are then equipped with powerful tools for the future. These goals rarely are achieved in mathematics courses, where the expectation traditionally has been that these skills belong elsewhere in the curriculum. As such, students do not see the connections between mathematics and their chosen disciplines; instead, they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings or see the importance to their future careers. Indeed, the mathematics many students are taught frequently is not the most relevant to their chosen fields. For these reasons, faculty members outside mathematics often perceive the mathematics community as uninterested in the needs of non-mathematics majors, especially those in introductory courses.

[1]Contact: Susan.Ganter@utpb.edu

The real question then becomes: How are "meaningful mathematical experiences" defined, and how can (and should) they be measured? Most educators would agree that many mathematics courses are not designed and executed in ways that create such experiences (Lederman, et al., 2013; Walker & Sampson, 2013; Blair, 2006). The lack of these experiences results in students who do not see the importance of mathematical thinking, creating an urge to finish with required mathematics courses as quickly and painlessly as possible.

This is a long-standing issue for the mathematics community—and for the discipline of mathematics broadly defined. Educators continue to agree that mathematical thinking is an important skill, as demonstrated by the continuation of mathematics requirements at all academic levels. The mathematics community ignores this situation at its own peril since approximately 95% of the students in first-year mathematics courses go on to major in other disciplines. However, what is not universally clear is exactly why—and in what ways—these mathematical skills are important to the vast number of students in first-year mathematics courses who go on to major in other disciplines. The challenge, therefore, is to determine—and then provide—the mathematical experiences that are true to the spirit of mathematics yet also relevant to students' futures in other fields. The question then is not *whether* they need mathematics, but *what* mathematics they need and in what context.

1.2 The Call for Integrated STEM Education

The teaching and learning of mathematics continue to dominate the national conversation, especially as integrated approaches to increasing science, technology, engineering, and mathematics (STEM) literacy at all levels are seen as a way to blend technical skill with creative problem solving. Developing global citizens with strong preparation in STEM is more essential now than ever. The National Science Board concurs, "Business and industry leaders, governors, policy makers, educators, higher education officials, and our national defense and security agencies have repeatedly stated the need for efforts to reform the teaching of STEM disciplines in the nation so that the United States will continue to be competitive in the global, knowledge-based economy" (National Research Council, 2007, p. 2). If pressing problems such as climate change, health care, and the sustainable use of natural resources are to be addressed, both a cadre of STEM professionals and a STEM-literate population are essential. Integrated STEM education is key to preparing future generations for increasingly data- and technology-driven careers.

A number of reports underscore the importance of mathematicians and colleagues in partner disciplines working together when making decisions about the mathematical preparation of students. These recommendations align with national discussions focused on STEM education, such as:

1. A 2005 National Academies committee, while focused primarily on policies to promote the hiring of STEM teachers in K-12 schools and the growth of STEM majors in colleges and universities, suggested that undergraduate mathematics instruction must move beyond lecture and embrace inquiry-based learning (National Research Council, 2007). A subsequent committee in 2012 added that it is important to "[have] students work in groups and include authentic problem solving activities" (National Research Council, 2012, p.3). In addition, it was noted that students struggle with problem solving and the use of multiple representations for mathematical concepts (such as graphs, symbolic expressions, numerical simulations, and verbal interpretations), and these challenges block students' ability to effectively learn in subsequent STEM courses (National Research Council, 2012).

2. The National Academies Committee on Prospering in the Global Economy of the 21st Century was given the charge "to enhance the scientific and technological enterprise so that the United States can successfully compete, prosper, and be secure … in the 21st century" (National Research Council, 2007). Science and mathematics education was one of four areas targeted by this committee, illustrating the centrality of these disciplines to the future of the nation. More recently, the President's Council of Advisors on Science and Technology (PCAST) stressed the urgency of improving education in STEM—including reform not only for courses serving STEM majors, but also those designed to prepare a more general "STEM-capable" workforce (PCAST, 2012, p.3). One of PCAST's five recommendations for action is to "Launch a national experiment in postsecondary mathematics education to address the mathematics-preparation gap" (PCAST, 2012, pp.27–30). The rationale for this recommendation includes the National Research Council's (NRC) observation that "Low-performing students with a high interest and aptitude in STEM careers often have difficulty with the math required in introductory STEM courses" (National Research Council, 2012, p.i). PCAST states that current introductory college mathematics courses are

"frequently uninspiring, relying on memorization and rote learning, while avoiding richer mathematical ideas" (PCAST, 2012, p.28). Unfortunately, mathematicians have not typically interacted with faculty outside of mathematics on curriculum development (Ewing, 1999; Ganter & Haver, 2011). As such, "Students do not see the connections between mathematics and their chosen disciplines; instead they leave mathematics courses with a set of skills that they are unable to apply in non-routine settings" (Ganter & Barker, 2004).

3. The Investing in the Next Generation through Innovative and Outstanding Strategies (INGenIOuS3) project was a 2013 collaboration among mathematics and statistics professional societies and the National Science Foundation (NSF), to identify and envision programs and strategies for increasing the flow of mathematical sciences students into the workforce pipeline. Six main action "threads" were identified; Thread 5—Develop alternative curricular pathways—focuses on the need for modernized mathematics programs that incorporate alternative curricular entry points, applications that reflect the complexity of problems typically faced in work environments, and appropriate uses of technology tools (Zorn, 2014).

4. *The Mathematical Sciences in 2025* is a 2013 NRC report that states, "The educational offerings of typical departments in the mathematical sciences have not kept pace with the changes in how the mathematical sciences are used. A redesigned offering of courses and majors is needed. Although there are promising examples, a community-wide effort is needed . . . to make undergraduate courses more compelling to students and better aligned with the needs of user departments" (National Research Council, 2013, p.124).

5. The 2015 report *A Common Vision for Undergraduate Mathematical Sciences Programs in 2025* builds upon previous findings (National Research Council, 2013; PCAST, 2012; Saxe & Braddy; 2015; Zorn et al., 2014) that found widespread concerns about the applicability of mathematics skills, the alignment of those skills with other disciplines, and the preparation of a STEM workforce. Faculty from five professional associations focused on undergraduate mathematics education, emphasizing that "the status quo is unacceptable" (Saxe & Braddy, 2015, p.35). *Common Vision* concludes that updated curricula should acknowledge the reality that those who are truly STEM-literate will need to work effectively in multidisciplinary teams. Such arguments in support of systemic institutional change to reinforce and sustain curricular reforms recently have gained traction. Unfortunately, most change efforts have focused on the faculty, course, or department level, while policies that have the potential to lead to success are primarily at the institutional level (Saxe & Braddy, 2015). Common change models include strategic planning and organizational learning, typically focusing on practices such as vision setting, benchmark identification, information gathering, and data analysis. A landscape analysis that provides deep understanding of the culture and politics of an institution also is an essential component for achieving buy-in and preparing for expected resistance to widespread change (Saxe & Braddy, 2015; Kezar, 2018).

6. The National Academies Committee on Barriers and Opportunities in Completing 2-Year and 4-Year STEM Degrees (National Academies of Sciences, Engineering, and Medicine, 2016) advocated in its 2016 report for acknowledgement of multiple student pathways to earning STEM degrees. The committee concluded that diversification of the STEM workforce is possible through a focus on student support that provides multiple opportunities to engage in high-quality STEM experiences. This commitment must be coupled with a more precise fit of STEM programs and teaching practices with the specific needs of the student populations served by an institution. Recognizing that a singular approach is impossible, the committee advocates for system change created by "a series of interconnected and evidence-based approaches" (National Academies of Sciences, Engineering, and Medicine, 2016, p.4). The resulting series of recommendations stresses the importance of gaining an understanding of student populations, their pathways, and their education goals, and recognizing that graduation rates are not the only institutional success metric (National Academies of Sciences, Engineering, and Medicine, 2016).

7. The National Science and Technology Council's Committee on STEM Education (2018) delineates a five-year plan for "lifelong access to high-quality STEM education" (frontispiece). The report discusses a notable shift from a group of disciplines that merely "overlap" to an "integrated and interdisciplinary approach to learning and skill development" (p. v). This shift has included a focus on real-world problem solving as well as competencies such as critical thinking (Committee on STEM Education, 2018). Specifically, the report includes four pathways to support its goals for building STEM literacy, increasing diversity in STEM fields, and ensuring a prepared future STEM workforce. One of these pathways is to "engage students where disciplines converge" to enhance

meaning (p.15). The report also establishes multiple necessary actions for achieving its goals, specifically increasing support for educator “upskilling” through professional development initiatives such as webinars and workshops (Committee on STEM Education, 2018; SUMMIT-P, 2020a; SUMMIT-P, 2020b). Recognizing the importance of success in mathematics and statistics, the report’s strategic plan promises to “make mathematics a magnet,” through more fully contextualized instruction and greater focus on understanding the language, methodologies, and problem-solving approaches associated with various disciplines (Committee on STEM Education, 2018, p.17).

If the U.S. is going to meet the challenges of the 21st century, the widely-recognized issues currently faced (e.g., Committee for the Undergraduate Program in Mathematics, 2004; Ganter & Barker, 2004; PCAST, 2012; National Academies of Sciences, Engineering, and Medicine, 2016; Committee on STEM Education, 2018) must be met by a mathematically-literate population with the necessary skills to solve them. Recognizing the importance of STEM skills and the generally low levels of student performance in STEM topics (even, in many cases, after significant coursework), government and private funding agencies around the world have focused significant attention on ways to improve STEM instruction (e.g., Project Kaleidoscope, 2006 and 2017). These efforts have resulted in knowledge about how students learn STEM subjects (e.g., National Research Council, 1999), as well as new instructional strategies and models for improving STEM learning (e.g., Handelsman et al., 2004 and Elrod & Kezar, 2016). What has yet to happen is the widespread implementation and institutionalization of these research-proven instructional methods and models (e.g., National Research Council, 2003). The core of this implementation problem is that change agents working toward STEM reform are operating in relative isolation using an inadequate set of change strategies. Examples that illustrate these findings include the following:

1. **Current change strategies have not been widely effective – new approaches are needed.** Recent decades have seen numerous calls for the reform of undergraduate instruction in STEM disciplines. Substantial time and money have been allocated to the creation of knowledge about instructional alternatives to traditional, lecture-based instruction and the development of instructional strategies and curricula consistent with this new knowledge. However, these new strategies and curricula are far from being the norm in undergraduate STEM courses.

2. **STEM change agents generally do not treat the development and testing of change strategies as a problem that can be addressed through research.** By synthesizing research-based knowledge, a starting point and impetus for developing more systematic approaches to the development and improvement of change strategies relevant to undergraduate STEM instruction can be created.

3. **Each STEM discipline has more or less independently developed its own set of implicit change strategies.** Change agents and curriculum developers within one discipline do not know much about change strategies used by change agents within other disciplines.

Scientifically tested instructional strategies and theories now exist that can be used to design undergraduate STEM instruction. These strategies and theories have been used in a variety of STEM disciplines at many institutions. There is notably a large degree of similarity and agreement between the STEM disciplines in terms of what curricular strategies are promoted (Ferrini-Mundy & Güçler, 2009; National Research Council, 2003; Elrod & Kezar, 2016; Committee on STEM Education, 2018). For example, the supporting materials to Handlesman et al. (2004) identify five widely-used scientifically-based instructional strategies that are used in multiple STEM disciplines: group brainstorming or problem solving in lecture, problem-based learning, case studies, inquiry-based labs, and interactive computer learning.

At the same time, there is weak coupling among the development efforts of scientific-based curriculum development within and across the STEM disciplines. For example, the use of conceptual inventories began in the field of physics and is now found in all of the STEM disciplines (Libarkin, 2008; Furrow, 2019 Brownell, 2014; Bursic, 2020; Madsen, 2017). Although there are a few examples of cross-discipline interactions, educational researchers within each discipline are not typically aware of thinking or developments within the other STEM disciplines. Much work is presented at discipline-specific conferences and in discipline-specific journals, but without substantive interdisciplinary dissemination and sharing.

1.3 Sample Change Strategies Employed Independently Across STEM Disciplines

In his synthesis of the National Academy of Sciences Workshop on Linking Evidence and Promising Practices in STEM Undergraduate Education, James Fairweather (2008) suggests that "the greatest gains in STEM education are likely to come from the development of strategies to encourage faculty and administrators to implement proven instructional strategies" (p. 26). Thus, he calls for the development of "models for implementation, dissemination, and institutionalization for STEM reforms where the relative roles of evidence-based research on teaching, leadership, workloads, rewards, and so on are clearly delineated" (p. 26). Fairweather (2008) also notes the important role that disciplines and professional societies should play in this change process.

Commonly-used change strategies in STEM disciplines involve telling faculty about innovations and presenting research evidence to document the efficacy of the innovations (Fairweather, 2008; Seymour, 2001). While these are important for all faculty members and may influence some faculty to change, they do not appear to be sufficient to promote widespread instructional change. It is hypothesized (e.g., Fairweather, 2008; Merton, Froyd, Clark, & Richardson, 2009; Seymour, 2001; Tobias, 2000; Henderson, Beach, & Finkelstein, 2011; Finelli, 2014) that widespread change will require work at multiple levels.

For the last few decades, communities have increasingly engaged in educational research within the disciplines (for example, physics has offered PhDs for work in education research since the 1970s, and the American Physical Society endorses physics education research and offers a dedicated journal within the field). Such work has received increasing funding from key agencies such as NSF and, as mentioned above, has resulted in a new generation of curricula that are based on research. Nonetheless, change strategies remain largely unstudied and underspecified. Few cross-disciplinary efforts exist and the STEM disciplines—and higher education in general—stand to benefit from studies of change strategy both within and across disciplines, as can be seen in the disciplinary examples below.

Mathematics. In the early 1980s, the mathematics community became concerned about decreasing enrollments, low completion rates, and the outdated curriculum and teaching methods for college mathematics (e.g., Dunbar, 2006). This concern motivated a host of programs, projects, conferences, and other efforts to rethink the organization and content of mathematics instruction at all levels (Tucker, 1995; Schoenfeld, 1995). For example, a major national effort focusing on the reform of calculus was spawned by several conferences in the 1980s, culminating in the development and funding of the NSF Calculus Program (1988-98). Over 120 funded projects changed the landscape of this course, including the development of new curricular materials and textbooks; use of alternative learning environments; and an emphasis on different student skills, such as facility with computers and conceptual understanding (Ganter, 2001). The Calculus Program was the first initiative in NSF's revived Division of Undergraduate Education (DUE) and led to numerous course-related funding initiatives, including targeting courses such as differential equations, linear algebra, precalculus, and college algebra.

One critical outcome from these efforts was a change in emphasis for the Curriculum Guide of the Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America (MAA). The CUPM Curriculum Guide is a document published once each decade to assist college mathematics departments in the on-going development and improvement of their undergraduate programs, traditionally focusing only on courses in the mathematics major. However, in the late 1990s, as CUPM began discussing the preparation of its next Curriculum Guide, it was clear that this important document could no longer ignore the wealth of new programs, courses, and materials for introductory college courses resulting from the reform movement in the undergraduate mathematics community. In particular, the dramatic changes being implemented in introductory courses such as precalculus, calculus, and differential equations needed to be studied and directly addressed in the recommendations of the Curriculum Guide. As a result, in 1999 CUPM initiated a major analysis of the undergraduate mathematics curriculum, including a series of disciplinary workshops known as the Curriculum Foundations Project. The project findings were published by MAA in the report *Curriculum Foundations Project: Voices of the Partner Disciplines* (Ganter & Barker, 2004) and influenced the content of the *2004 CUPM Curriculum Guide* (CUPM, 2004). This focus on introductory courses (in addition to major-level courses) represents a dramatic shift in emphasis for the Curriculum Guide, and hence the importance placed on these courses by the mathematics community.

Engineering. Despite criticism that Accreditation Board for Engineering and Technology (ABET) accreditation is a disincentive to innovation (Volkwein, Lattuca, & Terenzini, 2008), Engineering Criteria 2000 (EC2000) have been an important lever for change in engineering education programs (ABET, 2006). In 1996, ABET adopted these new criteria in response to industry leaders' calls to emphasize professional as well as technical skills (Prados, Peterson, & Lattuca, 2005). The outcomes-based system allows a great deal of flexibility but little guidance. Throughout the 2000s, this void was filled by enlisting the expertise of assessment professionals and educational researchers, which resulted in innovations that utilize research on learning. Many of the contributions are tied to the eight Engineering Education Coalitions, funded by NSF from 1990–2005 and involving over 40 universities (Froyd, 2005). These coalitions "were intended to catalyze systemic change across the engineering education community by developing and demonstrating the efficacy of new curricular models" (Froyd, 2002). Coalition personnel also are credited with driving the adoption of EC2000 criteria by ABET (Prados et al., 2005). Examples include bringing active-learning design projects into the freshman year, service learning such as Engineering Projects in Community Service (EPICS; Coyle et al., 2000) and other brands, and wider use of active learning techniques in large lecture classes. In an assessment of ABET EC2000 criteria, faculty and program chairs reported that the new system resulted in increased use of active learning methods such as group work, design projects, and case studies (Lattuca, Terenzini, & Volkwein, 2006). More recently, Borrego, Hall, and Froyd (2013) report an 82% awareness rate and 47% adoption rate of seven of these innovations, based on a survey of U.S. engineering department chairs.

Physics. Physics education research at the college level has traditionally been dominated by small groups of curriculum developers who research and develop their own curricular products (Beichner, 2009). Thus, there exist a relatively large number of named curricula (Redish, 2003). Examples include Peer Instruction (Mazur, 1997; Mazur & Crouch, 2001; Mazur, 2009; Zhang, Ding, & Mazur, 2017), Interactive Lecture Demonstrations (Sokoloff & Thornton, 1997; Sokoloff, 2016), Tutorials in Introductory Physics (McDermott & Shaffer, 2002), Cooperative Group Problem Solving (Heller, Foster, & Heller, 1996; Heller & Hollabaugh, 1992; Heller, Keith, & Anderson, 1992; Heller & Heller, 2010; Yereushalmi, Cohen, Heller, Heller, & Henderson, 2010), and Workshop Physics (Laws, 1991, 1997; Laws, Willis, M., & Sokoloff, D., 2015). The developers of these curricula have worked to disseminate their products through talks, workshops, and publications that have reached almost all of the roughly 13,000 physics faculty employed in four- and two-year colleges in the U.S. (Ivie, Guo, & Carr, 2005; Ivie, Stowe, & Nies, 2003; McFarling & Neuschatz, 2003; Shulman, 2001).

Henderson and Dancy (2009) completed the first phase of a study designed to document the knowledge of physics faculty about and use of research-based instructional strategies relevant to the teaching of introductory quantitative physics. Results of this web survey of a randomly selected sample of physics faculty indicated that the development and dissemination efforts by physics education reformers have made an impact on the knowledge and practice of many faculty. For example, faculty knowledge and attempted use of research-based instructional strategies appears to be relatively widespread. Almost all faculty surveyed (87.1%) were familiar with one or more strategies and approximately half of faculty (48.1%) said they currently were using at least one strategy. However, the survey also indicated that there is significant room for improvement; e.g., research-based instructional strategies are commonly modified during implementation; rarely are they used as recommended by the developer. Results also indicated that faculty frequently try strategies, then discontinue their use. In many cases, discontinuance appears to occur because faculty believe that the strategies do not work or fit well with their teaching situation. The authors concluded:

> It is important to investigate ways to better support faculty use of research-based instructional strategies once they have developed awareness and interest. This is likely to include ways to provide substantial support and guidance to faculty during the implementation and customization process as well as ways to provide flexible curricula that can be easily customized without losing the essential features (Henderson & Dancy, 2009, p. 15).

Another problem that has been noted with the development and dissemination change strategy is that faculty often see curriculum developers as disinterested in them and their students, simply promoting a particular curriculum (Henderson & Dancy, 2008). Instructors described what they saw as the sales or evangelist mentality, making their interactions with curriculum developers somewhat confrontational.

Quantitative Literacy. Attention to the need for widespread quantitative literacy (QL) is a relatively recent phenomenon, but it is an increasingly relevant concern. The citizens of the 21st century have access to vast amounts of technical information, with enormous personal rewards (or penalties) resulting from decisions made in light of that information. While the citizens of 100—or even 30—years ago relied on "experts" to interpret such data and present the public with limited options from which to choose, individuals today have greater control over decisions regarding their personal health, finances, and general welfare than ever before, and hence the risk of ignorance becomes much greater. With increased power for each citizen comes increased responsibility to provide education that will allow for informed decisions.

This relatively recent shift in thinking about QL informs the underlying philosophy of a wealth of initiatives to promote QL that became prominent in the late 1990s. The publication *Why Numbers Count* (Steen, 1997) was one of the first to receive widespread attention, making bold statements about the need to empower the public of the 21st century with QL, much the same way that citizens of past centuries learned to appreciate the power associated with verbal literacy. This publication was quickly followed by the formation of the Quantitative Literacy Design Team by the National Council on Education and the Disciplines of the Woodrow Wilson National Fellowship Foundation, resulting in the publication *Mathematics and Democracy: The Case for Quantitative Literacy* (National Council on Education and the Disciplines, 2001). These two publications motivated several action-oriented initiatives, including a forum convened at the National Academy of Sciences in 2001 (Madison, 2003); the development of the National Numeracy Network (NNN), incorporated in 2004 (Ganter, 2003; Steen 2004); and the publication of *Achieving Quantitative Literacy: An Urgent Challenge for Higher Education* in 2004 (Steen, 2004), setting an agenda for pursuing QL in the educational system. Simultaneously, many colleges and universities were developing courses and programs to address the issues at the local level.

But what is quantitative literacy? Although many individuals and groups have examined the concept of QL over the past two decades (e.g., Kirsch & Jungeblut, 1986; Madison, 2003; National Center for Educational Statistics, 1993; National Council on Education and the Disciplines, 2001; Organization for Economic Cooperation and Development, 1995; Secretary's Commission on Achieving Necessary Skills, 1991; Sons et al, 1996; Steen, 2004, 1997) there is no consensus about exactly what constitutes quantitative literacy (also known as numeracy). Perhaps the first extensive effort to form such a definition came from the Quantitative Literacy Design Team in 1999. However, the discussions of this group led to an understanding that the components of QL are so broad that a succinct definition is difficult and impractical. Instead, the Design Team developed descriptive elements of QL, with corresponding skills (National Council on Education and the Disciplines, 2001). These elements and skills form the basis of a case for QL for every citizen, and the need for continuing creation of educational systems and programs that make achievement of such a goal possible. Subsequent widespread discussions (see Madison & Steen, 2003) led to the development of educational goals for both K-12 and postsecondary education in the area of QL (Steen, 2004).

Specifically, a baseline level of QL for all adult citizens would include elements such as:

- ability to interpret public data
- application of quantitative information to decision making
- confidence with mathematics
- cultural appreciation of mathematics
- logical thinking in forming opinions
- practical problem-solving skills
- prerequisite mathematics for advanced quantitative study
- strong number and symbol sense
- use of mathematics in context (National Council on Education and the Disciplines, 2001).

These elements are expressed in countless ways throughout everyday life. The design team examined several areas in which QL plays an important role for all citizens, including citizenship, culture, education, professions, personal finance and health, management, and work (National Council on Education and the Disciplines, 2001). Basic skills such as arithmetic, modeling, statistics, and reasoning must be second nature if one is to attain confidence in dealing with a wide array of quantitative information.

The complexities of developing educational systems that insure a quantitatively literate public are similar to those tackled in the area of writing over the past several decades. Though no less important for all citizens than fluency in reading and writing, QL too often continues to be the province of the few. Indeed, for too long, our educational system has produced a scientific and mathematical elite while failing to nurture the literate citizenry required for both robust democracy and popular support for science and mathematics. As a result, the gap between expert and citizen has widened dangerously, most notably when numbers and data are brought to bear in deciding public and private issues. A similar trend was observed in writing literacy until the mid-1980s, when a movement was initiated to make writing a priority for all students. Now, "writing across the curriculum" has become a widespread and established practice in higher education (Lieberman & Wood, 2003)—including in courses like science and mathematics, which have traditionally attracted many students because they "wouldn't have to write." A comparable infusion of quantitative concepts across disciplines is necessary if students are to become quantitatively literate. As such, it is now generally accepted that QL is not the domain of mathematicians alone, but is a shared responsibility of all disciplines. A simultaneous goal must be a societal shift that supports competency with quantitative information at the same level as reading and writing, making such statements as "I hate math" or "I can't do mathematics" socially unacceptable. All Americans must have the opportunity to participate fully as citizens, through education that empowers them to succeed in an increasingly quantitative society.

Fortunately, QL now is acknowledged within various K–2 standards, including mathematics and science (National Council of Teachers of Mathematics, 2000; National Research Council; 1995). Many colleges and universities cite QL as among their foremost educational aims and some have undertaken specific programmatic initiatives to support this new expectation for student learning. Many of these collegiate initiatives are institution-wide and interdisciplinary, indicating the shared responsibility for QL education among all disciplines.

Therefore, as has been argued in similar discussions within the writing community, developing quantitative literacy for postsecondary students is possible. However, the effort must be a sustained one, extending over more than one course and for more than one semester. Continuous learning opportunities in QL across numerous courses and throughout a student's formal education are vital, and the ultimate success of most students in QL will depend on its priority within individual institutions. Specifically, the creation of a quantitatively literate public—even among college graduates—will require an enduring effort of cooperation and collaboration within and among institutions.

Opportunities for Cross-Disciplinary Synthesis. It is important that the STEM Education community compare and contrast the strengths and gaps in the reform efforts across disciplines, seeking out those strategies that appear to have been successfully implemented in multiple disciplines, those that have been implemented with lesser impact, and those stand-out strategies that may only appear in one discipline but may have important elements that can be applied more broadly. In this way, the proposed cross-disciplinary synthesis will be guided by questions of what worked well within disciplines, at what academic level (for example, many of the named curricula in physics and similar efforts in mathematics focus on introductory undergraduate courses, because the impact is seen to be highest and the content most uniform), and from what organizational level (e.g., individual faculty efforts, departmental efforts, discipline-sponsored efforts). Similar strategies can subsequently be identified and strengthened, while different strategies can be further examined to understand whether and how they might be implemented across disciplines. For example, if the focus on introductory courses is uniform across disciplines (a likely outcome), how can some of the differences in approaches be applied to these introductory course reform efforts to better ensure their success? With ABET motivating assessment approaches for engineering courses, engineering change agents may be more likely to create assessment instruments for others to use in a variety of courses (with a variety of teaching approaches) to measure key identified learning outcomes, rather than specific curricula to be followed by others. Does this assessment-oriented approach hold promise for introductory courses in other STEM disciplines, given that faculty so often significantly modify named curricula during adoption (Henderson & Dancy, 2008)? This is a small example of the kind of cross-disciplinary synthesis that is possible when the disciplinary reforms are thoroughly reviewed.

The complexities of the higher education system in the United States make it challenging to support predictive, sustained change. Current educational change strategies lack attention to the broad number of components that influence educational practices. While faculty autonomy often allows for individuals to enact curricular changes in their individual classrooms, sustained and productive change requires attention to the broader educational system. Faculty are influenced by their departments, disciplinary communities, institutions, and broader cultural factors that shape what and how educational transformation might occur. For sustained, productive change, all levels of the educational system need to be involved.

1.4 Motivating the Interdisciplinary Conversation: The Curriculum Foundations Project

As previously discussed, MAA's Committee on the Undergraduate Program in Mathematics (CUPM) began discussing the preparation of its next Curriculum Guide in the late 1990s. As the subcommittee of CUPM concerned with the first two years of the college mathematics program, CRAFTY (Committee on Curriculum Renewal Across the First Two Years) has had a major role in analyzing and formulating recommendations concerning the foundational undergraduate years in mathematics instruction. Moreover, given the impact of mathematics instruction on the sciences and quantitative social sciences—especially instruction during the first two years—there was a need for significant input from these partner disciplines. Therefore, CRAFTY was charged with gathering this necessary information for the "mathematics intensive" disciplines (e.g., physics, chemistry, and engineering). In 1999-2001, CRAFTY conducted a series of disciplinary workshops that launched the Curriculum Foundations Project.

Each Curriculum Foundations (CF) workshop consisted of 20-35 national participants, the majority chosen from the discipline under consideration, the remainder chosen from mathematics. Each 2-3 day workshop was a discussion among the representatives from the partner discipline, with mathematicians present to listen and serve as resources. The result of each workshop was a report, written and reviewed by partner discipline representatives and directed to the mathematics community. The CF Steering Committee supplied a common set of questions to guide the workshop discussions. These common questions provided uniformity and the ability to compare findings across workshops (i.e., across disciplines). The 17 workshops culminated with a conference to analyze and synthesize the workshop findings. This *Collective Vision* and the 17 disciplinary reports were published by MAA in the document *Curriculum Foundations Project: Voices of the Partner Disciplines* (Ganter & Barker, 2004) and contributed significantly to the content of the *2004 CUPM Curriculum Guide* (CUPM, 2004).

Based on these findings—as well as input from numerous focus groups, surveys, and research findings—the CUPM Curriculum Guide made the following bold recommendation: "All students meeting general education or introductory requirements in the mathematical sciences should be enrolled in courses designed to:

- Engage students in a meaningful and positive intellectual experience;
- Increase quantitative and logical reasoning abilities needed for informed citizenship and in the workplace;
- Strengthen quantitative and mathematical abilities that will be useful to students in other disciplines;
- Improve every student's ability to communicate quantitative ideas orally and in writing;
- Encourage students to take at least one additional course in the mathematical sciences." (CUPM, 2004, pg. 28).

In addition, the Guide states

> Unfortunately, there is often a serious mismatch between the original rationale for a college algebra requirement and the actual needs of the students who take the course. A critically important task for mathematical sciences departments at institutions with college algebra requirements is to clarify the rationale for the requirements, determine the needs of the students who take college algebra, and ensure that the department's courses are aligned with these findings (CUPM, 2004).

In response to these challenging recommendations from CUPM, CRAFTY set a goal of having every mathematics department review its general education and introductory offerings. CRAFTY worked toward this goal by offering a series of panel discussions at regional and national meetings; sponsoring special sessions highlighting innovative courses offered at many colleges and universities; sponsoring workshops to provide professional development for faculty offering introductory courses; conducting (with support from the Harvard Calculus Consortium and in conjunction with MAA's Committee on Mathematics Across the Disciplines) a second series of disciplinary workshops focusing on the social sciences and humanities; developing a detailed set of College Algebra Guidelines that were ultimately endorsed by CUPM; and conducting (with support from NSF) a program to support faculty teams from eleven institutions to change their college algebra courses so that they more closely aligned with the MAA College Algebra Guidelines.

The second series of CF workshops, focused on the social sciences and humanities, were conducted in 2007–09. Partner discipline participants re-confirmed the universal needs identified in the original 17 workshops—in spite of the vast differences between the disciplines represented. Specifically, all 22 disciplinary reports (Ganter & Barker, 2004; Ganter & Haver, 2011) indicate a need for mathematics courses that emphasize:

- **Conceptual understanding and problem solving**—communicating solutions to diverse audiences; precise and correct use of mathematics in presentations and reports

Colleagues in the partner disciplines confirm that applying mathematics to unfamiliar problems requires far more than computational skill. In the partner disciplines, they will have to make judgments about what mathematical techniques (and what technologies) are appropriate for specific problems. Therefore, they must develop their own processes for solving problems. Support must be given as students make this transition.

- **Arithmetic and basic mathematical equations**—relationships between variables; percentages, proportion, and measurement; translation of words into appropriate formulas and equations; graphical representations; unit conversions

Faculty should explore locally what topics can be omitted and teach the remaining topics in more depth. Topics can (and should) be eliminated to achieve a depth of conceptual understanding on a limited number of mathematical tools. Time can be gained for conceptual underpinnings by focusing only on calculations using the most basic functions.

- **Modeling and problems in context**—building analytical models and testing their viability; applying theory to real problems and evaluating alternative solutions; communicating and coordinating with disciplinary faculty to develop alternative problems; using context to inspire and create a need for mathematics (i.e., mathematics as a common technical language)

Modeling is a powerful problem solving process that helps students use their skills, knowledge, and creativity to produce results and products that can benefit society. Therefore, modeling can build student confidence, introduce them to useful and powerful elements of mathematics, and provide a mechanism for communication, expression, and reasoning that is cross-cultural and cross-disciplinary.

- **Estimation and approximation**—use of experimentation and exploration to discover mathematical concepts

A broader view of mathematics needs to be communicated to students. The continuous, linear, and exact perspectives in mathematics are those that are usually covered in the undergraduate mathematics curriculum, while the discrete, nonlinear, and approximate are, for the most part, not covered. Students should be exposed to a variety of views, although the depth, breadth, sequence, and methods of this exposure would depend on the nature of the local institution.

- **Statistics and quantitative data**—measures of central tendency and standard deviation; analyzing data to make inferences and draw conclusions; presenting data as pictures (such as bar graphs, line graphs, and scatter plots)

The importance of data analysis for so many of the partner disciplines argues strongly for an increased presence of basic statistical training in the first two years of undergraduate mathematics. The experience provided should primarily be concerned with descriptive statistics, needing only a brief introduction to probability. The material should be motivated by a variety of examples and real data sets, including data collected by students.

- **Appropriate use of technology**—spreadsheets; geometrical/graphical software

Technology should be used in introductory mathematics courses to provide students with tools for solving problems. Current technologies make possible the discussion of important problems that previously were inaccessible, such as problems without analytic solutions. Specifically, partner discipline colleagues universally state that spreadsheets are the most widely used technological resource. However, mathematics faculty must stress to students the importance of choosing the appropriate method of calculation for the desired task. Therefore, mathematics courses should stress intelligent and careful interpretation of results obtained from technology.

- **Communication**—development of reading, writing, speaking, and listening skills; explaining mathematical concepts, logical arguments, and the “hows and whys” in words

Though there are successful examples of instructors who teach writing and speaking in the mathematics classroom, there is still a need for more universal implementation of these activities. Many mathematicians view instruction in writing and oral presentation as time-consuming and foreign to their own training. However, these skills are critical to students, and faculty members in all disciplines have a responsibility to incorporate them into class instruction. Such activities can take the form of written lab assignments, technical reports, group projects, professional presentations in class, and short essays on exams. Communication skills are related to logical reasoning: if you can't explain it, you don't understand it.

Although a line-by-line comparison between the stated needs of social sciences and humanities students versus those from the more traditionally mathematically intensive disciplines yields some subtle differences in the specifics, the broad categories outlined above are virtually the same for the 22 disciplines investigated. And even more amazing is that this list of priorities was consistently and independently developed across multiple disciplines. Whether the workshop focused on physics, engineering, economics, or the arts, the message from these partner disciplines was repeated again and again: introductory collegiate mathematics courses should focus on giving students an appreciation and understanding of fundamental mathematical topics while grounding the discussions in context. The specific topics are not as important as 1) technical confidence; 2) the application of mathematics to a variety of contexts; and 3) the ability to choose appropriate tools for modeling, evaluating, and communicating mathematical results.

1.5 Implementing the Curriculum Foundations Recommendations

The recommendations initiated by the conversations with partner disciplines still present a big challenge, coming from the large and diverse set of disciplines that are making ever greater use of mathematics. It certainly is understood that no mathematics department can possibly offer a different mathematics course for majors in each of the 22 different disciplines represented—and depending on the situation and circumstances at individual institutions, there may be a need for a greater focus on the needs of certain disciplines (Liston & Getz, 2019; Hartzler & Blair, 2019; Dana Center, 2020; Krueger, 2017; Dana Center, n.d.). This makes the need to rethink and revise the most popular introductory mathematics courses (such as College Algebra and Calculus) even more critical. Since broad categories such as conceptual understanding, problem solving, mathematical modeling, and communication cut across the recommendations from all partner disciplines, it makes sense to re-develop introductory mathematics courses in ways that incorporate these universal needs.

In considering the most effective way to revise these introductory mathematics courses, it is important to dispel the common belief among mathematicians that the users of mathematics care primarily about computational and manipulative skills, artificially forcing mathematicians to cram courses full of algorithms and calculations. Perhaps the most encouraging discovery from the CF Project is that this stereotype is largely false. Though there certainly are individuals from partner disciplines who hold the more strict algorithmic view of mathematics, the disciplinary representatives at the CF workshops were unanimous in their emphasis on the overriding need to develop in students a conceptual understanding of the basic mathematical tools.

Partner disciplines also value the precise, logical thinking that is an integral part of mathematics. They would like to see these habits of mind emphasized in early collegiate mathematics instruction in ways that enhance understanding of the underlying concepts. However, the partner disciplines vary widely in the level of rigor and formal reasoning their students need to master. For instance, business students often do not need significant training in formal mathematical proof, while students in computer science or electrical engineering must develop the skill to apply formal logic and construct simple, but rigorous, proofs (Ganter & Barker, 2004). It is therefore important that each institution carefully consider their unique student body and the corresponding major disciplines when planning how to incorporate mathematical reasoning into the curriculum.

The recommendations for departments as outlined in the *Collective Vision* (Ganter & Barker, 2004) resulting from the CF Project still hold true today. Namely, in order to address the changes needed in introductory mathematics courses and more broadly in the undergraduate STEM curriculum, departments across each institution should:

1. *Promote professional development*—the recommended modifications to the undergraduate curriculum will not come easily, and will require faculty from across the disciplines who not only are committed to the change, but who also are familiar with current understandings of curricular change and the implications for student learning.

2. *Establish collaborative efforts between mathematicians and partner disciplines that result in innovative instructional materials*—collaboration across department lines must be deliberate and supported at the institutional level; experimental courses taught by interdisciplinary teams should be encouraged.

3. *Encourage institutional assessment of programmatic changes*—the fundamental and far-reaching recommendations from the CF work require that institutions engage in sustained efforts to collect and analyze information about the influence of the recommended changes on the undergraduate curriculum (Ganter & Barker, 2004, p. 8).

The CF workshop reports (Ganter & Barker, 2004; Ganter & Haver, 2011) provide the details from individual discussions at the disciplinary workshops as interpreted by leaders from the partner disciplines. The words are their words, the recommendations are their beliefs about their students. These individual disciplinary reports, combined with the *Collective Vision* (Ganter & Barker, 2004) plus references to QL and its enormous variety of cross-disciplinary demands, can serve as resources for the critical multi-disciplinary discussions that need to occur regularly at individual institutions in order to revise the mathematics curriculum (and corresponding partner discipline courses) in ways that will promote student success. Because the reports were written by the partner disciplines, they are more likely to be received by colleagues in those disciplines as credible ideas from like-minded colleagues—not a mandate handed down by mathematicians. Supporting informed interdepartmental discussions about the undergraduate curriculum, including necessary standards and benchmarks that define what knowledge students need in order to understand, solve, and discuss applied problems that rely on mathematics, might ultimately be the most important outcome of the CF Project.

Increased interdisciplinary collaboration is likely to help students overcome the transfer problem from mathematics courses to partner discipline courses. Specifically, students often have difficulty seeing the relationships between problems in non-mathematics disciplines and material studied in mathematics courses. Mathematicians should seek out projects from partner disciplines to be used in mathematics courses, exposing students to discipline-specific contexts for various mathematical topics. Contextualized problem solving, defined as "the concept of relating subject matter to real world situations" (Accelerating Systemic Change Network, 2020, p.1), has been shown to a) motivate students to make connections and transfer learning as well as increase how much students learn in a finite amount of time, b) improve intrinsic motivation to learn, c) deepen engagement with learning, and d) improve students' confidence in their ability to learn (Accelerating Systemic Change Network, 2020; Beisiegel et al, 2020). While the research is not yet definitive, evidence suggests that contextualization has the potential to accelerate and improve student success in college, particularly for underprepared students (Cwikla, 2017). Studies on active learning methods, such as inquiry-based learning, typically show that students' abilities to solve conceptual mathematics problems increases without any loss in performance on procedural or algorithmic problems (e.g., Chappell, 2006; Houseman, 2003; Kwon, Rasmussen, & Allen, 2005; Rasmussen & Kwon, 2007; Smith, 2005). In addition, active learning techniques have a sizeable and persistent positive impact on the performance of low-achieving students in subsequent mathematics courses (Kogan & Laursen, 2014). Such interdisciplinary class sessions—supported by team teaching opportunities in both mathematics courses and courses in the partner disciplines—will have a positive effect on the ability of students to transfer knowledge between courses.

1.6 Conclusion

This discussion has come full circle to the question initially posed: How are "meaningful mathematical experiences" defined, and how can (and should) they be measured? Several decades of national conversations and extensive investment in the development of impactful curricula in undergraduate mathematics have confirmed the fact that mathematics content cannot be successfully developed, taught, and learned in isolation—especially at the introductory level. The power and relevance of mathematics for almost all students lies in the connections between these concepts and the many applications to virtually every other discipline.

Chapter 2 will explore a successful model for improving the transfer of knowledge from mathematics courses into the disciplines. The consortium for Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P) is a multi-institution, multi-disciplinary collaboration that was developed to define and drive disruptive transformation in the way postsecondary institutions teach mathematics. The objective is to develop substantive mathematics courses that utilize well-researched educational paradigms to structure mathematics instruction for non-mathematics majors in a way that reduces intimidation and increases outcomes such as critical thinking, transfer of knowledge, and the building of sustainable skills.

Through the work of SUMMIT-P, a model for the establishment of sustainable partnerships across disciplinary lines is being developed and tested at a wide variety of institutions. These partnerships consist of collaborative curriculum development between mathematicians and colleagues from across the disciplinary spectrum of STEM, social sciences, business, education, and the humanities. Through these collaborations, a wealth of information and materials is being developed, some of which will be shared in the remaining chapters of this volume. Specifically, Chapter 2 will focus on the SUMMIT-P model and corresponding activities that have been used to solidify the productive interdisciplinary faculty partnerships across 15 institutions. Chapter 3 presents ready-to-use activities that have been developed, tested, and used in the affected courses at those 15 institutions. The materials and resources in the subsequent two chapters of this volume—together with the information presented in Chapter 1—provide a starting point for institutions looking to create this rich interdisciplinary culture among faculty, resulting in seamless and impactful learning experiences for all students.

1.7 References

ABET. (2006). *Engineering change: A study of the impact of EC2000* `https://www.abet.org/wp-content/uploads/2015/04/EngineeringChange-executive-summary.pdf`

Accelerating Systemic Change Network. (2020). *Accelerating systemic change in STEM higher education.* `https://ascnhighered.org/171349`

Beichner, R. J. (2009). An Introduction to Physics Education Research. In C. Henderson & K. A. Harper (Eds.), *Getting Started in Physics Education Research.* College Park, MD: American Association of Physics Teachers.

Beisiegel, M., & Doree, S. (2020). Curricular change in institutional context: A profile of the SUMMIT-P institutions. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1): 192–201, Virginia Mathematics and Science Coalition, Richmond, VA.

Blair, R. (Ed.) (2006). *Beyond crossroads: Implementing mathematics standards in the first two years of college.* American Mathematical Association of Two-Year Colleges.

Borrego, M., Hall, T. S., & Froyd, J. E. (2013). Diffusion of engineering education innovations: A survey of awareness and adoption rates in U.S. engineering departments. *Journal of Engineering Education 99*(3), 185–207.

Brownell, S.E., Freeman, S.; Wenderoth, M.P., & Crowe, A.J. (2014) BioCore Guide: a tool for interpreting the core concepts of vision and change for biology majors. *CBE Life Sciences Education 13*(2):200–11

Bursic, K.M. (2020) An engineering economy concept inventory. *The Engineering Economist 65*:3, pp. 179–194.

Chappell, K.K. (2006). Effects of concept-based instruction on calculus students' acquisition of conceptual understanding and procedural skill. In F. Hitt, G. Harel, and A. Selden (Eds.), *Research in collegiate mathematics education, VI,* American Mathematical Society: Providence, RI, 27–60.

Charles A. Dana Center (2020). *Systemically aligned: Guided pathways and mathematics pathways working together for student success.* The Charles A. Dana Center at the University of Texas at Austin. `http://dcmathpathways.org/sites/default/files/resources/2020-03/guided-pathways-brief_march-2020.pdf`

——— (n.d.). *Guide to aligning mathematics pathways to programs of study.* The Charles A. Dana Center at the University of Texas at Austin. `https://bit.ly/3vqh3zS`

Committee for the Undergraduate Program in Mathematics (CUPM; 2004). *Undergraduate programs and courses in the mathematical sciences: A CUPM curriculum guide.* Mathematical Association of America: Washington, DC.

Committee on STEM Education (2018, December). *Charting a course for success: America's strategy for STEM education.* National Science and Technology Council.

Coyle, E. J., Foretek, R., Gray, J. L., Jamieson, L. H., Oakes, W. C., Watia, J., et al. (2000). *EPICS: Experiencing Engineering Design Through Community Service Projects.* Paper presented at the American Society for Engineering Education Conference.

Cwikla, J. (2017). *Cross institutional synergy for women scientists.* `https://d32ogoqmya1dw8.cloudfront.net/files/ASCN/smti_2017_university_southern.pdf`

Dunbar, S. R. (2006). The enrollment flow to and from courses below Calculus. In N. B. Hastings (Ed.), *A fresh start for collegiate mathematics: Rethinking the courses below Calculus.* Washington, DC: Mathematical Association of America.

Elrod, S., & Kezar, A. (2016). *Increasing student success in STEM: A guide to systemic institutional change.* Washington, DC: Association for American Colleges and Universities.

Ewing, J. (Ed.,1999). *Toward Excellence: Leading a doctoral mathematics department in the 21st century.* American Mathematics Society: Providence, RI.

Fairweather, J. S. (2008). *Linking Evidence and Promising Practices in Science, Technology, Engineering, and Mathematics (STEM) Undergraduate Education.* Paper presented at the Commissioned Paper for National Academies of Science Workshop on Linking Evidence and Promising Practices in STEM Undergraduate Education. from `http://www7.nationalacademies.org/bose/Fairweather_CommissionedPaper.pdf`

Ferrini-Mundy, J., & Güçler, B. (2009). Discipline-based efforts to enhance undergraduate STEM education. *New Directions for Teaching and Learning, 117*(Spring), 55–67.

Finelli, C.J., Daly, S.R. & Richardson, K.M. (2014), Bridging the research-to-practice gap: Designing an institutional change plan using local evidence. *Journal of Engineering Education 103*: 331–361.

Finley, A. (2021). *How college contributes to workforce success: Employer views on what matters most.* Association of American Colleges and Universities. `https://www.aacu.org/sites/default/files/files/research/AACUEmployerReport2021.pdf`

Froyd, J. E. (2002). *The Foundation Coalition: An Agent in Changing Engineering Education.* Paper presented at the International Conference on Engineering Education.

——— (2005). The Engineering Education Coalitions Program. In National Academy of Engineering (Ed.), *Educating the engineer of 2020: Adapting engineering education to the new century.* Washington, D.C.: National Academies Press.

Furrow, R. E. & Hsu, J.L. (2019) Concept inventories as a resource for teaching evolution. *Evolution: Education and Outreach 12*(2).

Ganter, S. L. (2001). *Changing Calculus: A report on evaluation efforts and national impact from 1988–1998.* Washington, DC: Mathematical Association of America.

——— (2003). Creating networks as a vehicle for change. In B.L. Madison and L.A. Steen (Eds.), *Quantitative literacy: Why numeracy matters for schools and colleges.* National Council on Education and the Disciplines: Princeton, NJ.

Ganter, S.L. & Barker, W. (Eds., 2004). *Curriculum Foundations Project: Voices of the partner disciplines.* MAA Reports, Mathematical Association of America: Washington, DC.

Ganter, S.L., & Haver, W.E. (2020). The need for interdisciplinary collaborations. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 1–9, Virginia Mathematics and Science Coalition, Richmond, VA.

——— (Eds., 2011). *Partner discipline recommendations for introductory college mathematics and the implications for College Algebra.* MAA Reports, Mathematical Association of America: Washington, DC.

Handelsman, J., Ebert-May, D., Beichner, R., Bruns, P., Chang, A., DeHaan, R., et al. (2004). Education: Scientific teaching. *Science, 304*(5670), 521–522.

Hartzler, R. & Blair, R. (eds). (2019). *Emerging issues in mathematics pathways.* The Charles A. Dana Center at the University of Texas at Austin. `http://dcmathpathways.org/sites/default/files/resources/2020-04/)Emerging-Issues-in-Mathematics-Pathways.pdf`

Heller, P., Foster, T., & Heller, K. (1996). Cooperative group problem solving laboratories for introductory classes. In E. F. Redish & J. S. Rigden (Eds.), *The changing role of physics departments in modern universities: Proceedings of the International Conference on Undergraduate Physics Education* (pp. 913–934). Woodbury, NY: American Institute of Physics.

Heller, K. & Heller, P. (2010). Cooperative problem solving in physics: A user's manual. *Tersedia.* `http://www.aapt.org/Conferences/newfaculty/upload/Coop-Problem-Solving-Guide.pdf`

Heller, P., & Hollabaugh, M. (1992). Teaching problem solving through cooperative grouping. Part 2: Designing problems and structuring groups. *American Journal of Physics, 60,* 637–645.

Heller, P., Keith, R., & Anderson, S. (1992). Teaching problem solving through cooperative grouping. Part 1:Groups versus individual problem solving. *American Journal of Physics, 60,* 627–636.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching, 48*(8). `http://dx.doi.org/10.1002/tea.20439`.

Henderson, C., & Dancy, M. (2008). Physics faculty and educational Researchers: Divergent expectations as barriers to the diffusion of innovations. *American Journal of Physics (Physics Education Research Section), 71*(1), 79–91.

——— (2009). The impact of physics education research on the teaching of introductory quantitative physics in the United States. *Physical Review Special Topics: Physics Education Research 5*(2), 1–9.

Houseman, D. & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics,* 53, 139–158.

Ivie, R., Guo, S., & Carr, A. (2005). *2004 Physics & Astronomy Academic Workforce* (No. AIP Pub. Number R-392.6). College Park, MD: American Institute of Physics).

Ivie, R., Stowe, K., & Nies, K. (2003). *2002 Physics Academic Workforce Report* (No. AIP Pub. Number R-392.5). College Park, MD: American Institute of Physics).

Kezar, A. (2018). *How colleges change: Understanding, learning, and enacting change.* (2nd ed.). Routledge.

Kirsch, I.S. & Jungeblut, A. (1986). *Literacy: Profiles of America's young adults.* Educational Testing Service: Princeton, NJ.

Kohan, M. & Laursen, S. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education,* 39, 183–199.

Krueger, C. (2017). *A process for success: Developing and supporting student learning outcomes for multiple mathematics pathways.* The Charles A. Dana Center at the University of Texas at Austin. `https://bit.ly/3OsmsiE`

Kwon, O., Rasmussen, C., & Allen, K. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics,*105, 227–239.

Lattuca, L. R., Terenzini, P. T., & Volkwein, F. (2006). *Engineering change: A study of the impact of EC2000.* Baltimore, MD: ABET, Inc.

Laws, P. W. (1991). Calculus-based physics without lectures. *Physics Today, 44*(12), 24-31.

——— (1997). *Workshop physics activity guide.* New York: John Wiley & Sons.

Laws, P.W., Willis, M., & Sokoloff, D.R. (2015). Workshop physics and related curricula: A 25-year history of collaborative learning enhanced by computer tolls for observation and analysis. *The Physics Teacher 53*(401), `https://doi.org/10.1119/1.4931006`

Lederman, J., Lederman, N., Bartos, S., Bartels, S., Meyer, A., & Schwartz, R. (2013). Meaningful assessment of learners' understandings about scientific inquiry: The views about scientific inquiry (VASI) questionnaire. *Journal of Research in Science Teaching, 51*(1), 65 – 83. `https://doi.org/10.1002/tea.21125`

Libarkin, J. C. (2008). *Concept Inventories in Higher Education Science.* Paper presented at the Commissioned Paper for National Academies of Science Workshop on Linking Evidence and Promising Practices in STEM Undergraduate Education. `http://www7.nationalacademies.org/bose/Libarkin_CommissionedPaper.pdf`

Lieberman, A. & Wood, D.R. (2003). *Inside the National Writing Project: Connecting network learning and classroom teaching.* Teachers College Press: New York, NY.

Liston, C. & Getz, A. (2019). *The case for mathematics pathways.* The Charles A. Dana Center at the University of Texas at Austin. `http://dcmathpathways.org/sites/default/files/resources/201903/CaseforMathPathways_20190313.pdf`

Madison, B.L. (2003). The many faces of quantitative literacy. In B.L. Madison and L.A. Steen (Eds.), *Quantitative literacy: Why numeracy matters for schools and colleges.* National Council on Education and the Disciplines: Princeton, NJ.

Madison, B.L. & Steen, L.A. (Eds., 2003). *Quantitative literacy: Why numeracy matters for shools and colleges.* National Council on Education and the Disciplines: Princeton, NJ.

Madsen, A., McKagan, S.B., & Sayre, E.C. (2017). Best practices for administering concept inventories. *Physics Teacher 55*(9):530–6.

Mazur, E. (1997). *Peer instruction: A user's manual.* Upper Saddle River, New Jersey: Prentice Hall.

——— (2009). *Disseminating Curriculum and Pedagogy: Peer Instruction.* Paper presented at the Joint AAPT/AAAS Winter Meeting, Chicago, IL, Feb 14, 2009.

Mazur, E., & Crouch, C. H. (2001). Peer instruction: Ten years of experience and results. *American Journal of Physics, 69,* 970–977.

McDermott, L., & Shaffer, P. S. (2002). *Tutorials in introductory physics* (First ed.). Upper Saddle River, NJ: Prentice Hall.

McFarling, M., & Neuschatz, M. (2003). *Physics in the Two-Year Colleges: 2001-02* (No. AIP Pub. Number R-436). College Park, MD: American Institute of Physics).

Merton, P., Froyd, J. E., Clark, M. C., & Richardson, J. (2009). A case study of relationships between organizational culture and curricular change in engineering education. *Innovative Higher Education, 34,* 219–233.

National Academies of Sciences, Engineering, and Medicine. (2016). *Barriers and opportunities for 2-year and 4-year STEM degrees: Systemic change to support students' diverse pathways.* The National Academies Press.

National Center for Educational Statistics (1993). *Adult literacy in America: Report of the National Adult Literacy Survey (NALS).* U.S. Department of Education: Washington, DC.

A National Consortium for Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P). (2020a). *Annual reports.* `https://bit.ly/SUMMITPReports`

——— (2020b). *Products.* `https://www.summit-p.com/resources/products`

——— (2016). *SUMMIT-P.* `https://www.summit-p.com/`

National Council on Education and the Disciplines (2001). *Mathematics and democracy: The case for quantitative literacy.* The Woodrow Wilson National Fellowship Foundation: Princeton, NJ.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* NCTM: Reston, VA, 193–195.

National Research Council (1995). *National Science Education Standards.* NRC: Washington, DC.

——— (1999). *How people learn: Brain, mind, experience, and school.* Washington, DC: National Academies Press.

——— (2003). *Improving undergraduate instruction in science, technology, engineering, and mathematics: Report of a workshop.* Washington, D.C.: The National Academies Press.

——— (2007). *Rising above the gathering storm: Energizing and employing America for a brighter economic future.* The National Academic Press: Washington, DC.

——— (2012). *Discipline-based education research: Understanding and improving learning in undergraduate science and engineering.* The National Academic Press: Washington, DC.

——— (2013). *The mathematical sciences in 2025.* The National Academies Press: Washington, DC.

Organization for Economic Cooperation and Development (1995). *Literacy, economy, and society: Results of the first international adult literacy survey.* OECD: Paris, France.

——— (1998). *Literacy skills for the knowledge society.* OECD: Washington, DC.

Prados, J. W., Peterson, G. D., & Lattuca, L. R. (2005). Quality assurance of engineering education through accreditation: Engineering criteria 2000 and its global influence. *Journal of Engineering Education, 94*(1), 165–184.

President's Council of Advisors on Science and Technology (PCAST, 2012). *Engage to Excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Report to the President, Washington, DC.

Project Kaleidoscope. (2006). *Transforming America's scientific and technological infrastructure: Recommendations for urgent action.* Washington, DC: Project Kaleidoscope.

——— (2017). *Strategic plan.* `https://www.aacu.org/sites/default/files/files/Strategic%20Plan_PKAL_FINAL_2017.pdf`

Rasmussen, C. & Kwon, O.N. (2007). An inquiry-oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior,* 26, 189–194.

Redish, E. F. (2003). *Teaching physics with the Physics Suite.* Hoboken, NJ: John Wiley & Sons.

Saxe, K. & Braddy, L. (2015). *A common vision for undergraduate mathematical sciences programs in 2025.* Mathematical Association of America.

Schoenfeld, A. H. (1995, January). A brief biography of calculus reform. *UME Trends: News and Reports on Undergraduate Math Education 6*(6). 3–5.

Secretary's Commission on Achieving Necessary Skills (SCANS,1991). *What work requires of schools: A SCANS report for America 2000.* U.S. Department of Labor: Washington, DC.

Seymour, E. (2001). Tracking the process of change in US undergraduate education in science, mathematics, engineering, and technology. *Science Education, 86,* 79–105.

Singapore Ministry of Education (2018, October 11). *21st Century competencies.* `https://www.moe.gov.sg/education/education-system/21st-century-competencies`

Smith, J.C. (2005). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior,* 25, 73–90.

Sokoloff, D.R. (2016). Active learning strategies for introductory light and optics. *The Physics Teacher 54*(18). DOI: `https://doi.org/10.1119/1.4937966`

Sokoloff, D. R., & Thornton, R. K. (1997). Using interactive lecture demonstrations to create an active learning environment. *The Physics Teacher, 35*(6), 340–342.

Sons, L., et al. (1996). *Quantitative reasoning for college graduates: Report of a CUPM subcommittee on quantitative literacy requirements.* Mathematical Association of America: Washington, DC.

Steen, L.A. (Ed, 1997). *Why numbers count: Quantitative literacy for tomorrow's America.* The College Board: New York, NY.

——— (Ed.) (2001). *Mathematics and democracy: The case for quantitative literacy.* The Woodrow Wilson National Fellowship Foundation.

——— (2004). *Achieving quantitative literacy: An urgent challenge for higher education.* Mathematical Association of America: Washington, DC.

Tobias, S. (2000). From innovation to change: Forging a physics education reform agenda for the 21st century. *American Journal of Physics, 68*(2), 103–104.

Tucker, A.C. (1995). *Assessing calculus reform efforts: A report to the community.* Mathematical Association of America.

Volkwein, J. F., Lattuca, L. R., & Terenzini, P. T. (2008). Measuring the Impact of Engineering Accreditation on Student Experiences and Learning Outcomes. In W. E. Kelly (Ed.), *Assessment in engineering programs: Evolving best practices* (Vol. 3). Tallahassee, FL: Association for Institutional Research.

Walker, J., & Sampson, V. (2013). Argument-driven inquiry: Using the laboratory to improve undergraduates' science writing skills through meaningful science writing, peer-revision, and revision. *Journal of Chemical Education, 90*(10), 1269–1274. `https://doi.org/10.1021/ed300656p`

Yerushalmi, E., Cohen, E., Heller, K. Heller, P., & Henderson, C. (2010). Instructors' reasons for choosing problem features in calculus-based introductory physics course. *Physical Review Physics Education Research 6,* 020108.

Zhang, P., Ding, L., & Mazur, E. (2017). Peer Instruction in introductory physics: A method to bring about positive changes in students' attitudes and beliefs *Physical Review Physics Education Research 13*(010104), 1–8.

Zorn, P. et al (2014). *The INGenIOuS Project: Mathematics, statistics, and preparing the 21st century workforce.* MAA Reports, Mathematical Association of America: Washington, DC.

2

SUMMIT-P Processes for Interdisciplinary Faculty Collaboration: Transforming the Undergraduate Experience

Afroditi V. Filippas,[1] *Virginia Commonwealth University*
Debra Bourdeau, *Embry-Riddle Aeronautical University*
Stella Hofrenning, *Augsburg University*
Rosalyn Hargraves, *Virginia Commonwealth University*

As you continue to explore this volume, you may wonder what is unique about this work and the resources presented. Chapter 2 introduces a revolutionary model for the way in which undergraduate students' connections with elements of the curriculum can be impacted by interdisciplinary collaboration. SUMMIT-P has developed a scalable, sustainable, and transferable model for interdisciplinary collaborations that will work in any setting and for any purpose. This chapter describes the leadership structure, the protocols that were used, and how they were modified to meet the needs of a variety of institutions and disciplines. The collaborations themselves are unique at each institution. While the need for interdisciplinary collaborations are highlighted in product development and research settings, their value to the educational arena has been underemphasized. In fact, it will be shown that establishing sustained, trust-driven collaborations in educational settings leads to systemic, transformational institutional change that drives improvements to the entire curriculum.

The protocols described in this chapter can be scaled, customized, and executed in any team setting. More importantly, the systemic use of these protocols leads to sustained and expanded collaborations within an institution as well as between institutions. The implementation of regular, curated conversations between partner disciplines and institutions serves to provide a more nuanced approach to cross-pollinating the curricula with methods, programs, and initiatives that have been developed within each consortium institution. In addition, this chapter explores how the organic but thoughtful implementation of the protocols emphasize the specific needs of each discipline within an institution. The relationships, developed over time and built on trust, are thus more meaningful and have led to substantive changes in the curriculum at each institution.

This chapter, therefore, provides a necessary context for the active learning resources presented in Chapter 3. While those resources can stand on their own, it is important to recognize the context in which they were developed. Chapter 2 provides a robust, adaptable model for establishing interdisciplinary collaborations in any setting. Successful collaboration then leads to more substantive change, prompting an evolution within the institution that will create an alliance of disciplines working together to improve outcomes for the next generation of learners.

[1]Contact: avfilippas@vcu.edu

2.1 Introduction

In conjunction with CRAFTY, the consortium for Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P) was formed in 2016, comprising 15 institutions that are utilizing the CF recommendations to revise and improve the lower division undergraduate mathematics curriculum (see Table 2.4). The key element of these innovations is interdisciplinary partnerships, with partner disciplines directly involved in decisions and ongoing conversations about curricular needs and related application content. SUMMIT-P has established a national interdisciplinary faculty network—via online discussions and learning communities—in order to promote changes in faculty culture (Lane et al, 2019) and institutional transformation, through shared experiences and ideas for successfully creating functional interdisciplinary partnerships within and across institutions. SUMMIT-P has received support from NSF (Lead # DUE1625771/1822451), including a comprehensive evaluation of the project's outcomes for each institution and the consortium as a whole.

The vehicle of change is the collaboration (Ganter & Barker, 2004). Within each institution, collaborations are set up between mathematics and STEM or non-STEM disciplines that require at least one mathematics course for an undergraduate degree. The active cooperation between mathematics and the partner disciplines sets the stage for increased relevance of the course to the students, increased engagement by partner discipline faculty in the longitudinal application of the acquired skills in their own courses, and the development of discipline-relevant examples by partner discipline faculty that are implemented in the mathematics courses under review.

The multi-institutional collaboration ensures a systemic and systematic application of the hypothesized best practices, collaboration protocols, and developed active learning tools at a wide range of institutions. It provides a larger backdrop from which different teams can observe each other's successes and challenges, learn from each other, and help sustain engagement in this significant initiative. It also provides a mechanism for ensuring the collaboration protocols and educational paradigms are applicable over a wide range of institutions with differing size, composition, partner discipline focus, and other parameters that might impact the outcome.

Thus, the collaboration across departments and institutions needs to be driven by the discovery and affirmation of common goals and agreed-upon methods by each institution and between each mathematics department and partner discipline (Jay Dee & Zamel, 2009). This chapter describes the initiation, development, and final structure of this collaboration, investigates its application at a number of institutions, reviews representative work outputs and student outcomes, and discusses the viability, sustainability, and scalability of this work.

The SUMMIT-P team's initial priority was to establish a general collaboration framework to enable:

1. collaboration across institutions, managed by the project management team (PMT);
2. collaboration between mathematics and partner discipline(s) at each institution, managed by a local leadership team consisting of mathematicians and partner discipline colleagues;
3. cross-institutional exchange of ideas, processes, success stories, and challenges;
4. data collection through an independent evaluation, to assess activities and outcomes of SUMMIT-P.

The participating institutions are diverse, each boasting a distinct culture, focus area(s), faculty research and teaching expectations, and size and diversity of partner disciplines. However, the affected SUMMIT-P mathematics courses for every institution are at the lower undergraduate level, often inhibiting students from progressing in their degree or preventing timely graduation.

2.2 The SUMMIT-P Model and Theory of Change

SUMMIT-P's work is an important step toward understanding institutional transformation on a large scale that will support the mathematical development of STEM majors while increasing mathematical literacy among all college graduates nationwide. The SUMMIT-P consortium has had success with collaborative processes, resulting in ongoing partnerships across many disciplines and institutions nationally. Such partnerships help to immediately reinforce mathematical concepts in a variety of disciplinary settings, creating context across courses (Apkarian et al, 2018; Beisiegel & Doree, 2020; Beisiegel et al, 2020; Bishop, Piercey, & Stone, 2020; Bowers et al 2020; Brucall-Hallare, Moosavidzadeh, & Deo, 2020; Ganter & Haver, 2020; Hofrenning et al, 2020; Lai et al, 2020; Venkatesh & Militzer, 2020).

The SUMMIT-P theory of change is based on development of new ideas within and across institutional teams (Angelo, 1997; Dee & Daly, 2009; Eckel & Kezar, 2003; Kezar, 2018; Sanders et al, 1997). This development is supported and made more efficient through national-level coordination that, for example, allows institutions to more readily share ideas (Beisiegel & Doree, 2020; Hargraves et al, 2020; Piercey et al, 2020). Once developed locally, changes will only be sustained if they are institutionalized through structural and cultural changes (see Figure 2.1).

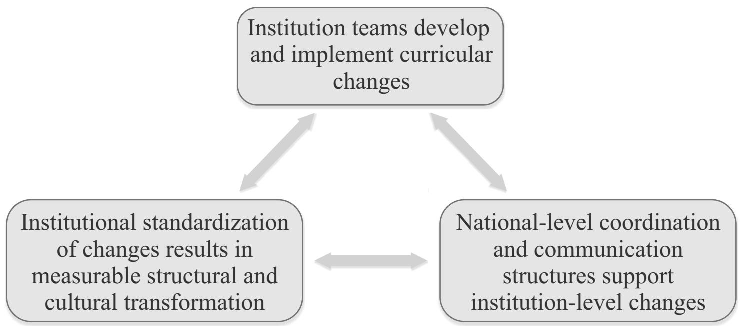


Figure 2.1. SUMMIT-P Theory of Change

The SUMMIT-P theory of change is further detailed in the SUMMIT-P model that has developed through the consortium's collective work (see Figure 2.2). Using teams at the consortium and institution levels, implementation sites progress through the model components at various speeds, depending upon their local project and other institutional factors. Participating institutions benefit from being part of the consortium-level group learning model (Clear Impact, 2016) through:

- Campus site visits between participating institutions, using a common protocol and observing implementation "in action"
- Extensive opportunities for mentor-mentee pairings within and across institutions
- Professional development and support, through project meetings, webinars, and virtual communications and gatherings
- Use of Descriptive Consultancy and Success Analysis protocols (Hargraves et al, 2020) for consistent and effective group sharing of experiences, ideas, and progress within and across institutions
- Focused, organized, and goal-driven "business meetings" that require institutions to be frequently accountable for their progress
- Extensive consistently collected data via faculty evaluation portfolios that can be used to make conclusions about the effects of innovative partnerships and resulting curricular changes
- Momentum as a consortium that has fostered a wide variety of dissemination outlets.

At the institution level, numerous activities adapted to the specific context lead to the integration of content across courses, forming a seamless STEM curriculum. These include faculty learning communities (FLCs), "wish lists" from partner discipline colleagues, and "fishbowl style" activities in which an outer-circle group carefully observes active discussion by an inner circle of colleagues. FLCs, site visits, and professional development serve as the bridge between the consortium and the institutions while facilitating the showcasing of success and sharing best practices (Bishop, Piercey, & Stone, 2020; Brucal-Hallare, Moosavizadeh, & Deo, 2020; SUMMIT-P, 2020a, 2020b) (see Figure 2.2).

The SUMMIT-P theory of change is well-aligned with and informed by the broader framework for how change occurs known as the Networked Improvement Community (NIC) approach (Bryk et al, 2015; Martin & Gobstein, 2015). This broader, umbrella vision of how change occurs has been used by multiple projects in higher education, and was developed under the guidance of the Carnegie Foundation for the Advancement of Teaching to help groups accelerate progress toward shared goals. Table 2.1 outlines the four essential NIC characteristics (Bryk et al, 2015) and how SUMMIT-P embraces each feature.

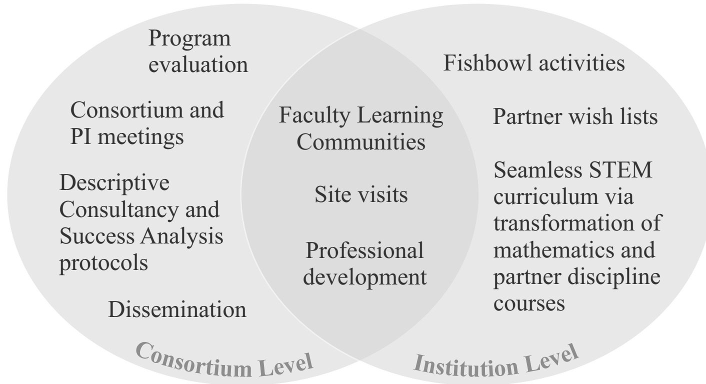


Figure 2.2. SUMMIT-P Model Components

The enactment of the SUMMIT-P theory of change occurs differently at different institutions based on local contexts, such as who is involved, what collaboration structures currently exist, and what resources are available. Lessons learned and expanded use of online technologies allow for adoption at more and varied institutions. As continuing research yields more data, the model will be adjusted for institutional context. This design will allow the consortium to expand the impact of the SUMMIT-P work from change within a select set of interdisciplinary faculty and courses to an expansion of the knowledge base for institutional transformation, created by studying SUMMIT-P components that result in sustained institutional change.

Table 2.1. Networked Improvement Community (NIC) Characteristics

NIC Characteristics	SUMMIT-P instantiation
1. Focused on a well-specified aim	Creating and sustaining substantive collaborations that result in lasting curricular change in the first two years of mathematics and in the partner disciplines served, creating an undergraduate curriculum—from mathematics to partner discipline—that is cohesive and seamless for students.
2. Guided by a deep understanding of the problem, the system that produces it, and a theory of improvement relevant to it	For the past five years, SUMMIT-P has nurtured and critically investigated the contexts and interventions that promote interdisciplinary partnerships within and across institutions and the resulting impacts on curricular change.
3. Disciplined by the rigor of improvement science	Moving forward, the work of the SUMMIT-P partners will leverage the Change Dashboard. The Dashboard articulates the key tactics of an action plan to get from the current state to the desired state and serves as a key planning and accountability tool.
4. Coordinated to accelerate the development, testing, and refinement of interventions and their effective integration into practice	SUMMIT-P's future work will be coordinated and accelerated across three structural levels: consortium, institutional cluster groups, and individual institutions.

2.3 Faculty Learning Communities

Curricular innovations like those of SUMMIT-P can improve student learning only if they are consistently put into practice by faculty over long periods. Henderson and Dancy (2011) advocate that the dissemination of these innovations is one of the most significant agents of positive change in student learning and participation in STEM. One of the most effective models for disseminating such innovations is faculty learning communities (FLCs) (Dee & Daly, 2009), defined as a group of faculty from various disciplines "who engage in an active, collaborative, yearlong program with a curriculum about enhancing teaching and learning and with frequent seminars and activities that provide learning development, the scholarship of teaching, and community building" (Cox & Richlin, 2004, p.8). As part of the consortium's organizational structure, SUMMIT-P has embraced the practice of FLCs and, in particular, two features of FLCs are critical to SUMMIT-P's work: 1) seeing innovations in action, and 2) having a safe space for experimentation (Bickerstaff & Edgecombe, 2012). The work of SUMMIT-P is built on over twenty years of FLC scholarship and practice (Lane et al, 2002; Sanders et al, 1997; Shapiro & Levine, 1999; Cox, 2001; Cox & Richlin, 2004), uniquely targeting the CF recommendations by forming teams at each participating institution that include faculty from partner disciplines together with mathematics faculty. As the CF recommendations state, input from partner disciplines is critical in the development of authentic contextualized problem scenarios.

The SUMMIT-P institutions have been motivated by the voluntary nature of FLCs. FLC research, combined with results from the SUMMIT-P work (Beisiegel & Doree, 2020; Bishop, Piercey, & Stone, 2020; SUMMIT-P, 2020a, 2020b), show that strategic use of FLCs can result in increased faculty interest in teaching and learning as well as a heightened sense of support for the investigation of new, innovative methods of instruction. Central in importance is a sense of safety and trust offered by FLCs as well as the creation of an environment where community members feel respected, challenged, and empowered as change agents (Borrego & Henderson, 2014).

2.4 Collaborative Structures

SUMMIT-P collaborations occur on multiple levels.[2] Overseeing the consortium is the project management team (PMT), a leadership group that determines priorities, general strategies, and timetables around which all institutions align. The PMT also ensures compliance with reporting requirements, and interfaces with the evaluation team to track progress and outcomes.

At the institution level are collaborations between mathematicians and partner disciplines colleagues. These collaborations have brought about significant changes in multiple sections of at least one mathematics course, with implications for the related partner discipline courses. Additionally, there have been unexpected changes and collaborative interactions resulting from the original partnerships that will be described here.

There are many best practices described as part of a successful collaboration. Establishing these best practices, while also minimizing associated challenges, is critical. These challenges (e.g., full or conflicting schedules, a clear articulation of common benefits, the teaching/research reward structure of the institution) can be overcome through planning, time management and the establishment of complementary priorities and shared expectations. By applying SUMMIT-P's structure and norms, while implementing reasonable adjustments to accommodate diverse needs and maintaining realistic expectations, any institution can establish, sustain, and grow interdisciplinary collaborations that align with the academic priorities of the institution.

Partnering Across Disciplines While there are many questions regarding the exact combination of factors that lead to student success in STEM courses, one fact is well-documented: a large majority of students enrolled in the traditional STEM curriculum have difficulty applying knowledge from mathematics courses to problem solving in other disciplines—in both STEM and non-STEM fields (CUPM,2004; Ganter & Barker, 2004; Ganter & Haver, 2011; National Research Council, 2012, National Research Council, 2007; National Research Council, 2013; PCAST, 2012). **The goal of SUMMIT-P is to study and substantiate the levers that lead to sustainable interdisciplinary and inter-institutional collaborative efforts that support the dissolution of traditional disciplinary silos.** The result is the creation of substantive collaborations that culminate in lasting curricular change in the first two years of mathematics and in the partner disciplines served, creating an undergraduate curriculum—from mathematics to partner discipline—that is cohesive and seamless for students.

Full participation from partner discipline faculty is a key ingredient in successfully redeveloping introductory mathematics courses in a way that incorporates the contextual needs of other disciplines. As such, the consortium's first task was to find ways to best engage colleagues in the partner disciplines. Initial conversations at SUMMIT-P meetings led to activities that experimented with a variety of mechanisms for that purpose.

[2] Additional resources for each of the collaborative structures discussed here can be found on the SUMMIT-P website: https://www.summit-p.com/resources/collaboration-tools.

Specifically, because this collaborative approach for curriculum development is being implemented at a variety of institutions, each institution has 1) used locally appropriate strategies, 2) engaged faculty from locally-selected partner disciplines, and 3) focused on mathematics courses selected by that institution. However, all SUMMIT-P institutions have made a commitment to utilize the following processes:

- Faculty have studied the CF recommendations and the relevant disciplinary reports (Ganter & Haver).
- Opportunities have been provided for partner discipline faculty to describe to mathematics faculty which of these CF recommendations are most important to their students, often through “fish bowl” activities (Hoffrening et al, 2020).
- Partner discipline faculty have developed “wish lists” of mathematical topics and experiences their students need, and mathematics courses are being changed in response to these lists.
- Faculty have participated in local professional development experiences (through seminars and learning communities) and SUMMIT-P professional development (through webinars, poster presentations, panel discussions, and the development of published volumes and journal special issues; Bishop, Piercey, & Stone, 2020).
- Courses are being developed, piloted, and refined in a collaborative fashion involving faculty and students from mathematics and partner disciplines (Bowers et al, 2020; Lai et al, 2020).
- Mathematics and partner discipline faculty have visited courses offered outside of their departments (Venkatesh & Militzer, 2020).
- Faculty have participated in SUMMIT-P “course clusters” that frequently bring together institutions that are working on similar mathematics courses to discuss implementation strategies and outcomes.
- Teams from other SUMMIT-P institutions as well as project leadership and evaluation personnel have participated in site visits at each SUMMIT-P institution to 1) attend classes, 2) interact with faculty and administrators, and 3) talk with students; these site visits are organized using a protocol designed to help the host institution plan for and conduct the visit in a way that engages the broader college community (Piercey et al, 2020).

Fishbowls The success of interdisciplinary, multi-institutional collaborations hinges upon how well the institutions engage with and support each other—both within and across institutions. With such a large and widely-dispersed national group, it is necessary to establish an effective means of sharing successful practices and facilitating peer learning. One protocol used effectively by SUMMIT-P is the fishbowl. Fishbowls help to foster conversation between mathematics and partner disciplines, and are very effective in allowing each side to be heard and to engage in discussions that lead to a deeper understanding of the needs of each discipline. The fishbowl protocol assigns the role of facilitator and observer to one group (mathematics), while the second group (partner discipline) responds to prompts in a conversational format. The conversations are conducted entirely by the partner discipline, while the mathematics group carefully observes without interruption. In effect, the mathematicians are watching and listening to the partner discipline members as if they were fish in a bowl. In this way, mathematicians gain a more nuanced understanding of the needs of partner disciplines. The fishbowl primarily serves to mitigate “specialist bias;” i.e., the tendency to dictate what is necessary for students to learn without adjusting for specific needs of the partner disciplines. Through these discussions, the two groups can come to collaborative decisions that benefit students’ breadth and depth of understanding for the combined disciplines.

The initial fishbowl was conducted at the opening SUMMIT-P meeting in 2016, and included all participating institutions. The purpose of this first fishbowl was to establish the continuing validity of the Curriculum Foundation (CF) reports (Ganter & Barker, 2004), and to help the institutional teams better understand how to use the fishbowl protocol. Subsequent to this consortium-level fishbowl activity, each institution organized one or more local fishbowls, to establish the partnership between mathematics and the chosen partner discipline(s) and to initiate the creation of “wish lists” from the partner disciplines. The institutional fishbowls set the stage for the SUMMIT-P curriculum work to be conducted collaboratively by multiple disciplines, to develop interdisciplinary learning activities for selected mathematics course(s).

To prepare for a fishbowl discussion, faculty read the CF report relevant to their discipline and review the CF recommendations in the context of their institution. Additionally, guiding questions for the fishbowl discussion are provided to participants as follows:

General

1. Do the recommendations of the CF report still ring true?
2. Do you believe there are topics unique to your discipline that are not reflected in the summary report?

Understanding and Content

1. What conceptual mathematical principles must students master in the first two years?
2. What mathematical problem-solving skills must students master in the first two years?
3. What broad mathematical topics must students master in the first two years?
4. What priorities exist between these topics?
5. What is the desired balance between theoretical understanding and computational skill?
6. How is this balance achieved?
7. What are the mathematical needs of different student populations and how can they be fulfilled?

Technology

1. How does technology affect what mathematics should be learned in the first two years?
2. What mathematical technology skills should students master in the first two years?
3. What different mathematical technology skills are required of different student populations?

Instruction

1. What instructional methodologies relative to teaching mathematical concepts in your discipline would you like to be made aware of?
2. What kinds of mathematical technologies do you use in your discipline?
3. What mathematical technologies should students develop? (Hofrenning et. al., 2020)

SUMMIT-P institutions use both the relevant CF reports and faculty surveys to modify fishbowl questions to address specific local needs. An analysis of survey responses creates a common starting point for the fishbowl discussion, which is especially valuable when the mathematics course is a prerequisite for a disparate group of partner disciplines that have different needs. A key difference between the SUMMIT-P fishbowl discussions and those conducted as part of the CF workshops is that these discussions take place at individual SUMMIT-P institutions. In this way, the format for each institutional fishbowl discussion can be tailored to match the individual institution's culture, curriculum, and academic needs.

The fishbowl leads to the creation of partner discipline wish lists and syllabi mapping, in which skills identified in the fishbowl are mapped onto the syllabus of the course(s) planned for revision. An important outcome is the eventual creation of exercises, problems, and examples from partner disciplines that model or utilize the mathematical concepts identified as important in the fishbowl discussion. These activities are then implemented in pilot sections of the targeted course(s), and ultimately result in ongoing changes to the relevant syllabi.

Additional Collaboration Protocols The PMT identified two additional protocols that provide a format for effective and fruitful discussion during quarterly virtual and annual face-to-face project meetings. The two protocols, the Descriptive Consultancy protocol and the Success Analysis with Reflective Questions protocol, were chosen because of their documented effectiveness in educational settings (Yau & Lawrence, 2020; McDonald et al, 2013; Mindlich & Lieberman, 2012; Bryk, 2010; Hargraves et al, 2020) and the experience of select consortium members in their use.

These protocols were modified to provide SUMMIT-P a structured format for giving feedback to institutional teams about implementation challenges, and to provide opportunities to share success stories. Specifically, the modifications include changes to the reflective questions while adjusting the time allotted for each protocol activity to give the presenter and their consultant colleagues an opportunity to think about the context and environment in which the successful practice was executed and also how the practice could be adapted and implemented at other institutions.

The Descriptive Consultancy protocol (Mohr, Parrish, & Taylor, 2019; McDonnough & Hanschel, 2015) helps presenters think more expansively about a particular, concrete dilemma and obtain advice from colleagues on how to resolve it. The protocol allows a problem to be framed and then reframed to enable and empower the presenter to move towards a focused solution. As stated by Mohr, Parrish, and Taylor (2019), the protocol "recognizes that the best advice is the least advice, and that helping to define and set the problem is what is truly helpful in reaching resolution.. . . It asks us to practice being more descriptive and less judgmental" (para. 1). During the modified Descriptive Consultancy protocol used in SUMMIT-P, the "presenter" and the "consultants" follow a timed, eight-step process (see Table 2.2).

Table 2.2. Modified Descriptive Consultancy Protocol

	Presenter	Consultants	Time
1	Describes the problem with all dimensions. How has this been addressed to date?	Listen and take notes if needed	3 min
2	Listens and takes notes, if needed.	Ask questions of the presenter. Considers what information is missing.	2 min
3	Responds to questions.	Listen and take notes.	3 min
4	Listen and take notes.	Each of the consultants addresses the presentation of the problem: • What I heard you say was ... • It is still unclear because ... • I would like to know more about ... • May pass if your reflection has already been offered	5 min
5	Responds to consultants' expressed understandings and provides further clarification of the problem.	Listen and take notes	3 min
6	Listens and takes notes	Brainstorm possible solutions or next steps: • What if . . . ? • Have you thought about . . . ? • Would . . . be a possible solution? • I heard/read about . . .	10 min
7	Reflects on advice. • How might you be thinking now as a result of what has been said? • Did you gain any new insights? • You do not need to answer any questions		5 min
8	Did anyone learn something from the MDC that will be useful to their challenge?		5 min

The Success Analysis with Reflective Questions protocol (Johnson, 2019) was created to help groups or teams learn from successes and share best practices through structured discussion. As modified for SUMMIT-P, the protocol gives the presenter an opportunity to share a success that proved to be highly effective in achieving an important outcome. As stated by Grove (2019), protocols like this one work "in the spirit of appreciative inquiry," to allow presenters to "share professional successes with colleagues in order to gain insight into the conditions that lead to those successes, so participants can do more of what works" (para.1). During the modified Success Analysis Protocol, the "presenter" and the "consultants" follow a timed, four-step process (see Table 2.3).

These two protocols differ in their format and function; however, together they create a comprehensive framework for structured engagement. The Descriptive Consultancy protocol helps address issues faced by an institutional team, while the Success Analysis protocol allows for best practices to be shared and potentially adapted and implemented at other institutions. Both protocols give participants an opportunity to reflect on the information shared and consider how that knowledge can be adapted to or used directly in their own institutional context.

Table 2.3. Modified Success Analysis Protocol

Presenter:	Consultants:	Time
Shares successful practice in the implementation of the grant (e.g., activity, lab, project, collaboration, assessment)	Take notes	5 min
Encouraged to participate Prodded through questioning	• How may the practice have directly contributed to success? • What other factors may be involved?	5 min
Compile a list of characteristics of the practice: • Specific successful behaviors • Underlying principles		5 min
How might lessons learned in this protocol be applied to other parts of our work?		5 min

2.5 Site Visits

An important part of any collaboration is to establish methods that catalyze the exchange of ideas and methods between all participants. It is therefore important to provide an immersive environment for participants to experience the innovative paradigms implemented by other teams (Bishop, Piercey, & Stone, 2020; Piercey et al, 2020). This provides a mechanism to adapt and adopt these methods, as well as to provide feedback to the partner institution.

As with the PMT and PI meetings, these visits follow a prescribed protocol, with some flexibility to adapt to individual institutions. The site visits are approved at the PMT level, and each institution is required to visit and host at least one other institution. In general, site visits are conducted according to the following:

1. Lead responsibility for organizing and conducting the site visit will reside with the host institution.
2. Consistent with the premise of SUMMIT-P, the visiting team includes at least one mathematics faculty member and one faculty member from a partner discipline. At least one evaluator and one other member from the PMT also participates.
3. The site visit is 1.5 days in duration.
4. Prior to finalizing plans for the site visit, all participants from both institutions and the PMT clearly outline their specific goals for the visit.
5. The host institution prepares a report on the current status of their work.
6. The participant goals and the host institution's status report are distributed to all participants prior to the site visit.

After all site visit participants have reviewed the goals statements and the current status report, the host institution prepares the site visit schedule. Visitors pay their own transportation, meals and hotel expenses.

During the site visit All site visits include common components that have been tested throughout SUMMIT-P, both for maximizing the value to participants and enabling the collection of common data. The specific way in which these components are implemented can vary, but the following are important parts of the site visit experience:

Project Highlights: Lasting 90-120 minutes, this session is the centerpiece of the visit. Ideally, it takes place near the beginning of the visit and includes all visitors plus many individuals from the host institution, including mathematics and partner discipline faculty, course developers, pilot instructors, department chairs, deans, the provost and other executive leadership, and students from impacted courses. The session includes a carefully planned set of presentations and/or facilitated discussions, structured to accommodate the specifics of the local project. An appropriate representative presents the overall work of SUMMIT-P. The host team should be creative and flexible in developing this session so that it is unique to the institution and involves as many people as possible from across campus.

Classroom Observations: Visits to see the affected mathematics or partner discipline courses "in action" is an important part of understanding how the interdisciplinary partnerships are impacting students. Visitors observe 2-3 class sessions, while local faculty and administrators are encouraged to visit classes before or after the site visit.

Student Focus Groups: The goal of these sessions is for visitors and evaluators to learn about the impact of course innovations from the student perspective. Students are selected from the affected course sections and are prepped in detail by the host project leaders about portions of the course that have been changed, since they will have no way of knowing what is new. A protocol with questions to be asked during the sessions is developed in advance.

Faculty Discussions: These have several goals and happen throughout the site visit. Faculty teaching the affected courses will share particular content or instructional approaches. Additionally, project leaders will discuss issues related to project management and recruiting/engaging faculty. Partner discipline faculty will be particularly interested in discussing how the interdisciplinary partnerships have been initiated and supported, as well as content ideas particular to their discipline.

Social Activities: Opportunities for visiting and host teams to discuss ideas and observations in a non-structured setting is important. Such conversations often reveal things—both positive and negative—that would not come up in the more formal parts of the visit. Meals are the most common way to accomplish this, but campus tours and drives to/from the airport are other examples of times for informal discussion.

Course Materials: The host team is encouraged to share new course materials with the visiting team, either as a special portion of the agenda or for inclusion in site visit preparation materials. In addition to actual course materials, faculty also appreciate having an informal document that describes the change process and how the materials are used in the classroom.

Sustainability Plans: A candid discussion between the host and visiting teams about the challenges and opportunities related to the long-term viability of the SUMMIT-P work is a valuable addition to the site visit protocol.

Subsequent to the site visit The host PI, visiting PI, project evaluator, PMT representative, and other site visit participants each complete a post-site visit survey. The post-visit survey is an important part of the continuous improvement process, and ideally is completed prior to the conclusion of the site visit. Important lessons learned include the following:

1. To avoid transportation problems, arrange visitor schedules and planned time of arrival.
2. Include presentations from both sets of partner disciplines (host and visitor).
3. An overview of the Curriculum Foundations Project and SUMMIT-P should be presented by the PMT representative to demonstrate the role of the host institution in this national curriculum effort.
4. Consider visiting partner discipline classes and/or labs as part of the site visit.
5. A campus tour can be helpful in providing context.
6. It is helpful for visitors to receive course syllabi and possible classroom examples before visiting classes.
7. Time should be incorporated for visitors to process what they have learned and to consider impact on their own work.
8. Host PI needs to have a clear calendar to handle unanticipated situations.
9. Hosts appreciate opportunities for debriefing/informal feedback; this could be in person at the end of the site visit and/or in a virtual discussion within a week of the visit.
10. Create a detailed schedule of the visit ahead of time with dates, times, rooms and people who will be attending. Distribute the schedule to everyone who will be attending any part of the site visit.
11. Have hosting team and administrators meet before the site visit to discuss goals and the role of each person; if someone will be presenting, make sure they know what they should be talking about and if there will be any unusual circumstances.
12. Create a "portfolio" of each person involved in the site visit (including local administrators, grants office administrators) with pictures and distribute before the visit. This will help to facilitate introductions.

Feedback allows the site visit protocol to be continuously improved, providing a robust mechanism for exchange of ideas between SUMMIT-P leads, conversations with faculty and students, and immersion into the specific implementation of SUMMIT-P protocols at the host institution. Both the host and the visiting teams gain insight and discover techniques to be implemented at their respective institutions. And while the information presented here has discussed a site visit involving an external visiting team, the protocol can easily be modified to be used for internal visits between academic units.

2.6 Learning Activities Developed Through Interdisciplinary Partnerships

Faculty partnerships across departments that result in interdisciplinary learning activities, for the purpose of facilitating student learning and driving transference, are one major achievement of the SUMMIT-P consortium (Housman & Porter, 2003; Perin, 2011). Chapter 3 of this volume includes ready-to-use activities that have been developed, tested, and used in the affected courses at SUMMIT-P institutions. Table 2.4 shows the partner disciplines that are participating in the project at each institution, as well as the affected mathematics course(s).

Table 2.4. Nature of SUMMIT-P Interdisciplinary Partnerships

Institution	Partner Disciplines	Mathematics Courses
Augsburg University	Chemistry, Economics	Calculus I, II, and III
Embry-Riddle Aeronautical University (ERAU)	Humanities, Communication	Quantitative Reasoning
Ferris State University (FSU)	Business, Nursing, Social Work	Quantitative Reasoning for Professionals I/II
Humboldt State University (HSU)	Biology, Mechanical Engineering, Physics	Calculus for Biology, Trigonometry, Calculus I
LaGuardia Community College (LAGCC)	Economics	College Algebra, Trigonometry
Lee University	Education, Natural Science, Social Science	Algebra for Calculus, Concepts of Mathematics, Introduction to Statistics
Norfolk State University (NSU)	Electrical Engineering	Calculus, Differential Equations
Oregon State University (OSU)	Biology, Chemistry, Zoology	Calculus I, II
Saint Louis University (SLU)	Business, Economics	College Algebra for Business, Business Calculus
San Diego State University (SDSU)	Biology, Physics	Precalculus, Calculus for Life Sciences
University of Tennessee - Knoxville (UTK)	Anthropology, Geography	Data Science, Mathematical Reasoning
Virginia Commonwealth University (VSU)	Engineering, Physics	Calculus I, Differential Equations

A primary motivating factor that drives the collaboration at each institution is the personal desire of the team to improve student experiences. The structure of the collaboration varies across institutions due to numerous factors, including size of the institution and nature of the partner discipline and disciplinary departments. The mathematicians and partner discipline colleagues work together to determine the type and scope of learning activities to be developed. Other factors that can affect the collaboration process are diversity of degrees offered, research expectations of the institution, and administrative support for institutional change. For this reason, the approach of SUMMIT-P intentionally allows each institution to:

1. develop strategies tailored to local needs;
2. engage faculty from partner discipline(s) selected by their institutional team;

3. focus on mathematics courses chosen collaboratively with the partner discipline and taught as part of the required curriculum for the partner discipline major.

In order to ensure that the process will be effective over a wide range of institutions (Beisiegel & Doree, 2020) and institutional types, SUMMIT-P institutions intentionally were selected to include broad national representation of higher education institutions (see Table 2.5).

Table 2.5. Attributes of SUMMIT-P Institutions

[3]

Institution	**Size**[4]	**Private/ Public**	**MSI Status**[5]	**Carnegie Class**	**Incoming Transfer Rate (High: $>$ 10%)**	**Pell-Grant (High: $>$ 40%)**
Augsburg	S	Private		Masters	10%	40%
ERAU	M	Private		Special	11%	26%
FSU	M	Public		Doctoral	10%	35%
HSU	M	Public	HSI	Masters	13%	53%
LAGCC	M	Public	MSI	Associates	9%	41%
Lee	S	Private		Masters	6%	30%
NSU	S	Public	HBCU	Masters	8%	74%
OSU	L	Public		R1	7.6%	23%
SLU	M	Private		R2	4%	17%
SDSU	L	Public	HSI/AANAPISI	R2	14%	33%
UTK	L	Public		R1	6%	25%
VCU	L	Public		R1	7%	29%

It is not expected that these attributes will impact outcomes at all institutions, but it is likely that they play a role in how the interdisciplinary collaboration is initiated, how it evolves, and its ultimate sustainability. This information also is intended to assist other interested institutions in identifying similar institutions within SUMMIT-P and emulating or adapting that institution's processes.

In collaboration with the institutional teams, the PMT established a clear and methodical process for creating and sustaining partnerships between mathematics and the partner discipline(s) at each institution. Primary elements of these partnerships, consistent within and across all institutions, are:

1. setting up parameters for the partnerships, predicated on a clear understanding of the motivating factors that would drive and sustain the collaboration.
2. ensuring that certain milestone activities are executed, including:
   (a) fishbowls as previously described, to develop a relevant, dynamic, and sustainable experiential learning process;
   (b) participation of team members in annual face-to-face and quarterly virtual consortium meetings;
   (c) participation in data collection activities, led by the evaluation team.
3. communicating regularly within and across institutions.
4. engaging multiple faculty members from all affected disciplines, to grow a deeper awareness of the scope of the project, to gather feedback for improvement, and to generate ideas for new classroom activities.
5. providing continuous feedback to the PMT on all aspects of the institutional team's activities, including the work's impact on faculty and students.

[3]Data from IPEDS Data Site accessed June 15, 2021

[4]S: $UG \leq 6k$; M: $6k < UG \leq 20k$; L: $20k < UG$

[5]MSI: Minority Serving Institutions, HSI: Hispanic Serving Institutions; AANAPISI: Asian American Native American Pacific Islander Serving Institution; data from `https://inclusion.uci.edu/msi/msi-directory-2/`.

6. organizing and hosting site visits, including opportunities for the PMT to engage with students affected by SUMMIT-P's work.

7. participating in site visits at other SUMMIT-P institutions to:

   (a) gain an understanding of how other institutional teams are implementing the curricular changes achieved through the interdisciplinary partnerships;

   (b) exchange ideas and information in a more nuanced way than can be achieved through other means.

2.7 Institutional Process

Institutional adjustments were made to the SUMMIT-P collaboration protocol based on individual circumstances that included the size of the institution, the diversity of disciplines that took the mathematics course(s) under review, and the existence of prior or current interdisciplinary collaborations on educational research and development efforts. Typical first steps involve setting up one-on-one meetings among the local leadership team, involving at least one mathematics professor and one partner discipline professor. Initial communication, consisting of in-person meetings and follow-up emails, can clarify priorities and identify opportunities, challenges, and key stakeholders.

At the conclusion of these meetings, the institutional SUMMIT-P team decided if they had enough information to plan the fishbowl or if more information was needed. Typically, the smaller institutions, teams with prior collaborations, and teams that already had decided which mathematics and partner discipline courses would be impacted would plan the fishbowl at the conclusion of the first meeting. The larger institutions, where it was necessary to get feedback from diverse stakeholders, usually needed additional discussion to design the best approach for the fishbowl. Often helpful is the development of a pre-fishbowl survey, administered to all partner discipline faculty, to provide an initial assessment of student needs. This information helps develop a more cohesive image of the common needs among diverse disciplines as well as a mechanism to rethink the structure of the affected mathematics course. It also allows the team to hold more effective fishbowl discussions, capturing all voices and leading to substantive information that allows the team to optimize their long-term strategy for the specific course and, in some cases, the distribution of the material in connected courses (see Hofrenning et al 2020).

The pre-fishbowl survey can be used to identify potential complications surrounding such issues as large student enrollments, wide range of student backgrounds, and the diversity of partner disciplines to be served. For some institutions, the pre-fishbowl survey will reveal the need for more fundamental changes, such as the establishment of more sections with a cap on the number of students, trial sections taught by faculty who were early adopters, or assessment of student impact (with results used to incentivize other faculty to adopt the SUMMIT-P paradigm in their courses). Faculty feedback from different disciplines via the survey also can be used to determine the mathematics course(s) to be targeted.

Once the fishbowl is conducted, institutions usually follow similar patterns for establishing any necessary changes to the selected course(s), developing active learning modules, evaluating the results, and continuing the improvement cycle. Table 2.6 illustrates the typical continuous improvement model utilized by SUMMIT-P institutions.

Feedback from institutional teams indicates that use of the fishbowl protocol is critical in promoting interdisciplinary conversation and often is used to encourage other collaboration efforts. However, being empowered to make adjustments to the protocols for site visits and other SUMMIT-P tools that address individual idiosyncrasies is a very important aspect that leads to team success.

The main purpose of the fishbowl activities is to identify the mathematical knowledge and skills that students need to be successful in their chosen partner discipline. Some outcomes from the SUMMIT-P fishbowl activities across institutions augmented the Curriculum Foundations reports, including the following:

1. Necessary mathematical skills that need more attention, such as proportions, fractions, and probabilities, tend to be ones not taught at the collegiate level. Fishbowl participants stated that students often struggle with knowledge that is an expected part of the secondary curriculum.

2. The differences in nomenclature and symbols used by partner disciplines are significant enough that students have difficulty linking the partner discipline activities to previously learned mathematical concepts.

Table 2.6. Sample Timeline for Continuous Instructional Improvement

Timeline		Milestones
Year 1	Period 1	Project definition: Goals, Strategy, Timeline
	Period 2	Climate study: survey partner discipline and/or mathematics faculty to assess course/student needs. Use survey outcomes to develop steering questions for fishbowl activity and conduct the fishbowl.
	Period 3	Use fishbowl activity and follow-up survey outcomes to develop interactive course activities via interdisciplinary mathematics/partner discipline faculty team.
Year 2	Period 1	Implement activities in mathematics/partner discipline classroom; for institutions with multiple course sections, begin by piloting in 1–3 sections. Half-day meeting of interdisciplinary mathematics/partner discipline faculty team to discuss activities, engage in classroom observations, and discuss any challenges. Provide feedback from classroom observations, emphasizing successes and offering actionable ideas for improvement. Review partner discipline, mathematics faculty, and student feedback. Implement any necessary changes to activities or course.
	Period 2	Implement revised activities or course; engage more faculty to include more course sections.
	Period 3	End of first continuous improvement cycle; engage all faculty teaching affected course(s), to discuss how to better engage students. Survey students and mathematics/partner discipline faculty; establish assessment baseline for evaluating outcomes.
Year 3	Period 1	Options: Expand to all course sections, develop common syllabus, develop common student achievement metrics. Host interdisciplinary mathematics/partner discipline faculty workshop, to discuss initiative expansion to other courses and/or disciplines.

3. The time between foundational mathematics courses (secondary or postsecondary) and the need to apply the resulting mathematical knowledge to a discipline-specific topic is an important factor for students in making connections between the concepts. This time lapse has a significant effect—not only in retention of skills, but also in ability to relate partner discipline courses with correlated mathematics courses.

These points argue for a more consistent and methodical longitudinal application of mathematical skills in a variety of courses, spanning across the entirety of secondary and post-secondary education.

2.8 Sustainability of the Collaborations

As SUMMIT-P looks to the future, the focus turns to sustainability of the collaborations. It is necessary that mathematics faculty continue to teach the renovated courses, revising materials as needed. In general, it is believed that if the course is easy to teach ("in a box"), future instructors will use it. Institutional culture requires continuous improvement through regular course revisions, so it is anticipated that such revisions will be part of the instructor's normal work load. Additionally, it is important to maintain connections across disciplines, to continue to adapt courses as needed and to help partner disciplines teach the mathematical concepts in their disciplines in a way that supports transference. This aspect is an important focus of the on-going SUMMIT-P work.

Another main element of continued sustainability that has been identified by SUMMIT-P is that of institutional recognition and support. The impact of an institutional team's work will wane unless it is expanded into the general curricular structure, with inter-departmental collaborations encouraged and supported. Expanding existing interdisciplinary teams to continue updating course methods and materials is critical if the activities are to remain relevant.

One method for gaining cost efficiencies necessary for long-term administrative buy-in is to utilize a virtual environment when it enables equally productive and successful outcomes. For example, shifting some or all site visit activities online not only reduces costs, but also reduces the carbon footprint by minimizing faculty travel. SUMMIT-P has experimented with virtual venues and found them to be a viable alternative for many team-based and multi-institutional activities. Online environments also can be more inclusive, allowing for broader faculty participation especially when travel budgets are tight. Additionally, the elimination of travel time means that more faculty—especially when including a broad range of partner disciplines—will be able to attend and contribute vital information about the disparate requirements across disciplines. For SUMMIT-P, this has resulted in a new understanding—especially among partner disciplines—of the ubiquitous need for mathematics in all aspects of life. This advantage from the online components of the site visit was further confirmed when the post-visit survey elicited a wider variety of responses from a large and diverse group of faculty participants.

2.9 Outcomes

As part of the SUMMIT-P consortium, each institution has:

1. Set up and maintained an active collaboration between mathematics and a partner discipline.
2. Used fishbowls to engage a wider partner discipline audience and to develop a more holistic determination of the changes necessary for students from the partner discipline.
3. Determined ways to continue engaging partner discipline faculty, such as yearly meetings to report out on SUMMIT-P activities and ensure the partner discipline needs are met and satisfaction surveys to faculty and students to gauge the effectiveness of the changes to mathematics instruction.
4. Used the information gathered from multiple modalities to design a course sequence that includes active learning modules relevant to a partner discipline.
5. Maintained a connection with other SUMMIT-P institutions, learning from each other and continuing to evolve.
6. Reported regularly to the PMT and evaluation team on outcomes from their activities.
7. Maintained an outlook toward sustaining the interdisciplinary collaboration and engaging in continuous improvements across disciplines.

To accomplish these goals, institutional teams need to be flexible with the number and timing of outreach activities between partner discipline and mathematics faculty, setting up fishbowls, and forming of relevant active learning modules. Thus, a strict timeline for individual institutions is not possible, emphasizing the importance of setting institutional goals and milestones. Teams also must be rigorous about implementing collaboration protocols, participating in regular meetings and reporting activities, and facilitating site visits. However, if diligently followed, the SUMMIT-P model can result in a productive and sustained team of faculty from a wide range of disciplines and institutions, focused on a vital educational change initiative. The structure and methods outlined here provide a solid framework for implementing such change at any institution.

2.10 Overcoming Barriers to Curricular Reform

SUMMIT-P has embraced both a top-down and bottom-up approach to overcome the traditional barriers to curricular reform. As described in Henderson, Beach and Finkelstein (2011), both an individual/emergent approach (FLCs) and an emergent/environmental approach (SUMMIT-P consortium) are used to change the work of individuals and related policy. One strategy alone cannot suffice in overcoming some of the intractable challenges faced when trying to affect curricular reform or ensure that these practices are enacted throughout the curriculum. However, the FLCs have facilitated the need to have major CF recommendations embraced by a group of instructors and implemented in several courses at each of the institutions. Furthermore, the FLCs have allowed for sharing of materials, experiences, ideas, and progress within and across institutions through regular and frequent meetings and on-line discussions.

By using the collective influence and voice of the SUMMIT-P consortium, the CF recommendations have been used to demonstrate how the practices can be implemented at various types of institutions and across multiple disciplines. Specifically, curricular reform often is not enacted because there is no collective will or adequate motivation to make the change. However, the mere size and breadth of the SUMMIT-P consortium, the extensive research and evaluation plans, and the anticipated outcomes have engendered a desire in others to see these changes endure and expand. Similar results to those described in (Bush et al, 2016) have been observed; i.e., individuals were able to affect change in curricula because of their ability to secure federal grant dollars and involve many institutional stakeholders. Curricular change also resulted when the investigators could demonstrate impact on improving student success. Specifically, if given the option of potential success (as demonstrated by others) when faced with the reality of current failure, most individuals will choose change:

> We took a course that has historically about a 50% success rate, and now has about a 75% success rate for first time through. We spent 18 months with three instructional specialists, an instructional designer, a leader of the center for teaching and learning, a multimedia specialist... redesigning it into a blended course ... We are now doing it in (other courses) with remarkable success... (Bush et al, 2016).

Also critical to SUMMIT-P's success has been the explicit commitment of presidents/provosts/deans/department chairs at each of the consortium's institutions, demonstrating a willingness to embrace the work of the project. The consortium, in partnership with these allies and advocates, has demonstrated and championed the effectiveness of the practices and curricular changes, while also providing examples of ways in which these practices can be infused into existing departments. Specifically, change has been accomplished, in part, by:

1. Providing access to institutional teams about well-tested teaching and assessment methods.
2. Understanding the values and goals of each institution via institutional teams that consist of leaders in the vested departments.
3. Demonstrating how the proposed changes align with institutional values and practices.

2.11 Importance of the Consortium Model

The SUMMIT-P consortium institutions were intentionally selected to constitute a wide variety of institutional types as well as a broad national geographical representation. SUMMIT-P developed a model for this wide variety of institutions that has been instrumental in ensuring the consistency and sustainability of the project as it gains national prominence. Institutional teams have been further strengthened by site visits, professional development, and interactions among institutions with common areas of focus (SUMMIT-P, 2020a, 2020b). The project evaluation includes shared metrics and is yielding extensive common data across the SUMMIT-P institutions, leading to more accurate information about the effects of innovative partnerships and resulting curricular change (SUMMIT-P, 2016). Preliminary results indicate that students are seeing the connections between mathematics and partner disciplines, resulting in significant changes in attitude about the importance of various components within the undergraduate curriculum. Faculty are taking more ownership of developing course materials that assist with these student transfers of information. Additionally, the project is generating a greater understanding among faculty of the interdisciplinary nature of mathematics (SUMMIT-P, 2016).

While the use of FLCs to support the work of the multi-institutional SUMMIT-P consortium (as opposed to institutions working independently) requires some additional resources, the benefits of the consortium far outweigh the costs. This multi-institutional consortium shares curricular improvements across institutions while collectively reflecting on active learning pedagogical practices. There also is significantly greater potential for sustaining and expanding the consortium's work, based on research that FLCs provide support networks not found in a typical single institution project. In addition, several expenses incurred (e.g., evaluation, dissemination, advisory board) are ones that ordinarily would be distributed among the individual institutions at a higher total cost—and with far less impact and transferability than a national consortium provides.

Additionally, a key strength of the consortium model is that it allows each institution's personnel to collectively work in a variety of dimensions, but with different collaborators—some on course content, others on applied contexts, still others on pedagogy. A wide range of institutions allows for the study of these interactive effects. For example, both

LaGuardia Community College (LAGCC) and San Diego State University have extensive experience working with Hispanic students; Norfolk State University and LAGCC have extensive experience working with African-American students; and Augsburg University brings extensive experience working with students with disabilities. These institutional teams collectively ensure the development of culturally-responsive contexts and universally-accessible materials.

2.12 Strength in Numbers

When formulating the SUMMIT-P work, it was tempting to have all institutions in the consortium focus on one course, one disciplinary partner, one pedagogical change, or one student support shift. After all, scientists are trained to control all but the one factor being investigated. But the research indicates that improving conceptual understanding of the content, setting the content in applied contexts, using active and engaging pedagogies, teaching in ways that reach diverse student populations, and supporting students outside the classroom are not independent factors. Quite the opposite is true—it is the interactive effects of these factors that hold the power to dramatically and sustainably reshape the STEM curriculum. SUMMIT-P deliberately has studied the process of curricular change across a range of courses, with a variety of partner disciplines, and incorporating pedagogy and student support in a carefully structured interdependent system.

Studying interdisciplinary collaboration and institutional change via FLCs through the lens of teaching mathematics addresses two pressing national needs: 1) student improvement in mathematics, leading to their ability to engage in further STEM studies, and 2) understanding and formalizing the process of impactful institutional change that leads to meaningful and sustainable outcomes. The highly interdisciplinary model adopted by SUMMIT-P has worked to erode disciplinary silos, which contribute to a "fragmented approach to teaching and learning [which] often results in a student experience that lacks coherence and relevance" (Bishop, Piercey, & Stone, 2020, p.70). Specifically, mathematics faculty have worked carefully with partner discipline colleagues to identify mathematical concepts and skills needed for success in discipline-specific courses. The subsequent creation of "wish lists" from the partner disciplines ensures that problem scenarios remain authentic and appropriate to the discipline, resulting in a mathematics curriculum that delivers mathematical content in a more meaningful, context-rich environment. These interdisciplinary teams not only have modeled effective collaboration, but also have benefited from the diverse perspective of team members to produce high-quality materials (Bishop, Piercey, & Stone, 2020; Brucall-Hallare, Moosavizadeh, & Deo, 2020; Hofrenning et al, 2020; Lai et al, 2020; May et al, 2020; Piercey et al, 2020; Robinson et al, 2020; Venkatesh & Militzer, 2020). For example, Saint Louis University has integrated a business approach to spreadsheets in their college algebra courses. This approach takes advantage of the functionality of spreadsheets while illuminating algebraic thinking (see Sections 3.11, 3.12, 3.17 of this volume). Oregon State University has developed a series of calculus problems in which students study the absorption of glucose and other issues related to diabetes (see Section 3.16 of this volume and SUMMIT-P, 2020b).

The unifying theme remains: **how to best implement lasting curricular change in the first two years of mathematics and throughout partner discipline courses, using findings from the Curriculum Foundations Project to support substantive collaborations across disciplines. Such a system requires clear collaborative structures, allowing for continuous sharing of in-class activities, projects, data, examples, and experiences.** SUMMIT-P has created such a structure through numerous cross-institutional activities such as site visits. Site visits allow curriculum developers and implementers to narrow the focus of their efforts, contribute to community building, support cross-pollination of ideas, and provide dedicated time to reflect on the ongoing work. These multi-institutional visits (conducted both face-to-face and virtually) allow participants to observe their peers in action while also providing the opportunity to collaborate on the creation and sharing of faculty development ideas, which subsequently are shared with other SUMMIT-P institutions via discussions and workshops.

To that end, a broader network of institutions has been formed as part of the long-term sustainability plan of the SUMMIT-P work. To date, more than 50 institutions have had faculty involvement in SUMMIT-P activities. And while many of these institutions have not formally joined the consortium, the faculty colleagues at these institutions are highly interested in maintaining their interactions with this national network. SUMMIT-P is planning for a partnership with the National Numeracy Network (NNN), which will serve as a long-term mechanism for housing the SUMMIT-P consortium. Specifically, NNN promotes education that integrates quantitative skills across all disciplines and at all levels. To this end, NNN supports faculty development, curriculum design, assessment strategies, educational research, and systemic change. The many common goals of SUMMIT-P and NNN make a long-term partnership ideal.

2.13 Conclusion

The work of SUMMIT-P has a dual objective: to establish a flexible and sustainable framework for educational collaborations among faculty from different disciplines and, through that collaboration, to drive improvements in student attainment—either in the mathematics course itself, or in subsequent classes for which the mathematics course is a prerequisite. Such collaboration provides a longitudinal and latitudinal aspect to the study and practice of mathematics. For the many students in introductory mathematics courses who find mathematics intimidating, the interdisciplinary faculty partnership can help to remove the intimidation factor by establishing relevancy through active learning paradigms established with the partner discipline. Thus, the mathematics course can provide more depth and, through the discovery of areas of interaction with the partner discipline, can offer opportunities for a continued application of mathematics principles to the student's area of study. Establishing early connections can help build a bridge from mathematics to the partner discipline through a more direct approach. At a minimum, the examples provided by the partner discipline introduce the mathematical concepts in a language and a form that students will encounter later in their studies. This can widen the students' perspective from the standardized mathematical language to the more diverse forms that mathematical structures can take.

In order to ensure relevance and transference of knowledge and skills, it is necessary to provide a forum that allows the partner discipline to discuss the challenges and corresponding needs of their students. The fishbowl technique is one example of an agile framework for such discussions across many disciplines. The SUMMIT-P model extends this technique to create a foundation for change as follows:

1. Survey partner disciplines using the Curriculum Foundations reports to craft questions along with topics covered in the specific mathematics course targeted for revision.
2. Analyze survey to create a new common starting point for the fishbowl discussion, which is shared with the partner discipline fishbowl participants prior to the fishbowl.
3. Conduct fishbowl using the structure provided.
4. Create partner discipline wish lists and a syllabus mapping exercise in which the topics identified in the survey and fishbowl are mapped onto the syllabus of the course slated for revision.
5. Compile and create exercises and examples with partner discipline input that utilize the mathematical concepts identified in the fishbowl activities. In this way, collaborations among partner disciplines and mathematics can lead to substantive changes that benefit student learning.

Once necessary changes are developed and implemented, assessment is crucial for continuous improvement and continued relevancy and sustainability. If an institutional team is able to demonstrate a correlation between SUMMIT-P modules and student success, it will ensure engagement of faculty, sustainability, and scalability. By following the protocols outlined here and incorporating the ready-made activities to follow in Chapter 3, a collaboration can be initiated, maintained, and sustained over a long period, providing a framework for successful collaboration that leads to systemic and positive change.

2.14 References

Angelo, T.A. (1997). The campus as a learning community: Seven promising shifts and seven powerful levers. *AAHE Bulletin, 49*(9), 3–6.

Apkarian, N., Bowers, J., O'Sullivan, M., & Rasmussen, C. (2018). A case study of change in the teaching and learning of Precalculus to Calculus 2: What we're doing with what we have. *PRIMUS, 28*(6), 528-549.

Beisiegel, M. and Doree, S. (2020). Curricular change in institutional context: A profile of the SUMMIT-P institutions. *Journal of Mathematics and Science: Collaborative Explorations, 16*(1): 192–201.

Beisiegel, M., Kayes, L., Quick, D., Nafshun, R., Lopez, M., Dobrioglo, S., & Dickens, M. (2020). The process and a pitfall in developing biology and chemistry problems for mathematics courses. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1): 92–106, Virginia Mathematics and Science Coalition, Richmond, VA.

Bickerstaff, S., Edgecombe, N., & the Scaling Innovations team (2012). Pathways to faculty learning and pedagogical improvement. *Inside Out, 1*(3).

Borrego, M., & Henderson, C. (2014, April 25). Increasing the use of evidence-based teaching in STEM higher education: A comparison of eight change strategies. *Journal of Engineering Education, 103*(2), 220–252.

Bowers, J., Williams, K., Luque, A., Quick, D., Beisiegel, M., Sorensen, J., Kunz, J., Smith, D., & Kayes, L. (2020). Paradigms for creating activities that integrate mathematics and science topics. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 58–68, Virginia Mathematics and Science Coalition, Richmond, VA.

Brucal-Hallare, M., Moosavizadeh, S., & Deo, M. (2020). Promoting partnership, cultivating colleagueship: A SUMMIT-P project at Norfolk State University. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 136-146, Virginia Mathematics and Science Coalition, Richmond, VA.

Bishop, R., Piercey, V., & Stone, M. (2020). Using a faculty learning community to promote interdisciplinary course reform. *Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 69–83. `https://doi.org/10.25891/c0m8-6v73`

Bryk, A. S. (2010). Organizing schools for improvement. *Phi Delta Kappan, 90*(7), 23–30.

Bush, S., Rudd, J.A. II, Stevens, M., Tanner, K., & Williams, K. (2016). Fostering Change from Within: Influencing Teaching Practices of Departmental Colleagues by Science Faculty with Education Specialties. *PLOS.* `http://dx.doi.org/10.1371/journal.pone.0150914`

Clear Impact. (2016). *Achieving collective impact.* `https://bit.ly/3JZnvmY`

Cox, M.D. (2001). Faculty learning communities: Change agent institutions into learning organizations. In D. Lieberman and C. Wehlberg (Eds.), *To Improve the Academy.* Anker Publishing: Bolton, MA, 69–93.

Cox, M.D. & Richlin, L. (Eds, 2004). *Building Faculty Learning Communities: New directions for teaching and learning.* Jossey-Bass: San Francisco.

Dee, J. R., & Daly, C. J. (2009). Innovative models for organizing faculty development programs: Pedagogical reflexivity, student learning empathy, and faculty agency. *Human Architecture: Journal of the Sociology of Self-Knowledge, 7*(1), Article 2.

Eckel, P., & Kezar, A. (2003). *Taking the reins: Institutional transformation in higher education.* Greenwood Publishing.

Ganter, S. & Barker, W. (Eds.). (2004). *The curriculum foundations project: Voices of the partner disciplines.* Mathematical Association of America.

Ganter, S.L., & Haver, W.E. (2020). The need for interdisciplinary collaborations. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 1-9, Virginia Mathematics and Science Coalition, Richmond, VA.

Grove, G.T. (2009). *Success analysis protocol for individuals.* School Reform Initiative. `https://bit.ly/3rEXzGC`

Hargraves, R., Hofrenning, S., Bowers, J., Beisiegel, M., Piecey, P., Young, E. (2020). Structured engagement for a multi-institutional collaborative to tackle challenges and share best practices. *Journal of Mathematics and Science: Collaborative Explorations, 16*(1): 43–57.

Henderson, C. & Dancy, M. (2011) *Increasing the impact and diffusion of STEM education innovations.* Commissioned by the Characterizing the Impact and Diffusion of Engineering Education Innovations Forum. `https://bit.ly/38dVMBS`.

Hofrenning, S.; Hargraves, R.H.; Chen, T.; Filippas, A.V.; Fitzgerald, R.; Hearn, J.; Kayes, L. J.; Kunz, J., & Segal, R. (2020). Fishbowl discussions: Promoting collaboration between mathematics and partner disciplines. *Journal of Mathematics and Science: Collaborative Explorations, 16*(1): 10–26.

Housman, D. & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics 53*(2), 139–158.

Jay Dee, M. H. T. & Zamel, V. (2009) Editors' Note : NECIT , CIT , and teaching transformations 2009. *Human Architecture: Journal of the Sociology of Self-Knowledge 7*(1).

Johnson, V. (2019, August 19). Success analysis protocol with reflective questions. School Reform Initiative. `https://bit.ly/37BAAWd`

Lai, C., Henshaw, G., Chen, T., & Kone, S. (2020). Integrative and contextual learning in college algebra: An interdisciplinary collaboration with economics. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 120–135, Virginia Mathematics and Science Coalition, Richmond.

Lane, J., Froyd, J., Morgan, J., and Kenimer, A. (2002). *Faculty Learning Communities.* ASEE/IEEE Frontiers in Education Conference Proceedings. American Society for Engineering Education: Washington, DC.

Lane A.K., Skvoretz, J., Ziker, J.P., Couch, B.A., Earl, B., Lewis, J.E., McAlpin, J.D., Prevost, L.B., Shadle, S.E., & Stains, M. (2019). Investigating how faculty social networks and peer influence relate to knowledge and use of evidence-based teaching practices. *International Journal of STEM Education, 6*(28).

Martin, W. G., & Gobstein, H. (2015). Generating a networked improvement community to improve secondary mathematics teacher preparation: Network leadership, organization, and operation. *Journal of Teacher Education, 66*(5), 482–493.

May, M., Segal, R., Piercey, V., & Chen, T. (2020). Good teachers borrow, great teachers steal: A case study in borrowing for a teaching project, *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 180-191, Virginia Mathematics and Science Coalition, Richmond, VA.

McDonald, J. P., Mohr, N., Dichter, A., & McDonald, E. C. (2013). *The power of protocols: An educator's guide to better practice.* Teachers College Press.

McDonnough, J. M., & Hanschel, M. (2015). Professional learning community-based induction: creating support for new teachers of science. In J. A. Luft, *Newly hired teachers of science: A better beginning* (pp. 145–154). Rotterdam: Sense Publishers.

Mindich, D., & Lieberman, A. (2012). *Building a learning community: A tale of two schools.* Stanford Center for Opportunity Policy in Education.

Mohr, N., Parrish, C., & Taylor, S. (2019, August 16). Descriptive consultancy. School Reform Initiative. `https://www.schoolreforminitiative.org/download/descriptive-consultancy/`

A National Consortium for Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P). (2016). *SUMMIT-P.* `https://www.summit-p.com`

/ National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* NCTM: Reston, VA, 193–195.

National Research Council (1995). *National Science Education Standards.* NRC: Washington, DC.

National Research Council. (1999). *How people learn: Brain, mind, experience, and school.* Washington, DC: National Academies Press.

Perin, D. (2011). Facilitating student learning through contextualization: A review of evidence. *Community College. Review 39*(3), 268–295.

Piercey, V., Segal, R., Filippas, A., Chen, T., Kone, S., Hargraves, R., Bookman, J., Hearn, J., Pike, D., & Williams, K. (2020). Using site visits to strengthen collaboration. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 27–42, Virginia Mathematics and Science Coalition, Richmond, VA.

Robinson, J., McClung, P., Maher-Boulis, C., J. Cornett, & Fugate, B. (2020). Counting on collaboration: A triangular approach in the educator preparation program for teachers of mathematics. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 107-119, Virginia Mathematics and Science Coalition, Richmond, VA.

Sanders, K., Carlson-Dakes, C., Dettinger, K., Hajnal, C., Laedtke, M., & Squire, L. (1997, June). A new starting point for faculty development in higher education: Creating a collaborative learning environment. In D. DeZure (Ed.), *To Improve the Academy 16*, 117–150.

Shapiro, N.S. & Levine, J.H. (1999). *Creating learning communities: A practical guide to winning support, organizing for change, and implementing programs.* Jossey-Bass Higher and Adult Education Series, Jossey-Bass: San Francisco, CA.

Shulman, L. (2001). *The Carnegie Classification of institutions of higher education, 2000 Edition.* Menlo Park, CA: The Carnegie Foundation for the Advancement of Teaching).

Venkatesh, A., & Militzer, E. (2020). From creative idea to implementation: Borrowing practices and problems from social science disciplines. *The Journal of Mathematics and Science: Collaborative Explorations, 16*(1), 161–170, Virginia Mathematics and Science Coalition, Richmond, VA.

Yau, J., & Lawrence, J. (2020, April 6). School reform initiative scope of work. School Reform Initiative. `https://www.schoolreforminitiative.org/research/school-reform-initiative-scope-of-work/`

2.15 Appendix: Meet the Institutions!

Augsburg University (Biology, Chemistry, Economics) The Augsburg team selected the calculus sequence, with the goals that students appreciate and enjoy the courses, understand the impact of calculus on their studies, recognize and apply the learned concepts in context, and seek out more mathematics courses. Increasing the number, breadth, and relevance of active learning modules was the primary change strategy. The entire calculus sequence was addressed to ensure optimal continuity between course units and with any subsequent relevant partner discipline course. The new course sequences, were designed and revised during the summer, with implementation and data collection each year in the subsequent fall semester.

Embry-Riddle Aeronautical University (Communication, Humanities) The ERAU team has developed a pair of new courses (Learning to Reason I and II), delivered asynchronously online, that deeply investigate quantitative reasoning through multiple perspectives to provide an alternative mathematics pathway for non-STEM students. The courses blend elements of rhetoric, logic, and history with mathematical computation, representation, and application to break through the perceived barriers between the well-defined world of mathematics and the ambiguous world of the humanities. The interdisciplinary course development team includes a mathematician, statistician, humanist, and communication scholar. As a result, the Learning to Reason courses bring together the diverse methods of inquiry and reasoning that present students with experiences in critical thinking that involve both numbers and words.

Ferris State University (Business, Nursing, Social Work) While most SUMMIT-P institutions began by examining where specific mathematical content was used in partner discipline contexts, the Ferris team first considered partner discipline scenarios and then identified the relevant mathematics. The target course is Quantitative Reasoning, so the context-first approach fits the ethos of the course. The wishlist developed from the fishbowl was built around big-picture learning outcomes rather than specific content needs. A variety of contexts and scenarios were considered, and those that were particularly rich in connections among mathematics and the partner disciplines have been used as case studies and for role-playing simulations. A faculty learning community supports the work through multi-disciplinary teams that develop course materials for both Quantitative Reasoning and partner discipline courses. Faculty teams were guided through a process that mirrored the "scenario-first" approach originally taken by the course. The faculty learning community was well-received by faculty, and is a vital part of the sustainability for the Ferris project.

Humboldt State University (Chemistry, Physics) Team members participated in a faculty learning community to create a shared repository for introductory mathematics course materials. The SUMMIT-P materials will be part of this Canvas resource. The long-term concept is to have both a Canvas repository of resources and to build out Canvas "shells" for introductory courses that all instructors can use and modify for their own purposes. This work is a broader departmental effort, but it will help with the dissemination of SUMMIT-P materials.

LaGuardia Community College (Economics) LaGuardia serves 50,000 students, of which 43% are Hispanic and 21% are Black. The LaGuardia team worked on the College Algebra course. This course serves a broad range of student majors across a large number of sections (60 sections per semester). Student responses to surveys of college algebra showed that they have negative attitudes towards the course since its usefulness cannot be recognized. As such, the team's approach was to use economics contexts to motivate the mathematics content. Given the wide variety of student interests and majors, the team mapped College Algebra content to contexts in microeconomics that would be of general interest, such as the use of data to construct demand curves (see 3.7).

Lee University (Chemistry, Education, Psychology) The Lee University team deviated slightly from the traditional format during the actual fishbowl exercise. During the exercise, partner discipline faculty were given the current Algebra for Calculus syllabus as well as relevant information from the Curriculum Foundations (CF) reports. These documents helped to keep the conversation focused and relevant. During the fishbowl, partner discipline faculty discussed the CF recommendations and generated a wish list based on those recommendations. In addition to the algorithmic algebra skills, partner discipline faculty wanted students to have a good understanding of the meaning of functions and relations. Fishbowl participants from all disciplines gained a better understanding of the language differences between mathematics and science courses, which also was used to focus the curricular changes.

Norfolk State University (Engineering) The primary goal for the Norfolk State team is to broaden participation of African-Americans in the STEM workforce. Mathematics and engineering faculty are partnering to redesign Calculus I–II because these two courses are identified as roadblocks for students wanting to major in science. A half-day faculty retreat, including a fishbowl activity, revealed there were a few new topics that engineering faculty believe are important, while others were deemed less important. This was surprising to many mathematics faculty. A main outcome of the SUMMIT-P work is that faculty from both disciplines now understand that they need to work together to ensure that courses have important activities and are aligned across disciplines so that students have the tools they need to be successful.

Oregon State University (Biology, Chemistry) The OSU team is working on discipline integration in Calculus I. Through fishbowls and related activities, biology faculty created a wish list with mathematics topics at a much lower level than expected for undergraduate students, and the team learned that very few biology faculty members use mathematics beyond introductory statistics in their courses. Broad faculty participation in the discussions was achieved by leveraging regular department meetings. By linking the fishbowl activity to improvements in the biology curriculum, the team was able to get buy-in from the department chair. The team was pleasantly surprised by the desire of biology faculty to contribute to decisions about the mathematical skills required for their students, and modifications to the biology major also were an important outcome of these discussions.

San Diego State University (Biology, Physics) All sections of Precalculus and Calculus for Life Sciences are using SUMMIT-P materials. The college algebra sections are piloting activities in one section. After hosting a virtual site visit, SDSU shared Desmos (Precalc) activities with Lee and HSU, with HSU also sharing solar panel lessons. The SDSU mathematics department received a grant to use the active learning modules developed in precalcalculus (as well as others developed for Calculus I & II) from the California Learning Lab. The modules will be shared with Grossmont Community College, one of the largest colleges that articulates with SDSU.

Saint Louis University (Business) Over the past year, a total of 1,170 students were impacted from Business Calculus, College Algebra for Business, Managerial Accounting, Finance, Economics, and Operations Management; faculty from across several business disciplines continue to partner with mathematics faculty on the SUMMIT-P work.

University of Tennessee-Knoxville (Anthropology, Geography) The course Mathematical Reasoning has been targeted at UTK. The team's goals are to (a) improve the perception of relevance of mathematics among students by including more authentic applications of mathematics, (b) equip instructors to teach partner discipline applications, and (c) instill mathematical confidence in students who are majoring in partner disciplines. A new textbook, targeted for campuswide adoption, has been written that includes interactive applications developed by collecting ideas from geography and anthropology faculty. Plans are in place for SUMMIT-P work in a second course that is part of a new undergraduate major in Data Science.

Virginia Commonwealth University (Engineering) VCU used a modified fishbowl approach, beginning with an online survey to gather information from as many engineering faculty as possible. The survey contained a list of all major topics typically in the Differential Equations course, and the faculty were asked to rate the importance of each topic. The ratings were compiled and circulated to mathematics faculty in preparation for the fishbowl. At the fishbowl, mathematics faculty were surprised to learn that streamlining course content would serve engineering students better than covering a wide range of topics. As a result, significant changes were made to the Differential Equations course, including the addition of active learning components based on relevant engineering problems. A team-teaching approach solidified the connection between Differential Equations and courses in several engineering subdisciplines. The team plans to continue the work by implementing similar changes in both calculus and engineering courses.

3

Interdisciplinary Classroom Modules from the SUMMIT-P Consortium

Edited by Victor Piercey,[1] *Ferris State University*
Jody Sorensen, *Augsburg University*
Jeneva Clark, *University of Tennessee-Knoxville*

3.1 Introduction

Victor Piercey, Jody Sorensen, and Jeneva Clark
Ferris State University, Augsburg University, and University of Tennessee-Knoxville

The final chapter in this volume includes classroom modules produced using the SUMMIT-P collaboration model described in Chapter 2. These modules provide examples illustrating what can be accomplished across disciplines.

The modules can be used directly in the classroom, and while we hope you do that, we also hope these modules inspire you to engage with your partner discipline colleagues. Perhaps you want to know more about one of the contexts. Perhaps your colleagues would like to build on your implementation of the module in their partner discipline courses. Even better – perhaps you and your colleagues would like to produce similar modules for your courses and programs. In addition to using the model described in Chapter 2, you and your colleagues might want to dive into the research described and cited in Chapter 1.

A key project described in Chapter 1, and indeed the basis for the SUMMIT-P project, is the Curriculum Foundations (CF) project. The 20 modules in this chapter represent the recommendations from the CF work in action, requiring conceptual problem solving, exploration of mathematical relationships between variables, and modeling in context. Some modules connect with content that STEM students will see in future courses, such as titration in Chemistry or circuits in Engineering, while some overlay mathematics with complex societal issues. For example, Module 3.4 requires students to analyze ethical issues arising out of using expected values to calculate insurance premiums under multiple circumstances.

The SUMMIT-P institutions have increased their use of active learning through oral communication using such methods as small group discussions, student presentations, and group problem solving activities. Team projects and presentations encourage and enhance interpersonal skills of students. The ability to work in a team and to effectively communicate the results of a problem are recognized as core skills in the workforce (see, e.g., Finley 2021). Furthermore, the experience of presenting to audiences outside one's discipline forces students to highlight major themes and not rely on jargon but to engage an audience with common narratives. For example, students in Ferris State University's quantitative reasoning course engage in informal in-class discussions, small group debates, and formal written

[1]Contact: piercev1@ferris.edu

presentations to diverse audiences across multiple partner disciplines in order to construct and reinforce their learning (e.g., see Modules 3.3, 3.6 and `http://bit.ly/RuralHealthClinic`).

Additionally, each institution has increased its emphasis on conceptual understanding within the targeted courses. Students demonstrate this conceptual understanding by applying models and relating graphs, symbols, and concepts (a key CF recommendation—e.g., see Modules 3.19, 3.20, and 3.21). Assessments of students' conceptual understanding is obtained through written communication such as lab reports, small group project reports, and open-ended questions on exams (e.g., see Modules 3.4, 3.6, 3.7, and 3.8). Collaborative writing and peer review are additional examples of strategies that allow students to think about a concept more deeply, to recognize others' viewpoints, and to give constructive criticism to peers.

Finally, several of the modules involve working with data in spreadsheets. Spreadsheets are one of the most common forms of technology used by professionals in the 21st century, and the ability to use spreadsheets is becoming an increasingly important quantitative skill (see, e.g., Vacher & Lardner 2010). Modules such as 3.3, 3.8, 3.13, and 3.14 incorporate basic spreadsheet skills such as entering a formula or generating a graph. Others can be used to teach more advanced skills, such as the use of mixed cell references in Module 3.6.

Three modules created at one institution (3.11, 3.12, and 3.17) strive to teach students to think deeper about using a spreadsheet - what the authors call "Thinking in Excel." These modules are derived from College Algebra and Calculus textbooks written by the authors, and students in their classes are eventually taught to design their own spreadsheets in a way that is user-friendly and easy to read, while assisting with solutions to complex problems. This represents what the partner discipline authors (from business) see as one of the more critical quantitative skills their graduates will need for success.

The final module in this volume, 3.22, integrates mathematics with the humanities. While the other modules in this chapter involve bringing partner discipline content into mathematics, this module is about bringing mathematics into the humanities. It can be applied as an enrichment resource in just about any mathematics course.

References

Finley, A. (2021) *How college contributes to workforce success: Employer views on what matters most*. American Association of Colleges and Universities and Hanover Research. 2021.

Vacher, H L, & Lardner, E. (2010) Spreadsheets across the curriculum, 1: The dea and the resource. *Numeracy 3*(2) Article 6. `http://dx.doi.org/10.5038/1936-4660.3.2.6`

3.2 How to Use These Modules

Victor Piercey, Jody Sorensen, and Jeneva Clark
Ferris State University, Augsburg University, and University of Tennessee-Knoxville

The modules consist of a standard introduction which includes items such as the institutional background information of the authors, partner discipline background necessary to facilitate the module, and an implementation plan. In most cases, the introductions are followed by the module, including (where appropriate) solutions. In some cases, it made more sense to include the module followed by solutions. This was the case for Modules 3.16 and 3.20. In other cases, such as Module 3.3, all of the solutions are in a spreadsheet, and you can contact the authors for the file. Finally, in some cases, such as 3.22, or an essay portion of a module, there are no fixed solutions.

What makes these modules special is that they are the result of the collaboration process described in previous chapters. As such, they integrate authentic applications of mathematics that one does not typically find in courses and textbooks developed solely by mathematicians.

This means that the focus is on the integration of mathematics and partner discipline content, not pedagogy. While the authors generally implemented the modules using some form of active learning, and the use of active learning was included in the curriculum foundation recommendations, this is not strictly necessary. One could use the modules as a basis to prepare lecture notes or assign them as homework.

We do want to encourage using the modules in some form of an active learning environment. To that end, each module includes implementation recommendations that describe how the authors used the modules in the classroom and the choices that they made. Those with less experience that are interested in using one or more of these modules with active learning are encouraged to consult Abell et al (2018), Ernst, Hitchman, & Hodge (2017), Smith, Rasmussen, & Tubbs (2021), and Braun et al (2017) for guidelines.

In addition, the reader implementing these modules should be attentive to equity. We recommend the reader consult Burton (2003), Feldman (2018), and Bonner (2021) to identify appropriate inclusive and equitable teaching practices for their implementation. Additionally, while equity is one of the four pillars of teaching with inquiry (Laursen & Rasmussen 2019) and inquiry has been shown to improve equity outcomes (Kogan & Laursen 2014), the reader should be cautioned that active learning and inquiry do not automatically lead to equity. See Louie (2017) and Johnson et al (2020) to identify harmful practices to avoid when implementing active learning.

Finally, several - but not all - of the modules were adapted for online learning during the COVID-19 pandemic. Those that were include descriptions of modifications. Others were either not used during the pandemic, or simply implemented with breakout rooms. The final module - STEM students as storytellers - was always designed for asynchronous learning.

For copies of sheets to print for classroom use, as well as supplementary excel files, handouts, or other documents, visit `https://bit.ly/SUMMITPNotesActivities`.

The modules available in this chapter are:

Course	Module	Mathematics Topic	Partner Discipline
Quantitative Reasoning	3.3: Refugee Camp Administration	Unit Rates and Percentages	Social Work
Quantitative Reasoning	3.4: Marshmallow Shooters	Quadratic Modeling	Education
Quantitative Reasoning	3.5: Health Insurance	Expected Value	Ethics
College Algebra/ Precalculus	3.6: Human Trafficking	Linear Functions	Social Work
College Algebra/ Precalculus	3.7: Demand	Linear Functions	Economics

Course	Module	Mathematics Topic	Partner Discipline
College Algebra/ Precalculus	3.8: Medicine Dosing	Exponential Functions	Health Science
College Algebra/ Precalculus	3.9: Population Dynamics	Percentage Change Rates	Social Science
College Algebra/ Precalculus	3.10: pH	Logarithm Laws	Chemistry
College Algebra/ Precalculus	3.11: Fitting Exponential Curves	Exponential Functions	Finance
College Algebra/ Precalculus	3.12: What-If Analysis	Solving for Variables	Finance
Calculus	3.13: New Product Adoption	Logistic Functions	Economics
Calculus	3.14: Titration	Derivatives	Chemistry
Calculus	3.15: Glucose Absorption	Limits, Derivatives, and Concavity	Biology
Calculus	3.16: ECG Signals	Curve Sketching	Biomedical Engineering
Calculus	3.17: Riemann Sums in Excel	Riemann Sums	General Business
Calculus	3.18: Solar Energy	Integration	Environmental Resources Engineering
Differential Equations	3.19: Second Order Circuits	Second Order Linear Differential Equations with Constant Coefficients	Electrical Engineering
Differential Equations	3.20: Wireless Power Transfer	Laplace Transforms	Electrical Engineering
Differential Equations	3.21: Nuclear Engineering	First Order Linear Differential Equations	Nuclear Engineering
Any	3.22: Media Journal	Mathematics in Popular Culture	Humanities

References

Abell, M.L., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *Instructional practice guide*, Mathematical Association of America.

Bonner, E.P. (2021). Practicing culturally responsive mathematics teaching. *Mathematics Teacher: Learning and Teaching PK–12 114*(1), 6–15.

Braun, B., Bremser, P., Duval, A.M., Lockwood, E., & White, D. (2017). What does active learning mean for mathematics? *Notices of the AMS 64*(2), 124–129.

Burton, L. (1996). A socially just pedagogy for the teaching of mathematics. In Gipps, C.V. & Murphy, P.F. *Equity in the Classroom*. Routledge: London. 145–154.

Ernst, D.C, Hitchman, T., & Hodge, A. (2017). Bringing inquiry to the first two years of college mathematics. *PRIMUS 27*(7), pgs. 641–645.

Feldman, J. (2018). *Grading for equity: What it is, why it matters, and how it can tranform schools and classrooms*. Corwin: Thousand Oaks, CA.

Johnson, E., Andrews-Larson, C., Keene, K. & Melhuish, K. (2020). Inquiry and gender inequity in the undergraduate mathematics classroom. *Journal of Research in Mathematics Education 51*(4) 504–516.

Kogan, M. & Laursen, S.L. (2014), Assessing long-term effects of inquiry-based learning: A case study from college mathematics. Innovative Higher Education. 39, 183–199.

Laursen, S. & Rasmussen, C. (2019). I on the prize: Inquiry approaches in undergraduate mathematics. *International Journal of Research in Undergraduate Mathematics 5*(1) 129–146.

Louie, N.L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education 48*(5) 488-519.

Smith, W.M., Rasmussen, C., & Tubbs, R. (2021). Introduction to the special issue: insights and lessons learned from mathematics departments in the process of change. *PRIMUS 31*(3-5) 239-251.

3.3 Genocide Refugee Camp Management

Victor Piercey, Mischelle Stone, and Rhonda Bishop
Ferris State University
Contact: pierceyv1@ferris.edu

3.3.1 About the Module

- Course: Quantitative Reasoning
- Partner Disciplines: Social Work
- Required Technology: Excel, or any other spreadsheet application

3.3.2 Institutional and Course Contexts

- Type/size of institution: Regional comprehensive university, about 13000 students
- Size of Class: 25
- Characteristics of Students:Non-STEM majors, developmental
- Mathematical Content: Proportional reasoning - units, unit rates, and percentages
- Purpose/Goal of the Activity: Students use proportional reasoning to assess the facilities at refugee settlement camps.
- After and Before: In our course, this module is completed after a chapter on proportional reasoning and before a chapter on constructing and deconstructing algebraic formulas. They need to know how to calculate unit rates and percentages.
- Other Prerequisites: Students also need to be able to enter a simple formula into Excel and copy the formula down a column as well as generate graphs in Excel. This is assumed in the module.
- Inspiration for the Module: This module was developed during a faculty learning community at Ferris State University led by Barry Mehler, Professor of History. The faculty learning community was built around the use of the SHOAH archive – a searchable database of video testimonies given by genocide survivors. Like other library databases, a subscription is required to access the content. For more information, visit https://sfi.usc.edu/what-we-do/collections.

3.3.3 Partner Discipline Background

Genocide, as defined by the United Nations Genocide Convention as well as the Rome Statute of the International Criminal Court, means any of the following acts committed with intent to destroy, in whole or in part, a national, ethnic, racial, or religious group as such:

1. Killing members of the group;
2. Causing serious bodily or mental harm to members of the group;
3. Deliberately inflicting on the group conditions of life calculated to bring about its physical destruction in whole or in part;
4. Imposing measures intended to prevent births within the group;
5. Forcibly transferring children of the group to another group.

The Rohingya, an ethnic Muslim group in the northwestern state of Rakhine in Myanmar, have been subjected to treatment that, since August 2017, human rights organizations such as Amnesty International have called genocide. By the end of 2017, over 600,000 of the nearly 1 million Rohingya had fled over the border to Bangladesh to seek refuge. Bangladesh set up refugee settlement camps in the city of Cox Bazar, near the border.

This module uses data from the REACH Initiative, an organization based in Geneva, Switzerland whose purpose is to gather and assess data in order to aid in humanitarian crises. The data outline conditions of bathroom and water facilities in the refugee camps. Students will analyze the data to assess the condition of the facilities and make recommendations.

The data include numbers of latrines, washrooms, and improved water sources. A latrine only has a toilet, whereas a washroom includes a sink and may have a shower or bath. An improved water source is a constructed water source that adequately protects the water from outside contamination.

3.3.4 Implementation Plan

Formal Learning Objectives

1. Use technology to calculate unit rates and percentages.
2. Interpret unit rates and percentages in context.
3. Use data-based evidence to assess conditions in a social context.

Materials and Supplementary Documents The module requires the use of the data in an Excel workbook or some other spreadsheet, and this will require access to computers.

The module spreadsheet is available on the volume's website. For a completely filled out spreadsheet with the answers, contact the authors.

Time Required The module takes one or two 50-minute class sessions. The specific amount will depend on how much time you give students to work in class and how much time you expect them to work out of class.

Implementation Recommendations It is important to frame the discussion before beginning. Start by introducing the Rohingya and the crisis. There are several news videos online that you could either show in class or ask students to watch before class. If feasible, you could invite a guest speaker to discuss the Rohingya crisis, or more generally about experience in a refugee camp.

Part of framing involves making sure students maintain respect and affirm the humanity and dignity of each refugee. If you have access to the SHOAH archive of video accounts by genocide survivors, you could ask students to watch one or more videos concerning conditions at a refugee camp and ask them to summarize what they learned. You could also play video testimonies of Rohingya survivors. Emphasize that the refugees come from diverse walks of life, from professionals to laborers. Finally, discuss with students what camp conditions are necessary to support the human dignity for each individual refugee.

You may need to come back to such discussions if you find the students are stereotyping the refugees.

Once you are ready to begin the module, couch it as an assignment from their supervisor at the U.N. They were called into their supervisor's office and asked to evaluate the conditions in the camps, including density in the shelters and access to bathroom facilities and water. Their staff collected the data in the spreadsheet. The goal of the task (the first portion of the handout) described what the supervisor asked for, and includes background information.

Typically students need one class session to complete the spreadsheet and another to write their analysis. While working with the spreadsheet, circulate and troubleshoot any issues. Most issues will concern the use of the spreadsheet. Encourage students to think about what they want the spreadsheet to calculate before typing anything, and even write some thoughts down on paper.

Note that the tasks in the guiding questions get repetitive. Encourage students to figure out what they need to do to fill out the spreadsheet without having to read all of the guiding questions.

This module was designed for students to apply what they already learned about proportional reasoning, specifically unit rates and percentages. It could be used as a vehicle to teach unit rates and percentages, or to introduce them. In such a case the instructor will need to consider places to pause for discussion about the underlying mathematics.

Finally, students may need help with the spreadsheet. If you haven't used Excel in class yet, be sure to help students understand how to enter and copy formulas in a spreadsheet as well as generate graphs. Show how graphs can be copied and pasted into text. This will enhance their recommendations. In addition, consider showing them how to highlight individual cells so that they can pick out individual camps with conditions that are concerning.

Cultural Contexts (if applicable) Since this module deals with potentially sensitive social issues, in addition to the discussion framing described above, it will be helpful to establish some norms ahead of time. Good norms include items like "listen to understand, not respond" and "let people complete their comments before speaking."

If you have students who have experienced life as a refugee, this module could trigger trauma. Consider an alternative assignment for those students if they do not think that they are emotionally prepared to process their trauma and work through this assignment.

Adaptations to Online Learning We found that the task as a whole was overwhelming for students in an online environment, so we split the task into portions to be completed weekly. The first portion involved filling in the columns involving unit rates, and the second involved filling in the columns involving percentages. The third and final portion involved assessing the conditions and writing the memo.

Genocide Refugee Camp Management

Module

The term **genocide** was established after the Holocaust (1933–1945). Examples of genocides occurred in Turkish Armenia (1915–1917), Cambodia (1975–1979), and Rwanda (1994).

You work for the United Nations High Commissioner on Human Rights. The commissioner is concerned about the treatment of the Rohingya. The Rohingya are a group of Muslims living in Myanmar (formerly Burma). They mostly live in Rakhine State, near the northwest border with Bangladesh. They have been subjected to treatment that human rights organizations such as Amnesty International [ref], believe qualifies as genocide since August 2017.

Most Rohingya are now refugees at camps in Cox Bazar, Bangladesh. Refugee camps are expensive and difficult to maintain, and the commissioner has put together a team to provide support to improve the conditions in the camps. While others are working on the budget, your task is to look at selected facilities to identify needs.

Before getting into your assignment from your supervisor, find both Rakhine State in Myanmar as well as Cox Bazar in Bangladesh on Google Maps to orient yourself!

The Goal of the Task Your task is to evaluate conditions in the refugee camps in Cox Bazar. Your staff has gathered data[2]. Analyze the data as described in the guiding questions. Create a case to improve conditions, either across all of the camps or in individual camps.

Once you complete the guiding questions, write a memo of one or two paragraphs to the commissioner making a case to improve conditions for the refugees. Illustrate the supporting data with graphs, and explain the significance of each recommendation in human terms.

Guiding Questions Open the spreadsheet called "Cox Bazar Data." This spreadsheet has the raw data for each camp in Cox Bazar. The file includes data on availability of latrines, washrooms, and improved water sources. A latrine only has a toilet, whereas a washroom includes a sink and may have a shower or bath. An improved water source is a constructed water source that adequately protects the water from outside contamination.

Leave all the results of your calculations in the spreadsheet and turn it in with your recommendations. You might find it helpful for your own thinking to highlight any cells with conditions you find unacceptable. Copy and paste any graphs you think are important into your recommendations.

1. Look at the data in Columns B and C.
   (a) In Column D, calculate the number of individuals per shelter in each camp.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column D, in cell D36, calculate the average number of individuals per shelter. This can be done by typing "=AVERAGE(D2:D34)" (which can be adapted for other columns).
   (d) Based on your calculations, what actions are necessary?
2. Look at the data in Columns B and E.
   (a) In Column F, calculate the number of individuals for each functional and safe latrine.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column F, in cell F36, calculate the average number of individuals for each functional and safe latrine.
   (d) Based on your calculations, what actions are necessary?
3. Look at the data in Columns C and G.
   (a) In Column H, calculate the percentage of shelters in each camp that are within 50 meters of a safe latrine block.

[2] Source: `https://data.humdata.org/dataset/cox-s-bazar-refugee-settlement-infrastructure`

   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column H, in cell H36, calculate the average percentage of shelters within 50 meters of a safe latrine block.
   (d) Based on your calculations, what actions are necessary?

4. Look at the data in Columns I and J.
   (a) In Column K, calculate the percentage of washrooms that are functional and safe.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column K, calculate the average percentage of washrooms that are functional and safe.
   (d) Based on your calculations, what actions are necessary?

5. Look at the data in Columns B and J.
   (a) In Column L, calculate the number of individuals for each functional and safe washroom.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column L, calculate the average number of individuals for each functional and safe washroom.
   (d) Based on your calculations, what actions are necessary?

6. Look at the data in Columns B and M.
   (a) In Column N, calculate the number of individuals for each functional improved water source.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column N, calculate the average number of individuals for each functional improved water source.
   (d) Based on your calculations, what actions are necessary?

7. Look at the data in Columns C and O.
   (a) In Column P, calculate the percentage of shelters that are within 200 meters of a functional improved water source.
   (b) Generate a bar graph of the data in this column. Does anything look concerning?
   (c) At the bottom of Column P, calculate the average percentage of shelters within 200 meters of a functional improved water source.
   (d) Based on your calculations, what actions are necessary?

Complete the Assignment Now that you have finished the guiding questions, complete the assignment as outlined in the "Goal of the Task": write a memo of one or two paragraphs to the commissioner making a case to improve conditions for the refugees. Illustrate the supporting data with graphs, and explain the significance of each recommendation in human terms.

Turn in your memo along with your completed spreadsheet.

3.4 Marshmallow Shooters

Jeneva Clark, Lynn Hodge, Jason Robinson,
and Caroline Maher-Boulis
University of Tennessee–Knoxville and Lee University
Contact: dr.jenevaclark@utk.edu

3.4.1 About the Module

- Course: Mathematical Reasoning
- Partner Disciplines: Teacher Education

3.4.2 Institutional and Course Contexts

- Type/size of institution: large public research university
- Size of Class: multiple sections with 35 students in each. Annual enrollment is approximately 1,000.
- Characteristics of Students: non-technical majors, many first-year students.
- Mathematical Content: After learning about several physics principles and equations, such as Newton's laws of motion, continuity, friction, and pressure, students make intentional modifications to a classic PVC marshmallow shooter design. The students are experiencing qualitative modeling, in which they look at equations and make predictions about how the quantities affect motion. For example, since the ammunition's mass is in the denominator of the acceleration expression, students can tell that a larger mass may decrease acceleration, even though students do not measure mass or acceleration.
- Purpose/Goal of the Module: The instructor's goal is to promote student thinking, while the students' goal is to make the ammunition fly further. Based on physics principles, students try to explain why the air, the marshmallow, and the PVC pieces interact in the way that they do.
- After and Before: This lesson can stand alone without a lot of prerequisite knowledge. For us, it serves as a capstone project for a unit about making decisions. Lessons about probability, which precede this one, may influence the students' reasoning somewhat, but for the most part, this lesson is independent from prior lessons in content. The prior lessons have emphasized using mathematical reasoning to construct viable arguments about decision-making. What normally follows this lesson is a unit in which students prepare and give presentations to the class about a mathematical topic of their choosing and how it relates to their chosen major field of study.
- Inspiration for the Module: Some disciplines (e.g., engineering, art, architecture) naturally lend themselves to a design cycle. Students brainstorm, create a prototype, run trials, analyze observations, revise their design, and try again. Math classes, where deductive reasoning dominates, are not traditionally venues for this creative process. We wanted to change that. With funding from the Department of Education, through the Tennessee Higher Education Commission, in the Improving Teacher Quality grant program, Lee University led two workshops for in-service K-12 teachers. In those workshops, hosted by Lee University, Drs. Clark, Maher-Boulis, and Robinson introduced marshmallow shooters to 60+ mathematics teachers of middle and high school. With funding from the National Science Foundation (DUE-1758325), through a Noyce grant at the University of Tennessee, Drs. Clark and Hodge developed this into a lesson that could be implemented by pre-service math and science teachers during summer camps for children and at a local STEM children's museum.

3.4.3 Partner Discipline Background

If implementing this with pre-service teachers, or in a math course that serves future K-12 teachers, then some idea of inquiry-based learning would be helpful. Instructors can encourage the students to think about this activity through the lens of an educator instead of only through the lens of a math student. Next Generation Science Standards or

Mathematics Common Core State Standards could simply be mentioned by the instructor. However, for students who are future teachers, it could be helpful to pose questions that provoke students to attend to these standards in more depth. The following are examples of such questions:

- Which Next Generation Science Standards (NGSS) or Mathematics Standards from the Common Core State Standards (CCSS) are most present in this activity? Which NGSS crosscutting concepts are present?
- Which Mathematical Practices or NGSS Practices are used? How so?
- To what extent does this activity involve inquiry-based learning? How so?

At the University of Tennessee, Mathematical Reasoning is not intended to be a course for physics majors; however, the lesson utilizes content knowledge from physics. Thus, some physics expertise is helpful for instructors. Topics include Newton's laws of motion, continuity, friction, and pressure.

While this course is also not intended to be a math course for engineering majors, this lesson does make use of the the engineering design process. This process may include steps like imagining a solution to a problem, building and testing a prototype, evaluating its performance, and making modifications. This is useful for future K-12 teachers.

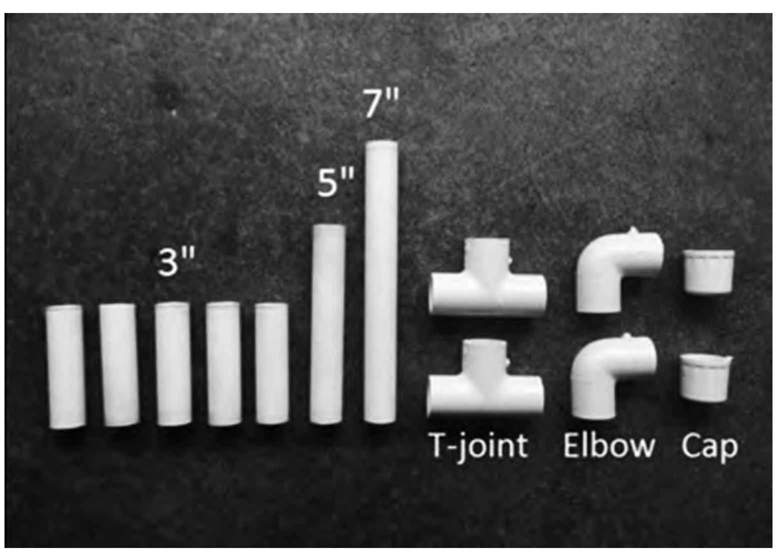
3.4.4 Implementation Plan

Formal Learning Objectives

- Students will use mathematical formulas from physics to describe effects on motion.
- Students will connect mathematics concepts to real-world contexts.

Materials and Supplementary Documents Each student or group of students will get the following pre-cut pieces of PVC and PVC connectors, all industry standard 1/2", in a quart-sized freezer bag.

- 2 caps
- 2 elbows
- 2 tees
- 1 PVC pipe of length 7 inches
- 1 PVC pipe of lenth 5 inches
- 5 PVC pipes of length 3 inches



Each student or group of students will get mini-marshmallows in a sandwich bag. Mini-marshmallows of various colors can be helpful if multiple students are shooting at the same time. One line of students could use green marshmallows, one pink, one yellow, and one white, and they can identify their ammunition easier, for measurement purposes. Alternatively, if each group is equipped with various colors of marshmallows, then students may run several trials at once and may use the colors to distinguish which trial was which.

During the follow-up class period, when trials are conducted, have a few tape measures that are at least 50 feet long. Trundle wheels could also work. Ideally each group could have their own tape measures to run trials outside of class sessions, but having a few to share between the class is sufficient. If tape measures are not available, find a courtyard or sidewalk that has equal units of measure marked on the pavement.

You can provide alternate ammunition for students to try, such as chapstick lids, bottle lids, pencil erasers, raisins, corks, cotton balls, different sizes of marshmallows, etc. You can also provide some alternate materials, such as duct tape, tissues, rubber bands, extra PVC joints and pieces, empty water bottles, modeling clay, tin foil, etc.

Time Required We devote one 50-minute class session to introducing the students' assignment. During a follow-up class session, students conduct trials, collect data, and draw conclusions. Because teams take turns conducting trials, we often have an additional review assignment that groups work on concurrently while taking turns with the tape measure.

Implementation Recommendations During the first class session, students are given supplies and are quickly taught a few physics principles that could influence their design decision.

Here are some precautions to consider:

- Inform campus security of the activity ahead of time. Tell them what to expect.
- Warn students to NOT inhale a marshmallow. Emphasize order of events as follows: (1) Inhale with mouth away from marshmallow shooter. (2) Put loaded marshmallow shooter up to their mouth. (3) Blow.
- Provide labels for shooters so students will not get theirs confused with another student's when lying them down.
- Have students avoid shooting marshmallows indoors unless there is an area where marshmallows will not get lost under furniture. A rogue marshmallow will attract ants.
- Encourage students to retrieve any non-biodegradable ammunition, so as not to litter.
- Find an area where few pedestrians pass, to avoid friendly fire.
- Choose a non-windy day for this activity, to avoid frustrations and chaotic data trends.

The above precautions could emerge from a discussion with students so that these guidelines are co-constructed community standards, enabling student ownership in the project.

Students' strategies vary widely. Often students are surprised by their results. For example, they might make predictions that a pencil eraser will fly much farther than a chapstick lid, but may be surprised to discover that the opposite is true for their particular design. Although a common question students ask is, "Why did this happen?," this question is one for students to try to answer themselves without much instructor intervention.

Cultural Contexts (if applicable) You may notice, from the images below, that the original prototype of the marshmallow shooters were designed to look like guns, assault rifles in particular. In order to de-emphasize any connection to violence, one could focus on how nature can inspire human design decisions. In particular, some plants disperse their seeds in explosive ways. Violet seeds and exploding cucumbers are a few examples. There are videos online that show exploding seeds in slow motion.

Adaptations to Online Learning We have not adapted this particular module to an online modality. We once did a similar activity online, however. Students were assigned to create a catapult at home from household items (rubber bands, plastic forks, popsicle sticks, etc.) and students were asked to upload a video of their catapult sending something flying.

3.4.5 Additional Information

Students appreciate getting to experience the design cycle of imagining, testing prototypes, implementing improvements, and retesting, as this type of experience is seldom experienced in math classes. Students appreciate doing mathematics in an atmosphere where there are no right or wrong answers. Instructors may add a competitive component, offering an incentive for the students that accomplish the longest shot.

In a physics class, one could adapt this module by considering the different forces that are acting on the marshmallow after it leaves the shooter, by calculating the speed at which the marshmallow leaves the shooter, and by predicting how far the marshmallow will fly. Students could mark their predicted landing distances with tape and could then test their hypotheses with trial runs. Teams could compete to see whose calculation was the closest.

This could also be modified for statistics classes by focusing on distance as a variable, by collecting data from multiple trials, and by describing the variable using statistics. In a statistics class, students could test the claim that a certain design causes the marshmallow to fly farther than the mean distance it flies with the traditional design.

In courses for preservice teachers, this lesson affords an opportunity to learn math and physics content and substantial pedagogy in the context of a design-based task. First, the design process, physical materials, and science content together provide a context for the math content to be meaningful. Math content serves a purpose and is a tool to understand the situation better. Second, the task illustrates one way to begin integration by offering connections between math, physics, and design thinking. Third, in terms of pedagogy, the lesson creates a space to explore a range of questioning from more closed questions about facts and computation to more open questions that inspire critical thinking.

One could allow preservice teachers to design a lesson plan that aligns with appropriate standards. Another productive activity is to engage preservice teachers in generating different kinds of questions to ask K–12 students during the lesson. Example questions might include: Where is the math in this lesson? How would you describe your design process? What is an effective shooter? What data can we or should we collect with the shooter? What did you notice? What do you have questions about? A related activity is to ask preservice teachers to put themselves in the shoes of K–12 students and to consider challenges they might experience as they engage in the lesson. This kind of consideration emphasizes the importance of designing instruction in light of students and the dynamic nature of teaching.

Marshmallow Shooters

Module with Solutions

1. Think about your current major or future career. Does *design* play any role? In some fields, like art or architecture, the *design* process is central to the discipline, and the outcomes are visible or tangible. In other fields, like political science, you might face the task of *designing* a new tax plan. Describe something below that you could be expected to *design* in your particular field of study.

   Solution: Responses may vary. This question works well as a starter activity to be posed at the beginning of the class session or to be completed prior to class.

2. The design cycle can be described by the following steps:

   - Determine a problem and its parameters.
   - Envision a potentially viable solution.
   - Build and test a prototype.
   - Evaluate its performance.
   - Revise the design and re-evaluate.

   Describe how this cycle may be applied to your future career or field of study.

   Solution: Responses may vary. This question works well as a starter activity to be posed at the beginning of the class session or to be completed prior to class.

3. Isaac Newton's three laws of motion can be summarized as follows:

   - An object at rest will remain at rest unless acted on by a force. An object in motion will remain in motion unless acted on by a force.
   - The acceleration of an object is directly proportional to the net force exerted on it and inversely proportional to its mass.
   - For every action, there is an equal and opposite reaction.

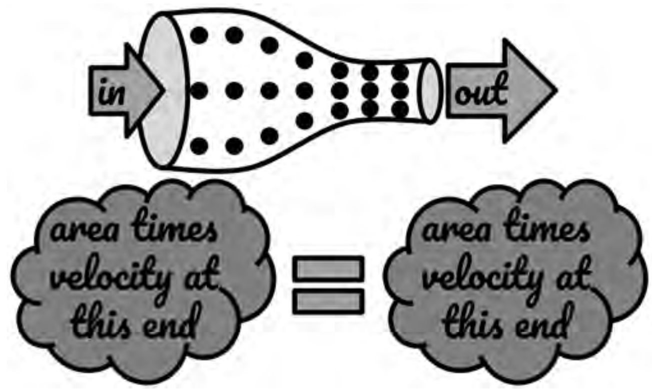
   Imagine you are designing a marshmallow shooter that will easily launch a marshmallow across a room if you blow into it. Choose one of the laws listed above and describe how it may influence your design choices.

   Solution: Responses may vary. A student may explain that because an object that is in motion tends to stay in motion unless acted upon by a force, they would choose to design a marshmallow shooter that has a long straight barrel with no bends and with few impediments in its path. A student could explain that because the marshmallow's acceleration should be directly proportional to the force exerted on it, then they would use a very strong puff of air to shoot the marshmallow. In the interest of classtime, this question could be addressed in whole-class discussion instead of a student-written assignment.

4. Think about what happens when you put your thumb part of the way over the end of a water hose. The water seems to come out of the hose faster. The principle of continuity that causes this phenomenon can be described using the following equation:

   $$A_1 v_1 = A_2 v_2$$

   where A_1 and A_2 are the areas of the hose's opening and v_1 and v_2 are the velocities of the water passing through. For simplicity, imagine a hose that were shaped like the following, with one end wider than the other.



The opening at the "in" end has a greater area than the opening at the "out" end. According to the formula $A_1 v_1 = A_2 v_2$, what must be true about the values of v_1 and v_2? How can you tell?

Solution: Responses will vary. Below are two sample responses.

Sample Response 1: Since the product on the left is equal to the product on the right, and since the area on the left is greater than the area on the right, then the velocity on the left must be less than the velocity on the right. Note that all values are positive in this context.

Sample Response 2: $A_1 v_1 = A_2 v_2$ but we know that $A_1 > A_2$, so $A_1 = A_2 + d$ for some positive difference d. $A_1 v_1 = A_2 v_2$ would become $(A_2 + d)v_1 = A_2 v_2$. We could divide both sides by A_2 and get $\frac{A_2+d}{A_2} \cdot v_1 = v_2$. Since $\frac{A_2+d}{A_2}$ is a fraction larger than one, v_1 must be less than v_2.

5. Examine the following equation that describes how friction forces can be calculated:

$$\text{force from friction} = (\text{some positive number}) \times (\text{the object's mass})$$

"Some positive number" in the above equation depends on what type of surfaces are touching each other and at what angles they are touching. From this equation, what can you tell about how mass influences friction? Can you name some forces of friction?

Sample Response: The greater the mass, the higher the force of friction acting on it. Sources of friction may include air and the walls of the pipes.

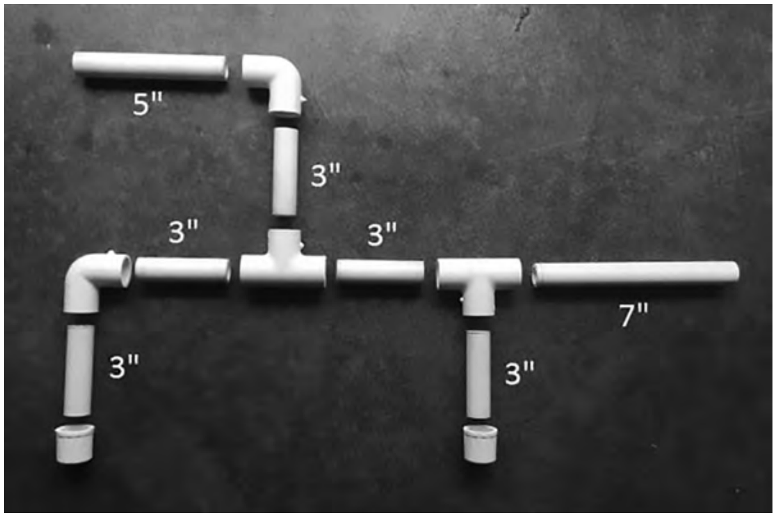
6. If you are to design a marshmallow shooter, air pressure may play a role in your design decisions. Pressure can be described using the following equation:

$$a = \frac{PA}{m}$$

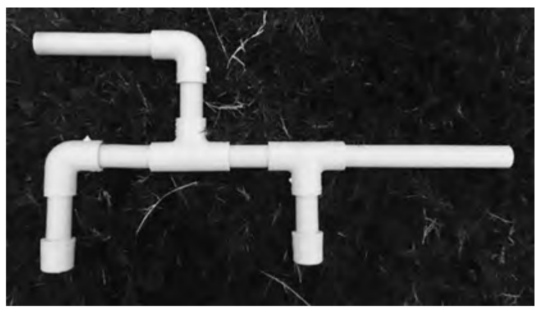
where a represents the acceleration, P represents the strength of the puff of air, A represents the area taken up by the ammunition, and m represents the mass of the ammunition. Notice that some values are in the numerator and some are in the denominator in this equation. What does this tell you about how you should design a marshmallow shooter that might maximize acceleration?

Sample response: The variables in the numerator are directly proportional to acceleration. The larger the puff of air, P, the greater acceleration, a. The larger the area A, the greater the acceleration. The variable in the denominator is inversely proportional to the acceleration. The higher the ammunition's mass, m, the lower the acceleration.

7. With the supplies in your bag, assemble this classic marshmallow shooter according to the following diagram.



After assembled, it should look like the following:



8. This classic design, with handles, was designed with aesthetics and comfort in mind. Our priorities are different however. Your job is to make modifications that improve the distance your ammunition will travel. You can modify the ammunition and try something other than a marshmallow. You can modify the shooter. You are allowed to add or remove material. The goal is to make modifications that will result in a longer distance than the original design.

Please show consideration of others as you run trials. Please do not inhale a marshmallow and choke. Please do not litter.

9. Brainstorm with your group. Sketch at least three hypothetical design strategies and discuss which ones you think will be most effective and why.

   Solution: Responses may vary.

10. Try out your design ideas and decide on your best contender. Describe what modifications you made to your design. Sketch a picture.

    Solution: Responses may vary.

11. Conduct three trials with your modified design and record the resulting lengths.

    Solution: Responses may vary.

12. What physics principles (e.g., Newton's laws, continuity, pressure, friction) had an influence on your design and its success or failure. Explain.

    Solution: Responses may vary.

3.5 Health Insurance in Imaginopolis

L. Jeneva Clark, Nicholas Nagle, Alex Bentley, and Vasileios Maroulas
University of Tennessee–Knoxville
Contact: dr.jenevaclark@utk.edu

3.5.1 About the Module

- Course: Mathematical Reasoning (Mathematics for Liberal Arts)
- Partner Disciplines: Anthropology and Geography

3.5.2 Institutional and Course Contexts

- Type/size of institution: large public research university
- Size of Class: multiple sections with 35 students in each. Annual enrollment is approximately 1,000.
- Characteristics of Students: non-technical majors, many first-year students.
- Mathematical Content: expected value, probability, Bayes' formula.
- Purpose/Goal of the Module: We aim to introduce the idea of expected value as a mathematical determination of what quantitative outcome should be anticipated on average, under typical circumstances. We anticipate that students will gain an understanding of how expected value influences decision-making.
- After and Before: This module falls in a unit called "Decision" and in a chapter called "Insurance." The two chapters before this one are "Probability" and "Percentages." The specific concepts that are assumed to be understood by students prior to this activity include the following:
  - $P(A)$ notation for the probability of an event A occurring.
  - The Law of Large Numbers
  - The probabilities of complementary events sum to one.
  - The product rule for finding the probability of independent events occurring.
  - The interpretation of percentages as probabilities.
  - The partitive model of division to determine equal amounts.

  After this module, the next two chapters are "Interest" and "Voting."
- Inspiration for the Module: In SUMMIT-P fishbowl-style discussions with partner discipline faculty from geography and anthropology, the concept of "expected value" was identified as a key concept for understanding the use of data science in their disciplines. A detailed description of this particular emergent process can be found in Short, Henson, & McConnell (2021). The ideas in this module had been used in prior years in this Mathematical Reasoning course; however, with SUMMIT-P-style collaboration, the lesson gained more depth in an ethical dimension and gained an extension to introduce the concept of Bayesian statistics. Parts of this lesson have been included in the textbook Clark & Clark (2021), which is the source for all images in this module.

3.5.3 Partner Discipline Background

The partner discipline faculty from geography and anthropology informed this activity by highlighting the need for making data seem more human. Data science, which is used in both of our partner disciplines, comes with ethical responsibilities. As we have discussed the best ways to make students think deeply about the ethics of using data, we arrived at the following principles that informed this activity.

- Author Patti Digh said, "The shortest distance between two people is a story." We believe that story-telling can be a powerful way to provoke meaningful thought, and is a profound communication style. In this lesson, we tell a story about Imaginopolis, an island with only five residents.

- We seek to "give data a face." Crunching numbers, without realizing what those numbers represent, can unintentionally lead to unethical data practices. There is not an easy recipe for practicing ethics, but a necessary precondition for ethical practice is a deep understanding of what numbers represent in the lives of humans and what mathematics may imply for human lives. This is why we develop characters within our story.

3.5.4 Implementation Plan

Formal Learning Objectives In this module, students will calculate expected values, will assess decision-making scenarios using principles of financial mathematics and laws of probability, and will reflect on ethical considerations related to mathematics-based decision-making.

Time Required One 50-minute class period is typical. Depending on whether the instructor wants to go into depth into Bayesian statistics, it could be expanded to two class sessions.

Implementation Recommendations We assign a short pre-reading for students to read before class. During the first 5 minutes of class, we ask all students to respond to the starter question. We then use slideshows to lead a whole-class discussion, which we call the "story," and this lasts approximately 20 minutes. During this discussion, we recommend asking students to explain the mathematics, to predict what they think will happen next in the story, and to make connections to real-life contexts. After that, we assign groups of students to work in groups on questions where they practice using their new ideas. In this module, the groupwork problems that are algebraic in nature can be completed during the class session, but the 'reflection questions,' which attend to the ethical considerations, are completed after class ends because they take more time for students to think about.

Cultural Contexts (if applicable) The Affordable Healthcare Act is relevant historical content to mention. Avoid referring to it as Obamacare, which is a nickname birthed during partisan political debate.

Consider asking students about any experiences they have living in non-U.S. nations. They often have information to share about how health insurance and healthcare works in other countries.

3.5.5 Additional Information

This module could be altered to include student role play. If the students could be partitioned into groups of six, then they could take on the roles of the six characters in the story: the insurer, William, Damien, Ace, Patrice, and Medaline. A bit more context could be developed that reveals details about the characters' lives, finances, health, happiness, etc. Rich details can more effectively remind students to consider data-driven decisions to have consequences on human lives.

An extension to this module which includes Bayes formula, can be found in Clark et al (2021). In this extension, we again invite students to Imaginopolis, whose population has now grown to 1,000 residents, and the infamous G.U.N.K. virus has infiltrated the island and has infected approximately 5% of the population. To help calculate the risk of each resident, you offer free G.U.N.K. screenings to everyone. You know the accuracy likelihoods for infected patients (90%) and non-infected patients (80%). A resident named Sadie tests positive and panics. What is the likelihood that Sadie actually has the G.U.N.K.?

References

Clark, L. J., Bentley, R. A., Nagle, N. & Maroulas, V. (2021). Expectations and disciplinary blends. In P. Short, H. Henson, & J. McConnell (Eds.), *Age of Inference: Cultivating a Scientific Mindset.* Information Age Publishing.

Clark, L. J., & Clark, J. M. (2021). *The beautility of math*. Great River Learning.

P. Short, H. Henson, & J. McConnell (Eds.), (2021). *Age of Inference: Cultivating a Scientific Mindset.* Information Age Publishing.

Expected Value in Imaginopolis

Module with Solutions

Pre-reading Material, from *The Beautility of Math* Plato, an accomplished math teacher and philosopher of Ancient Greece, believed that beauty is not simply pleasing poetry or painting, but instead, beauty leads humans to knowledge, to goodness, and to justice. If something nudges us to deeper deliberation, then that is beautiful. Sometimes mathematics, even basic calculations, can provoke us to consider serious issues with an enlightened perspective.

As we look closer at the mathematics of healthcare, consider what constructs are most valued in decision-making. Access to healthcare is a concern we all share. Understanding the dynamics of insurance, both quantifiable and ethical, helps us make shared decisions about shared risks and the policies that influence our shared well-being.

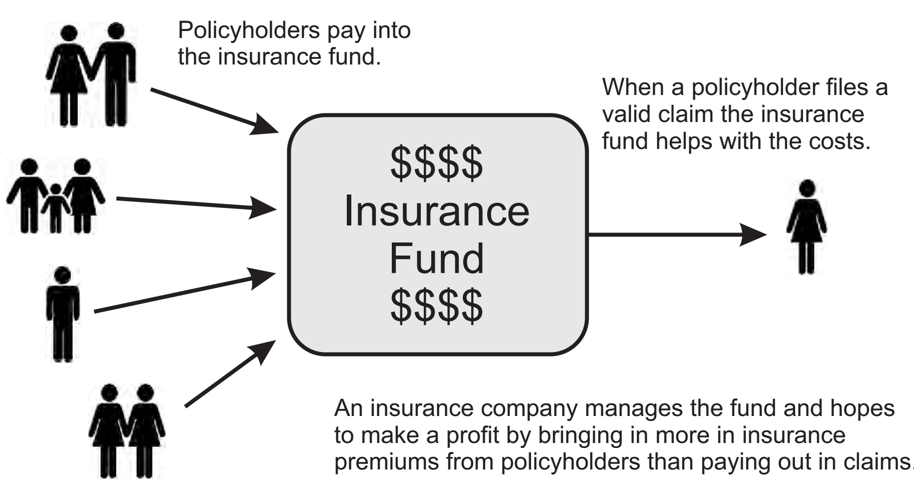
During many election cycles, healthcare is a topic that enters the candidates' debates. Many citizens simply listen for promises. However, more conscientious voters listen with scrutiny. For example, suppose a presidential candidate advocates for universal healthcare. Have they also thought through the web of consequences that such a change may weave? Using mathematics, we can run simulations of real-world scenarios in order to see what will happen in the face of complex changes.

At various points in history, lawmakers have made major reforms to healthcare. For example, the Affordable Care Act, signed into law in March 2010, extended health insurance coverage to uninsured Americans, prevented insurance companies from denying coverage due to preexisting conditions, required plans to cover certain essential health benefits, and implemented a tax penalty for those citizens who choose to go uninsured. There are advantages and disadvantages to each of the small changes. Consider what would have happened if this legislation had only made one change without the other changes.

Starter Question, from *The Beautility of Math* When we make decisions about healthcare, both as individuals and as communities, what constructs are most important to keep in mind when you consider the ethical implications of the choices? Justice? Finance? Fairness? Politics? Safety? Science? Economics? Happiness? Freedom? Explain why you made those choices.

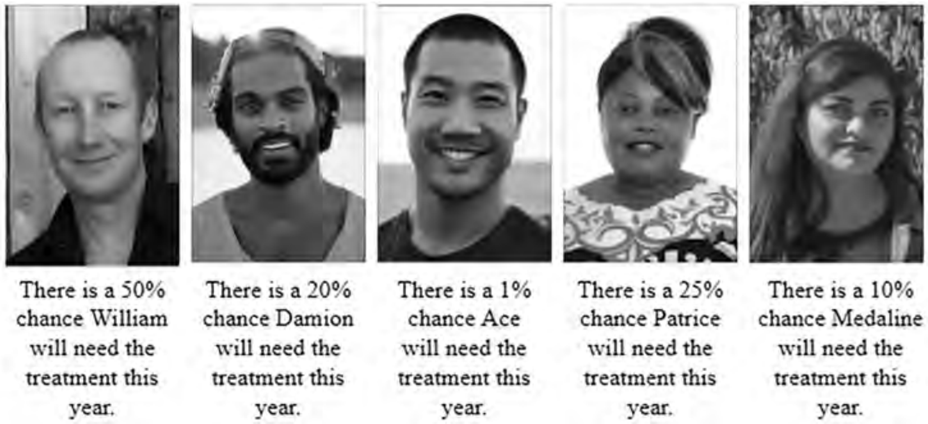
The Story The following image shows how shared risk of insurance plans work. Policyholders pay into an insurance fund. When a policyholder files a valid claim, the insurance fund helps with the costs. The insurance company manages the fund and hopes to make a profit by bringing in more in premiums than is spent in claims.

In this story, you will imagine being one of these stakeholders in one of these scenarios. There is an island called Imaginopolis, and there are exactly five inhabitants on this island. In this story, you are not one of them, but you are their insurance provider. Luckily you only have to worry about one type of illness in Imaginopolis. The only threat is the G.U.N.K. (Gross Unsanitary Nasty Killer) virus, which can be treated for $2,000. Each inhabitant has a



unique probability of contracting G.U.N.K. based on their risk factors. Luckily, as their insurance provider, you know precisely what those probabilities are.

There is a 50% chance William will need the treatment this year, 20% chance Damion will, a 1% chance Ace will, a 25% chance Patrice will, and a 10% chance Medaline will. In the images below, you can see each of your clients along with their individual risk.



William has a 50% chance of needing the $2,000 G.U.N.K. virus treatment this year. As William's insurer, you need to figure out what the average risk is to have William as a policyholder for a G.U.N.K. coverage policy. In order to do this, think about what an "average" or a "typical" William will cost you in a year. To conceptualize this, imagine for a moment that you had more than one of him.

Of course, we do not have six of him, and we cannot clone people, but let's pretend for a moment that had exactly six copies of William. Knowing that William has a 50% chance of contracting G.U.N.K. this year, how many of these six would you expect would need the treatment?

Since each of the six cloned Williams has a 50% chance, we would expect three of the six to catch it in a typical year. Granted, because this is based on chance, there could be two or four of the six that actually contract it instead. (In fact, there could be none, one, five or even all six that actually contract it!) However, the most likely case would be half of them contracting it, since each has a one half probability. Bernoulli's Law of Large Numbers helps us anticipate this as an average.

An insurance provider for the six Williams must have at least $6,000 in the fund because the three anticipated treatments would cost $2,000 each. Remember that insurance orchestrates a sharing of risk among all policyholders; thus, if the insurance provider wanted to collect premiums that equalled the needed $6,000, then they could charge each of the six Williams exactly $1,000. Therefore, the *expected value* for William's healthcare cost this year is $1,000.

Although we don't really have six of William on the island, the thought experiment of cloning him allowed us to imagine what a "typical" or "average" William would cost. $1,000 is the expected value for the amount William will cost the fund in future claims. It may sound odd to say that we expect William will cost $1000 in claims, when we know that there is no way William will cost $1,000. He will either cost $2,000, if he catches G.U.N.K., or he will cost $0, if he does not, each with 50% likelihood. However, the expected value is an average value after running many trials of an experiment. We cannot conduct many trials of William living out his year, unless we used some combination of Groundhog Day and Back to the Future phenomena. However, we can quantify what we would expect to happen if we could!

We can intuit the following formula for expected value from our calculation of William's expected value for the year.

$$\text{(probability)} \times \text{(value)} = \text{expected value}$$
$$(0.5) \times (\$2{,}000) = \$1{,}000$$

Applying this formula to all of our Imaginopolis residents, we can calculate the expected value for each one, as shown in the following image.

As their health insurance provider, you can expect to need $1,000 for the typical William, $400 for the typical Damien, $20 for the typical Ace, $500 for the typical Patrice, and $200 for the typical Medaline. Summing these, you can expect to need $1,000 + $400 + $20 + $500 + $200 = $2,120 in your fund. If you decide to charge everyone the same amount for their insurance premium, which is typical for health insurance, and if we ignore the desired profit margin for simplicity, then we need to calculate what an even and fair five-way split would be to meet this $2,120 needed total.

$$\frac{\$2{,}120}{5} = \$424$$

Dividing the needed sum of expected values by five gives you a policy price of $424 per person. If each of the five clients paid $424, then the shared fund would have $2,120, enough to fund the expected "typical" costs.

Ethical Consideration Health insurance policies premiums are usually the same for all policyholders in a certain group. Do you think this is fair? Why or why not?

Back to our story. You are in your insurance office, preparing to collect the $424 premiums, when you hear a knock at the door. It's Ace. He says, "I received the recent $424 bill, and I have decided that I don't want to buy the policy. While I can't be certain that I won't catch it, I know I am at a fairly low risk for the G.U.N.K. virus. So, I choose not to purchase insurance." Ace's intuition about his low risk is correct; your science and statistics had determined only a 1% probability of his catching G.U.N.K. this year. This plot twist will certainly change your numbers. You now need to recalculate to ensure you have enough in the fund.

After summing the four clients' expected values, you determine the fund will need $1,000 + $400 + $500 + $200 = $2,100 to cover expected costs, only $20 less than our previous need. There's a problem though. This risk needs to be shared among only four clients instead of five.

$$\frac{\$2{,}100}{4} = \$525$$

Four premiums of $525 would fund the account with $2,100, the level of expected values, while ignoring possibilities for charging extra for profit. However, that means you need to make four awkward phone calls about raising your rates.

Ethical Consideration While you have them on the phone, should you tell each client what probability they have of catching the G.U.N.K.? Do they have a right to know what your science predicts? Do they have a right to not know? How would you weigh well-being against democratic practices?

When you call Medaline, whose expected value was $200, she replies with, "I don't want to buy a policy for $525. I know my risk for the G.U.N.K. virus isn't too high. I could take the risk, and if I happen to get the G.U.N.K., I could just pay for the $2,000 treatment myself. Not everyone on the island has an extra $2,000 of disposable income, but I do, so I should take advantage of that. Please cancel my enrollment."

You didn't have to tell Medaline what her calculated risk was. Her intuition was correct. Since her expected value was only $200, the $525 premium is not worth it for her. Medaline is also right about not everyone being able to afford that risk. William's income, for example, is the lowest in Imaginopolis. You are starting to get concerned about how the cancellations of others are going to impact William's household budget. The power of insurance is in the sharing of risk. When some low-risk sharers opt out, the share of cost-per-person gets heavier for those who remain.

The sum of the three remaining expected values is $1,000 + $400 + $500 = $1,900. If that sum were evenly shared by three clients, each would pay approximately $633.33. No surprise that the rate increased again, to the detriment of William in his limited income.

Ethical Consideration Some individuals, like William, have both high health risks and low income. What is the right thing to do? Is there a situation where a person's financial means should influence their access to, or cost of, healthcare or health insurance? How do you weigh fair business practices differently from human well-being?

Spoiler alert: Imaginopolis is not real. In our real world, countries usually have more than only five citizens. Often, the low-risk citizens out-number the high-risk. In the following image are sixty citizens who are low-risk like Ace, twenty similar to Medaline, six like Damion, three like Patrice, and only one with high-risk like William. If all ninety of these citizens bought the G.U.N.K. insurance policy, then we could add up all of their expected values and divide the sum by ninety.

$$\frac{(60 \times \$20) + (20 \times \$200) + (6 \times \$400) + (3 \times \$500) + (1 \times \$1{,}000)}{90} \approx \$112.22$$

It turns out that the G.U.N.K. coverage could then be offered for approximately $112.22 each, which is a better price than the one for the five-client model. Although there is some safety in numbers, similar mathematical dynamics that we witnessed in the five-client model could happen here.

What would happen if all sixty citizens with the lowest risk opt out of coverage? Do you think the $112.22 per-person cost would increase, decrease, or stay the same?

$$\frac{(20 \times \$200) + (6 \times \$400) + (3 \times \$500) + (1 \times \$1{,}000)}{30} \approx \$296.67$$

If those with low risk chose not to purchase coverage, the price of the policy would go up from $112.22 to $296.67.

Let's assume, for simplicity, that having insurance meant guaranteed survival and that catching the G.U.N.K. when uninsured meant certain death. In this case, with all sixty low-risk customers opting out, what would be the likelihood that G.U.N.K. would claim a life? Either none of those sixty uninsured low-risk individuals will catch the G.U.N.K., or at least one of them will. These are complementary events. Thus, their likelihoods sum to one.

$$P(\text{None get G.U.N.K.}) + P(\text{At least one gets G.U.N.K.}) = 1$$

A rewrite of that equation would be the following:

$$P(\text{At least one gets G.U.N.K.}) = 1 - P(\text{None get G.U.N.K.})$$

The probability that none of these sixty low-risk citizens will contract G.U.N.K. can be found using the multiplication rule for probability. Each of the sixty represents the independent event of not catching the G.U.N.K. Since we knew that Ace's likelihood of catching the virus was only 1%, we know the probability that he will not catch it is 99%. Using the multiplication rule for all sixty people with the same risk as Ace, we can determine that the probability that none of these sixty will get the virus is $(.99)^{60}$.

$$P(\text{At least one gets G.U.N.K.}) = 1 - (.99)^{60}$$
$$P(\text{At least one gets G.U.N.K.}) \approx 1 - 0.547$$
$$P(\text{At least one gets G.U.N.K.}) \approx 45.3\%$$

There would be an approximate 45.3% chance that at least one of the 60 uninsured citizens would die from the G.U.N.K. This likelihood is important to consider when policies and laws are established for a group of people. The cost of the policies is not the only numeric influence on decision-making. Life, death, and the likelihoods of each are important ethical considerations for a community.

Ethical Consideration: Should insurance providers be able to deny high-risk customers?

In the same example of ninety clients of varying risks, there is only one William, who has the highest risk. What would happen if you deny coverage to William only? Do you think it would bring the price, which was \$122.22, up or down?

$$\frac{(60 \times \$20) + (20 \times \$200) + (6 \times \$400) + (3 \times \$500)}{89} \approx \$102.25$$

Denying coverage to William would bring the price down from \$112.22 to \$102.25.

However, price point is not the only consideration. What would be the likelihood that G.U.N.K. would claim a life in this scenario? If insurance inaccess would prevent William from treatment, and if G.U.N.K. were terminal when untreated, the likelihood of loss of life is simply 50%.

Reflection Questions These questions require more time for students to think and prepare responses.

1. As you can see, it's hard enough to be in charge of an insurance policy for a few fictional people. What if they were real? If you were in charge of other moving parts, like governing bodies or medical providers, what solution strategies would you champion? Could you come up with a plan that fixes everything? Which parts could you improve?

2. Choose one of the 'ethical considerations' in the narrative. Determine and explicitly define contrasting opinions on the topic. Organize a debate to be conducted in class. Each side should have two minutes to present their case, after which each side has two minutes to give rebuttals and summaries. Your classmates will determine a winner at its conclusion.

3. Compose a fictional conversation between the characters in the Imaginopolis story. In their dialogue, include discussion about the mathematics, ethics, and decision making of healthcare issues. Provide representation to a variety of opinions. Lead your classmates through a dramatization of your dialogue using roleplay.

Groupwork, "Investigations" from *The Beautility of Math* Four problems of similar difficulty are provided, as follows, so that different groups could work simultaneously on whiteboards during class.

1. Imagine you own a business that has exactly 160 employees. You want to set up a policy for your employees, for short-term disability coverage. If an employee has a valid claim, you would like this policy to pay them \$1,500. Here is how your employees are distributed:

	Group A	Group B
Under 30	30 employees	60 employees
30 and Over	50 employees	20 employees

For those in Group A under the age of 30, the probability is 5% they would file a claim. For Group A employees 30 and over, the probability is 3%. For those in Group B under 30, the probability is 2%. For Group B employees 30 and over, the probability is 4%.

How much money needs to be in this account? How much should each of the 160 employees pay each year to fund this account? Would you choose to make the policy a mandatory purchase for your employees? Would you retain the right to deny anyone coverage? Explain your decisions.

Solution:

$$\begin{aligned} 30(.05 \times \$1{,}500) + 60(.02 \times \$1{,}500) + 50(.03 \times \$1{,}500) + 20(.04 \times \$1{,}500) \\ &= 30(\$75) + 60(\$30) + 50(\$45) + 20(\$60) \\ &= \$2{,}250 + \$1{,}800 + \$2{,}250 + \$1{,}200 \\ &= \$7{,}500 \end{aligned}$$

\$7,500 needs to be in this account in order to cover expected expenses for all 160 employees each year.

$$\frac{\$7{,}500}{160} \approx \$46.88$$

If each of the 160 employees paid \$46.88 into the account each year, then it would sufficiently fund the expected costs.

Answers may vary concerning making the policy mandatory or retaining the right to deny coverage to individuals.

2. Imagine you are the health insurer for 190 people. You want to offer a policy to cover the treatment of a certain virus, which costs \$900. Although you have advised your customers to wear masks and practice social distancing, only 50 do both of these things, 80 practice social distancing only, 40 wear masks only, 20 do neither. Authorities report the following probabilities:

	Social Distancing	No Social Distancing
Mask	5% chance of getting it	10% chance of getting it
No Mask	15% chance of getting it	25% chance of getting it

How much money needs to be in this account? How much should each of the 190 customers pay to fund this account? Would you charge the same policy price to all customers? Would you deny coverage to anyone? Explain your decisions.

Solution:

$$\begin{aligned} 50(.05 \times \$900) + 80(.15 \times \$900) + 40(.10 \times \$900) + 20(.25 \times \$900) \\ &= 50(\$45) + 80(\$135) + 40(\$90) + 20(\$225) \\ &= \$2{,}250 + \$10{,}800 + \$3{,}600 + \$4{,}50 \\ &= \$21{,}150 \end{aligned}$$

\$21,150 needs to be in this account in order to cover expected expenses for all 190 customers each year.

$$\frac{\$21{,}150}{190} \approx \$111.32$$

If each of the 190 employees paid \$111.32 into the account each year, then it would sufficiently fund the expected costs.

Answers may vary concerning making the policy the same price for everyone or denying coverage to individuals.

3. You invented a new type of charger that charges phones with energy created by motion. You have designed one model that can attach to a person's shoe and one model that can be worn as a watch. You plan to make and sell 1,000 items for the year, 400 shoe models and 600 watch models. You want to set up a warranty fund to cover the cost of replacing items that break during their first year of ownership.

	Shoe Model	Watch Model
Cost to Replace	\$47	\$92
Probability it Will Break	30%	20%

How much money needs to be in this warranty fund? If you were to recuperate this cost by adding a certain fixed amount of money to the price of each item, how much would you increase the prices by? What information do you think you should provide to your customers about expected values, versus what information do you have a right to withhold from customers? Explain your stance.

Solution:

$$\begin{aligned}&400(.30 \times \$47) + 600(.20 \times \$92)\\&= 400(\$14.10) + 600(\$18.40)\\&= \$5{,}640 + \$11{,}040\\&= \$16{,}680\end{aligned}$$

\$16,680 needs to be in this account in order to cover expected expenses for all 1,000 products sold this year.

$$\frac{\$16{,}680}{1{,}000} \approx \$16.68$$

If you added \$16.68 to the price of each of the 1,000 items, it would recuperate the expected total cost.

Answers may vary concerning making information available to customers about expected values or withholding information from customers.

4. Noah is launching a new line of sunglasses, and he is interested in offering a one-year warranty, in which he replaces sunglasses that are returned with scratches or other damage. Noah plans to make and sell 2,000 pairs of sunglasses, 900 with plastic frames and 1,100 with metal frames. You want to set up a warranty fund to cover the cost of replacing items that break during their first year of ownership.

	Plastic Frames	Metal Frames
Cost to Replace	\$15	\$20
Probability it Will Break	20%	15%

How much money needs to be in this warranty fund? If you were to recuperate this cost by adding a certain fixed amount of money to the price of each item, how much would you increase the prices by? If you wanted to add a certain fixed amount to all plastic frames and a separate fixed amount to all metal frames, how much would you add to each? Would you rather provide the warranty to all customers or only to those who choose to purchase coverage? Explain your decision.

Solution:

$$\begin{aligned}&900(.20 \times \$15) + 1{,}100(.15 \times \$20)\\&= 900(\$3) + 1{,}100(\$3)\\&= \$2{,}700 + \$3{,}300\\&= \$6{,}000\end{aligned}$$

\$6,000 needs to be in this account in order to cover expected expenses for all 2,000 pairs of sunglasses sold this year.

$$\frac{\$6{,}000}{2{,}000} \approx \$3$$

If you added \$3 to the price of each of the 2,000 pairs of sunglasses, it would recuperate the expected total cost.

If you wanted to add a certain fixed amount to all plastic frames and a separate fixed amount to all metal frames, you would coincidentally add the same amount, \$3, to both types of sunglasses. The expected value for each plastic frame happens to be the same as the expected value for metal frames because $.20 \times \$15 = .15 \times \$20 = \$3$.

Answers may vary about providing the warranty to all customers or only to those who choose to purchase coverage.

3.6 A Monthly Budget for a Human Trafficking Shelter

Victor Piercey, Elizabeth Post, Mischelle Stone, and Rhonda Bishop
Ferris State University
Contact: pierceyv1@ferris.edu

3.6.1 About the Module

- Course: Quantitative Reasoning, College Algebra, or Precalculus
- Partner Disciplines: Social Work, Nursing, and Business
- Required Technology: Spreadsheets

3.6.2 Institutional and Course Contexts

- Type/size of institution: Regional comprehensive state university, primarily undergraduate, with about 13,000 students.
- Size of Class: 25
- Characteristics of Students: Students in business, nursing, and social work most of whom experience mathematics anxiety
- Mathematical Content: Linear functions.
- Purpose/Goal of the Module: Note the role of an assumption of linearity in budgeting, and the importance of budgeting when supporting victims of human trafficking.
- After and Before: This module comes at the end of a chapter on linear functions and before a chapter on exponential functions. Some use this activity in lieu of an exam.
- Other Prerequisites: Programming formulas in spreadsheets and using autosum. The ability to use absolute, relative, and mixed cell references would reduce the work students need to do, and reinforce the value of general models. An Excel Supplement is available on the volume's webpage. Instructors can provide direct instructions with any of these spreadsheet skills if they like.
- Inspiration for the Module: Human trafficking is a particular problem in Michigan, where Ferris State University is located. Faculty from social work and hospitality management have held conferences on campus about human trafficking, and it is a topic of concern to many students.

3.6.3 Partner Discipline Background

Human Trafficking is a form of modern day slavery and includes the buying and selling of people for profit. The International Labor Organization estimates that forced labor and human trafficking is a $150 billion industry worldwide, affecting an estimated 20.9 million people. The Federal Bureau of Investigation (FBI) believes that human trafficking is the third largest criminal activity in the world. Selling human beings is a lucrative business because unlike other products, humans can be sold again and again. This international crisis is one that many do not see because they do not know what to look for. It is also difficult to quantify due to under-reporting, as well as many variations in laws and prosecutions. There has been a recent paradigm shift in the United States from viewing a person as a prostitute to now understanding that these same people are victims and perpetrators of sex trafficking, respectively.

Human trafficking is a global epidemic that often goes unseen. It does not just happen to people from other countries or in the movies. It is not just a law enforcement issue or a social work issue. We may all interface with someone who is a trafficker or a victim—in hospitals, in restaurants, hotels, rest stops, sporting events etc. We all must be educated so we know what to look for and how to intervene. It may one day save someone's life.

3.6.4 Implementation Plan

Formal Learning Objectives Prepare and justify a monthly budget for a shelter under the assumption that the number of guests grows at a constant monthly rate.

Materials and Supplementary Documents Students will need the handout, access to a spreadsheet program along with the spreadsheet on the volume's webpage. The Excel Supplement at the same website may also be helpful, especially if you want students to use mixed cell references.

Time Required This module typically takes two 50-minute class periods: one to complete the spreadsheet, and another to write up the recommendation.

Implementation Recommendations The preparation questions are to be completed before the module is initiated in class. Responses can either be handwritten, typed, or submitted through a learning management system.

When we implement the module in class, we typically allow students to work on their own or in teams in class and circulate to answer questions. At Ferris, most students have their own laptops that they bring to class. The spreadsheets are available on our learning management system, and their spreadsheets and written justification are submitted through the learning management system. As students get into the assignment, it is fairly common for a few students to huddle together with or without the instructor to work out some of the mathematical modeling.

If your course does not typically include Excel, the activity could be modified so that students are tasked with finding the slope and intercept for each line item (different students or groups could be given different line items). From there, students could sketch the graph of each line item over time, as well as add the parameters to sketch the total. Additionally, an instructor could input the functions into Excel and share the file, and an instructor could total up each row to get an annual budget.

Alternative Solutions There are a couple of ways one could model each of the line items. One could either have Excel calculate the number of guests each month first, and then calculate the line items, or one could directly include the rate of change and the initial number of guests in the line item calculation. The former is actually simpler, and the spreadsheet could be modified to encourage this by adding a separate row for the number of guests each month.

Common Errors and Questions Students often have trouble noting that the line items depend on the number of guests and the number of guest is the quantity that grows linearly. They tend to leave out one or more factors in their model. But they are also quick to note that the results they get are unreasonable and then ask for help. In this case, they can be guided by working through one or two examples with "hand calculations."

Students are also hesitant to use mixed cell references, even when they have been taught. We don't force them, but once they are done we will show them how this kind of programming could have saved them some labor.

Cultural Contexts (if applicable) If you have students who have experienced human trafficking, this module may be difficult for them. Provide a trigger warning, and be prepared to have an alternative assignment if this is a problem for any of your students.

Adaptations to Online Learning We found that the task as a whole was overwhelming for students in an online environment, so we split the task into portions to be completed weekly. We took advantage of this adaptation to open the task up and provide more room for student thinking.

For the first portion, we gave students the previous year's guest data by month and asked them to come up with their own assumptions for how to anticipate guest numbers for the coming year. At this point in the course, students haven't been exposed to the assumptions behind any of the nonlinear families of functions, so they all estimated a slope and an intercept (the latter coming from December's numbers). They discussed this in breakout rooms and defended their decisions. Some students chose to average the changes between consecutive months, others looked to either the maximum or minimum change. This led to a discussion about the risks inherent in different choices for assumptions.

For the second portion, using their assumptions, students completed the budget sheet, and wrote their budget justification in the third portion.

One other way we took advantage of the online environment was by inviting the chair of the local human trafficking task force to come to speak with the students before they worked on any portion of the module. This provided a nice way to frame the work we did.

Those interested in implementing a similar approach can contact the authors for spreadsheets and handouts.

3.6.5 Additional Information

Students are impressed that something that to them appears abstract, such as linear functions (or $y = mx + b$), can be used to support victims of human trafficking. In an end of the semester reflection, one student noted that they learned how to "use math to help our neighbors when they are in trouble."

This module could be adapted to address any kind of shelter, from domestic violence to homeless. It could be followed-up with a visit to a local shelter, a conversation with the treasurer at the shelter, and even some volunteer time.

A Monthly Budget for a Human Trafficking Shelter

Module with Solutions

The United Nations Office on Drugs and Crime has put together a task force to deal with human trafficking. According to the United Nations, **human trafficking** is defined as the recruitment, transportation, transfer, harboring, or receipt of persons by improper means (such as force, abduction, fraud, or coercion) for an improper purpose including forced labor or sexual exploitation.[3]

We have joined the task force as a team representing North America, Central America, and the Caribbean. As a task force, we are interested in identifying the extent of the human trafficking problem and evaluating the costs involved in various preventative measures.

Preparation Assignment: Watch the following video about a shelter for sex trafficking victims in St. Paul, MN: `http://bit.ly/TraffickingShelter`.

List at least five expenses you expect a shelter for trafficking victims to incur.

Solution: Several are possible: food, personal supplies, employee wages, security, rent, clothes, medicines, transportation.

The Goal of the Task: The UNODC has asked us to prepare the 2020 budget for a shelter in San Bernadino, CA for human trafficking victims. We need to prepare budgets for each month of the upcoming year. As of December 2019, there are 23 guests. We assume the number of guests will grow by 3 each month.

Your supervisors are interested in how the total amount of the budget changes from one month to another. Is it steady growth or not? Why? How much more do we spend each additional month? What assumption(s) is/are the budget based on, and why is that important? This information will be helpful in submitting grant applications.

Prepare your budget by completing the spreadsheet in "Human Trafficking Data." Submit your spreadsheet together with a 1-to-2 paragraph budget analysis based on the questions in the previous paragraph. Be sure to use at least two different representations of the total budget in your analysis—think about your (very busy) audience!

In addition, do not use "math terms" like "slope" or "intercept" or "linear." Use practical interpretations. You CAN include an equation for the total costs, but use variables whose meanings are clear.

Guiding Questions: The spreadsheet "Human Trafficking Data" contains the budget information collected by your staff.

1. The spreadsheet contains the monthly base cost and monthly cost per guest for each line item in our budget. Row 1 numbers each month, which should make the computations easier.

   Program the spreadsheet to calculate the total cost for each line item in each month as well as the total annual amount for the line item and the total monthly costs. Don't forget about the current 23 guests!

   Note that the most efficient way to manage this is to use a **mixed cell reference**. This will allow you to use only one equation, then drag-and-drop!

   To use mixed cell references, recall that in the absolute cell reference, $B$3 fixes both the row and column. The first $ fixes the column, so $B3 will keep B fixed when you copy and paste downward, while the second $ fixes the row, so D$1 keeps the 1 fixed when you copy and paste sideways.

   Let's start in Row 3.

   (a) How would you figure out the cost in December 2019, when there are 23 guests? What would you tell Excel to do?

   Solution: Multiply the per-guest cost by 23, then add the base cost.

[3] Protocol to Prevent, Suppress and Punish Trafficking in Persons, Article 3(a). Dec. 25, 2003. Available at `https://www.osce.org/odihr/19223?download=true` (downloaded by Author on Sept. 22, 2017).

(b) How would you figure out how much to add to the December 2019 costs for the next month, when there are 3 new guests? What would you tell Excel to do?

Solution: Multiply per-guest cost by 3

(c) Adjust your Excel commends (using mixed references) so you can drag both downward and sideways to complete the budget.

Solution: Contact author for completed spreadsheet.

(d) Don't forget the totals!

Solution: Contact author for completed spreadsheet.

2. Analyze your results as follows:

(a) Have Excel generate a graph of the total by month. What do you notice?

Sample Respnse: The graph is a line, the costs are a linear function of time.

(b) Find the change in total costs every month. Is that constant? What does that tell you? Can you write an equation for total costs?

Solution: Yes, the monthly change is constant. The change is 6645 per month. Since December (time 0) costs are 64,095, the total cost function is: $C = 60,095 + 6645t$, where C is total cost by month t.

3. What are your estimated total costs for 2020 as of Tax Day (April 15, 2020)?

Solution: Add up the January, February, March, and half of April total costs. Result is 277,492.50.

4. Suppose your boss at UNODC tells you that you have 1.75 million available for the shelter in 2021. How many guests can you afford to add in 2021?

Solution: Our total 2020 costs are 1,287,450. Our additional monthly cost per guest is 2215 (sum Column C in the spreadsheet), and adds up to 26,580 for the year. If we bring in x guests in 2021, then we solve:

$$1,287,450 + 26,580x = 1,750,000$$

Solving gives us approximately 17 more guests for the year (rounding down to budget conservatively). Note that this can be solved intuitively and without algebra by figuring that we are given 462,550 over and above what we had for 2020, and dividing by the total annual cost per guest. Show students the connection between this reasoning and the algebra—it builds meaning to each step.

5. Think about another scenario in which victims are housed and cared for: refugee camps. Refugee camps need security. Since security will surround the camp, the number of guards (and hence the cost) will depend on the perimeter of the space.

Let's say a refugee camp adds a constant amount of square footage to their space every month to accommodate new refugees. Will the cost for security be linear or nonlinear? Why or why not? If not, what type of function would the security cost be? (You may assume that the configuration of the space is a square).

Suggested Response: This cost will be modeled by a nonlinear function, more specifically a square root function. We will need to figure the perimeter of the camp each month, and under the assumptions, that will be proportional to the square root of the area each month. The area is a linear function of the number of guests.

Complete the Assignment: Write a 1-to-2 paragraph budget analysis based on the questions above paragraph. Be sure to use at least two different representations of the total budget in your analysis—think about your (very busy) audience!

In addition, do not use "math terms" like "slope" or "intercept" or "linear." Use practical interpretations. You CAN include an equation for the total costs, but use variables whose meanings are clear.

Submit your analysis together with your completed spreadsheet.

3.7 Linear Functions and Demand in Economics

Glenn Henshaw, Tao Chen, Soloman Kone, and Choon Shan Lai
LaGuardia Community College
Contact: ghenshaw@lagcc.cuny.edu

3.7.1 About the Module

- Course: College Algebra
- Partner Disciplines: Microeconomics

3.7.2 Institutional and Course Contexts

- Type/size of institution: LaGuardia Community College is an urban community college with over $33,000$ students from more than 150 countries. Our students are 48% Hispanic, 23% Asian, 17% Black, and 12% White (2% other); 58% female and 42% male. Over 70% of our students come from families making less than $25,000 per year. We have 392 Full-Time faculty and 635 Part-Time faculty.
- Size of Class: 20
- Characteristics of Students: Students tend to be non-traditional and have a variety of majors. Most of them are mathematically under-prepared and experience mathematics anxiety.
- Mathematical Content: The module leads students to gain intuition about the demand curve using their understanding of linear functions.
- Purpose/Goal of the Module: Our goal is to re-enforce students' understanding of linear functions by tying it to the concept of demand in economics. Economics topics tend to be more motivating for students than stand-alone math topics because they are rooted in real-life applications. It's easier to generate meaningful class discussions when there is an economics topic from which to draw analogies.
- After and Before: This module comes at the end of a chapter on linear functions and before a chapter on quadratic functions. This assignment aligns with the assessment of LaGuardia's core competency: Inquiry and problem solving.
- Other Prerequisites: The student must be able to plot points on a coordinate plane.

  Note that it's not necessary for students to have a specific model before participating in this assignment but people naturally have an intuition for demand. They can leverage this intuition to help them explore the concepts in the first section. The second section helps them develop a linear model of the demand curve. We hope that the work they did in the first section will help them.
- Inspiration for the Module: During our first meeting with our partner discipline members in the Economics and Social Science department we were struck by the anti-intuitive nature of some economics notation. In particular, in economics, price is thought of as a dependent variable–dependent on quantity demanded. This was counterintuitive for the math faculty but made quite a lot of sense to the economics faculty. While we sorted through this issue, we also discovered that many incoming economics students had trouble with linear functions. A project of this sort matched the needs of economics faculty.

  Our original assignment idea was "Supply and Demand" but we felt that the complex relationship between the two economics topics obscured the math topic, linear functions. We decided to handle the two topics separately. First a concise assignment on demand highlighting linear functions. Then, in a future assignment, we introduce supply and demand with linear systems to describe the equilibrium point.

  We also added a group work modality. The goal is for student groups to spend as much time working and thinking as possible and less time listening to lecture. We were inspired by the Carnegie Pathways platform and their use of various learning modalities to increase student motivation. We tried to follow their lead with respect to carefully scaffolding the assignment in order to facilitate group work.

3.7.3 Partner Discipline Background

Demand in economics refers to the amount of goods and services buyers are willing and able to purchase at each price. Demand is essentially determined by needs, wants, and ability to pay. The quantity of a good or service that consumers would purchase at a particular price is called the quantity demanded. The law of demand says that, all other things held constant, as price increases the quantity demanded decreases and vice versa.

The demand curve is the collection of points with x-values as quantity demanded and y-values as price.

Note: Quantity demanded is specific to one price and is illustrated by one point on the demand curve. On the other hand, demand refers to the relationship between each price and the corresponding quantity demanded. Demand is represented by the whole demand curve.

In this project we consider only linear demand curves. This is an idealization.

3.7.4 Implementation Plan

Formal Learning Objectives

1. Understand the concept of quantity demanded.
2. Review how to graph a linear function by plotting points.
3. Understand the distinction between quantity demanded and the demand curve as as whole.
4. Gain intuition about the slope and y-intercept by exploring the example of the demand curve.
5. Write about mathematics in context.

Time Required We estimate the assignment at about one hour of class time as well as a few hours of student time to complete the writing assignment at the end.

Implementation Recommendations While the module can be completed by students individually, we believe that the multiple choice sections of the assignment are well suited for a collaborative learning style. These sections are meant to stimulate discussion and help students realize that the idea of demand is something they can reason about. In other words we want them to see that they can lead the way in exploring these ideas. Sometimes it's hard to get individual students to get into that mode. But if they see a lively, productive discussion between their fellow students they are likely to join in.

While recommended solutions are provided in the module, several questions (especially the multiple choice questions) could have multiple answers. We also hoping that this ambiguity will lead to a deeper, more interesting discussion within groups and later when instructors lead the class discussion. We hope that students will get a sense of the fact that demand is something real, subject to many economic and social conditions. We hope that students will spend more time thinking critically and creatively if the answers are not so straightforward. We want to foster a more exploratory learning process for the students. Instructors should emphasize this during the class discussion.

In the second section of the module, we chose to represent the demand curve symbolically because it connects to students' prior knowledge about linear functions. We hope that connecting these two topics will help students remember and value the learning experience as a whole.

This module was implemented during a time of distance learning. We used the breakout room function in Zoom to facilitate the group work sections. We suspect that this assignment would still be valuable in a traditional class discussion format, even without group work.

Introduction (5 minutes) The assignment begins with a short class discussion introducing students to the concept of quantity demanded.

Group work for section one and share (20 minutes) Students are divided into groups (max members is 5) to do the first section of questions, "What is Quantity Demanded?" These questions are multiple choice. The purpose is to help students gain intuition about the concept of price and quantity demanded and stimulate group discussion.

Group work for section two and share (25 minutes) Students do group work for section two, "The Demand Curve." Each group reports on their answers. Question four in this section can be challenging for students.

Writing assignment The students are asked to complete a writing assignment outside of class. The assignment contains prompts that are related to the group work questions.

Alternative Solutions If there is limited time for class discussion, the assignment is scaffolded enough so that we think most students should be able to go through the assignment on their own.

Common Errors and Questions Students have a lot of trouble distinguishing between the concepts of quantity demanded and the demand curve. Remind them of the difference between a point on the graph of a function and the function as an entity.

Cultural Context (if applicable) When discussing prices and products an instructor must be aware that students come from different socioeconomic backgrounds. Be aware that things that sound low-cost to one person may be prohibitively expensive to another.

Adaptations to Online Learning If the project is done during a period of distance learning then students will need computers. The instructor can use the Zoom break-out rooms feature to facilitate group work.

3.7.5 Additional Information

This module gives students the chance see how they can take an abstract concept like linear equations, add some economic context, and generate a deep class discussion.

This module prepares them for a further activity on “Supply and Demand” which is associated with the college algebra topic, linear systems.

Most students respond well to the group work modality. It gives them a chance to lead the discussion and creates memorable learning experiences. Some students need more encouragement to participate in their group’s discussions. Students who tend to be quieter could be given leadership roles in the group. For example, they could be asked to be the one who “report out” their group’s ideas to the rest of the class.

Students who aren’t used to writing assignments in math class will need extra guidance on the essay. But written communication is an important and neglected way for students to organize their thoughts and solidify their understanding of a math topic.

Linear Functions and Demand in Economics

Module with Solutions

What is Quantity Demanded?

Suppose you are trying to sell 100 tickets to a student production of the play Romeo and Juliet for \$5 each. The **quantity demanded** is the amount of tickets people are able and willing to buy at the given price. No matter what the ticket price is, some people like Shakespeare a lot and are **willing** to pay the price. Others wouldn't want to sit through three hours of 16th century English even if the tickets were free. Aside from personal taste, some people can afford the tickets and some people are not **able** to afford the tickets. Both of these factors, personal taste and financial situations, affect your ticket sales. In order to get a feel for the concept of quantity demanded make your best guess for the following questions.

1. Suppose that on Tuesday afternoon, in the library, you were able to sell 15 tickets, each for \$5. How many tickets would you guess you might have sold if you charged \$3 each?

   (a) 12 tickets
   (b) 15 tickets
   (c) 0 tickets
   (d) 25 tickets

   Solution: Choice (d): At a lower price, you would sell more.

2. On an average weekday there are approximately 300 students in the library. On weekends there are only about 100 students. How many \$5 tickets do you think you would have sold at the library on the weekend?

   (a) 5 tickets
   (b) 15 tickets
   (c) 17 tickets
   (d) 0 tickets

   Solution: Choice (a): With fewer potential customers, you would sell fewer tickets.

3. To fill empty seats, the theater department decides to sell tickets for \$3 on the day of the play. How many \$5 tickets do you think you would have sold on Tuesday if the costumers knew about this option?

   (a) 10 tickets
   (b) 20 tickets
   (c) 5 tickets
   (d) 0 tickets

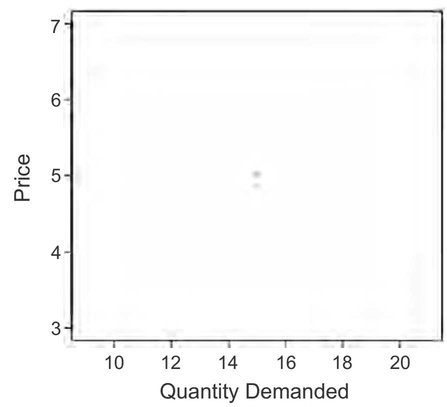
   Solution: Choice (d): If customers knew about this option and they knew there would be plenty of tickets available on the day of the play, they will wait to make their purchase.

4. The **law of demand** says that if you raise the price of a product then the quantity demanded will decrease. Can you think of a product that does not follow the law of demand?

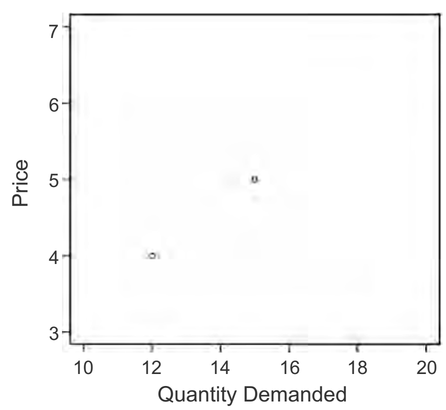
   Possible Response: Highly fashionable clothes, medical treatments, medications, baby formula, etc.

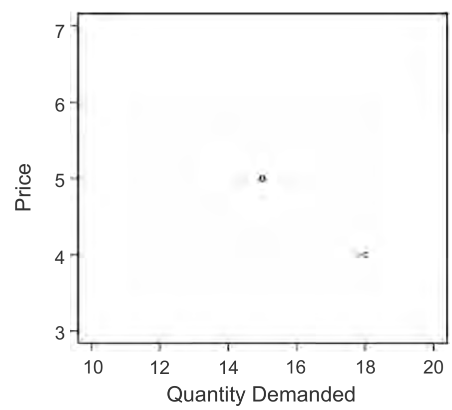
The Demand Curve

On Tuesday you sold 15 tickets for $5 each. So, for the price of $5, the quantity demanded is 15. Let's make a graph of the quantity demanded verses the price containing this single point. Notice the quantity demanded is on what is on the horizontal axis (usually called the x-axis) and price is on the vertical axis (usually called the y-axis). We use "natural lettering" to label the axes. Q is for quantity demanded and P for price.



1. Let's add another point to the graph corresponding to quantity demanded at the price $4. Which of the two graphs makes more sense? Why?

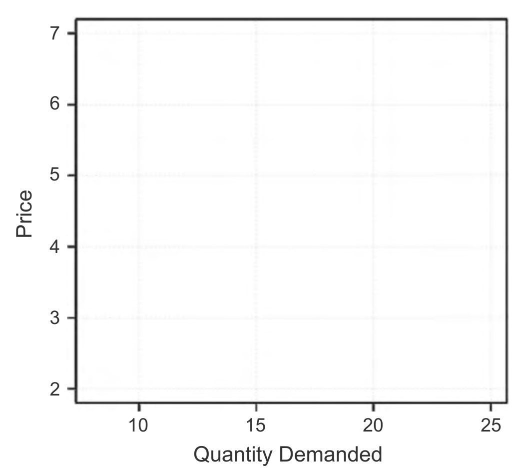




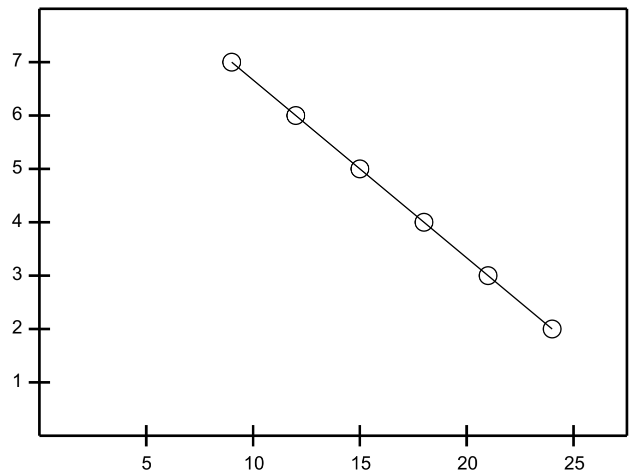
Solution: The graph on the right. At a lower price, you would sell more.

2. Graph the following points on the grid provided to show the quantity demanded (Q) for various prices (P).

Q	P($)
24	2
21	3
18	4
15	5
12	6
9	7



Solution: The graph (with a the demand curve filled in) is:



3. Notice the points happen to form a line. This line is called the **demand curve**. Is the slope of this demand curve positive or negative?

   Solution: The slope is negative. At a lower price, you would sell more.

   Note that the way we calculate slope as change in the dependent variable divided by the change in the independent variable would result in an interpretation more along the lines of "in order to sell more, lower the price" - which does not have as much universal meeting as the first interpretation. The difference comes from economists' convention of putting currency on the vertical axis

4. From the pattern, can you predict: 1) the price when the quantity of demand is 6; and 2) the quantity of demand when the price is 9?

   Solution: Based on this demand curve, assuming the pattern continues outside of the data, the quantity is 6 when the price is \$8, and the quantity is 3 when the price is \$9. As the price goes up by \$1, the quantity demanded goes down by 3.

5. Find an equation for the line that forms the demand curve. The variables should be Q and P instead of the usual x and y.

   Solution: The slope is -1/3, and following the pattern, the vertical intercept is at (0,10). The equation is:

$$P = -\frac{1}{3}Q + 10$$

6. Would you be surprised to find a demand curve with a positive slope? Why or why not?

   Solution: Yes, I would be surprised to see more people willing to purchase a product for a higher price than a lower price, unless it was a product like those we listed in #4.

In problem 5 we found the equation for the demand curve. It's important to remember that the demand curve and quantity demanded are two different things. The demand curve shows the relationship between the quantity demanded and price for many different prices. Whereas quantity demanded is for a particular price. When we change the price of a product we move along the demand curve but we don't change the demand curve. However there are many market factors that can have an effect on the demand curve.

For example, if there is an increase in population, the quantity demanded at each price increases, for most products. This would mean that the whole demand curve is shifted to the right. On the other hand, what if a new, popular competing product enters the market? In this case the quantity demanded at each price will decrease. This would shift the whole demand curve to the left.

For the following two questions, suppose Netflix increases their library of movies to include all of the Marvel movies. Currently, most of them are on Disney+.

7. Would that increase or decrease the quantity of Netflix subscriptions demanded at a particular price (quantity demanded)?

   Solution: The quantity demanded would increase.

8. How would that affect the demand curve? Would it shift to the left or the right?

   Solution: The demand curve would shift to the right.

Essay—minimum 3 paragraphs and 500 words

The United States consumes about 500 million gallons of gasoline per day. In this essay you will explore a demand curve for gasoline. The following table shows the quantity demanded (in millions of gallons) compared with the price (in US dollars).

Q	**P($)**
630	$2.00
460	$2.50
290	$3.00

Start your essay with a paragraph explaining the concept of quantity demanded and the demand curve. You should give an example using the numbers from the table above and include a sketch of the demand curve (Assume the curve is linear).

The second paragraph should discuss the law of demand. What is the law of demand? How does the law of demand relate to the slope of a demand curve? Assuming the demand curve for gas is linear, how high would gas prices have to be so that 0 gallons are sold (this is the y-intercept of the demand curve)?

Lastly write about factors that affect demand for gasoline.

- Suppose the number of electric car owners increases. What effect would that have on the quantity demanded for a specific price? Why?
- If there were an increase in population would the demand curve shift to the left or the right?
- What if there were greater access to public transportation? Would the demand curve shift to the left or the right?
- What other market forces might effect the demand curve for gasoline?

Solution: Answers will vary. Contact the authors for samples.

3.8 Medicine Dosage—Applying Dimensional Analysis/Graphing to Healthcare

John Hearn and Caroline Maher-Boulis
Lee University
Contact: cmaherboulis@leeuniversity.edu

3.8.1 About the Module

- Course: Algebra for Calculus
- Partner Disciplines: Biology, Chemistry and Health Science
- Required Technology: calculator/spreadsheet

3.8.2 Institutional and Course Contexts

- Type/size of institution: Liberal Arts University
- Size of Class: 30 or fewer
- Characteristics of Students: Undergraduate students who are seeking a degree which requires higher mathematics courses, such as Calculus I or trigonometry-based physics.
- Mathematical Content: Dimensional analysis, algebraic relations, graphs and their interpretation.
- Purpose/Goal of the Module: This module is designed to show students the application of algebra concepts and skills to real problems encountered in healthcare.
- After and Before: Students need to know how to set up proportions, how to plot points and how to graph.
- Inspiration for the Module: Faculty want to provide continuity between prerequisite courses and more advanced courses.

3.8.3 Partner Discipline Background

Prescribing medicine to patients must be done at the proper dose to maximize benefit, minimize side effects, and in some cases, avoid serious problems such as kidney failure. In extreme cases, overdoses can cause death. Medical dosages are based on body mass; for example, a dose of 1 mg/kg means that 1 mg of medicine is given per 1 kg of body mass. Medical practitioners, therefore, need to calculate a proper dosage for each patient based on body mass. Such a routine application of algebra suggests that everyone working in a medical profession needs to have mastery over these concepts.

One straightforward way of solving dosage problems is dimensional analysis. In dimensional analysis you find a way to combine the inputs (those quantities given in the problem) to give the correct units for the output (the unknown quantity). Units multiply and divide just like numbers. For example, $\frac{\mathrm{kg}}{\mathrm{kg}} = 1$, just like $\frac{3}{3} = 1$ or $\frac{x}{x} = 1$ (provided $x \neq 0$). Applying dimensional analysis may best be learned in the context of a problem.

Perry Odic is riding his bike at 20 miles per hour (mi/hr). How long will it take Perry to ride 8 miles?

Note that the question is asking for the amount of time, so the answer should be in time units, such as hours. The only place where a time unit shows up in the problem statement is in the speed, 20 mi/hr, but it is in the denominator. Inverting this quantity yields time in the numerator: 0.05 hr/mi, but then there is a length dimension (miles) in the denominator. Multiplying the inverted speed with the distance of 8 mi yields the desired units:

$$0.05\,\frac{\mathrm{hr}}{\mathrm{mi}} \times 8\ \mathrm{mi} = 0.4\ \mathrm{hr}.$$

We offer a word of caution. Dimensional analysis cannot be applied to every problem. Some mathematical relationships include unitless constants, such as the distance equation for accelerated motion, $d = \frac{1}{2}at^2 + v_0 t + d_0$, and the equation for kinetic energy, $KE = \frac{1}{2}mv^2$. Both of these equations contain a dimensionless $\frac{1}{2}$ that arises from calculus. The units for the variable quantities in the kinetic energy equation for example, m and v, can be kg and m/s respectively. So units for KE are kg $\frac{\mathrm{m}^2}{\mathrm{s}^2}$, with no units attributed to the dimensionless constant. In the numerical calculations, however, neglecting the $\frac{1}{2}$ yields the wrong answer. Thus dimensional analysis alone cannot be used to solve a kinetic energy problem. Even if dimensional analysis cannot be used to solve every problem, it can always be used to detect an error when the units do not match the expected units. In this activity, dimensional analysis can be applied to all problems.

3.8.4 Implementation Plan

Formal Learning Objectives

- Set up algebraic relationships by analyzing units.
- Generate a graph showing the relationship between independent and dependent variables
- Interpret graphical and/or tabular data to draw a qualitative conclusion

Time Required This module was assigned as a small project that students needed to complete. The module was introduced in class (about 5 to 10 minutes) and students were given a week to turn it in.

Implementation Recommendations The implementation of the module can be adjusted according to the instructor. It can be given as an in-class activity or as a group activity. Students can sketch the graphs on paper or, it if makes sense in your class, can use a spreadsheet.

Students will most likely generate a continuous graph when a piecewise graph is more appropriate. This is an opportunity to discuss the differences between the two graphs and the reason for the jump continuity, which connects the application to the mathematical representations.

We introduce this module after the students have worked with functions for quite a bit, so it comes later in the semester. They should be comfortable with the concept of functions, the notations, graphs and, particularly, piecewise functions.

We allow students to work in groups (2 or 3 at most) on the activity. Our intent is to have students work together and help each other solve the module, as different members of the group may have different ideas to bring to the table.

Common Errors and Questions Students who have weak algebra background stumble in the first two problems, not knowing ratios or proportions. Common mistakes in the third problem tend to be in miscalculating the dosages and representing the graph with a continuously increasing function with a limiting end-behavior, rather than a piecewise function. The instructor may want to have guiding discussion with the students prior to assigning the module, or add them in the module itself, that would help students to consider a graph with discontinuities.

3.8.5 Additional Information

The graphing questions can be used in a calculus course as an application to geometric sequences and discussion of convergent series. The graph can be a point of discussion about jump discontinuities. With proper adaptation, the module can also be used to discuss annuities in a financial mathematics class.

Medicine Dosage—Applying Dimensional Analysis/Graphing to Healthcare

Module with Solutions

Instructions to student: Answer the following problems 1 and 2 by first identifying the units for the answer, then combining the given quantities in a way that gives those units, and finally calculating the answer. Problem 3 guides you through solving the problem.

1. The tranquilizer valium is sold in 2.0 mL syringes that contain 50.0 mg of the drug per 1.0 mL of liquid (50.0 mg/mL). If 25.0 mg is prescribed, how many mL should the nurse administer to the patient?

   Solution:
   $$(25 \text{ mg})\left(\frac{1 \text{ ml}}{50 \text{ mg}}\right) = 0.5 \text{ ml}$$

2. A safe dosage of chloroquine is 3.5 mg/kg every 6 hours for malaria treatment. If a child's mass is 28 kg,

   (a) How much chloroquine should be administered to the child per dose?
   Solution:
   $$\left(\frac{3.5 \text{ mg/kg}}{\text{dose}}\right)(28 \text{ kg}) = 98 \text{ mg/dose}$$

   (b) How much chloroquine should be administered to the child over a 24 hour period?
   Solution:
   $$(24 \text{ hr})\left(\frac{1 \text{ dose}}{6 \text{ hr}}\right)\left(\frac{98 \text{ mg}}{\text{dose}}\right) = 392 \text{ mg}$$

3. A patient was admitted to the hospital with a ruptured appendix. After surgery, the doctors prescribed morphine at a dose of 30 mg every 4 hours to manage pain. The half-life of morphine is 4 hours, which means that the patient's body eliminates 50% of the medication after 4 hours.

   (a) What do you expect will happen to the amount of medication in the patient's body as time goes on?
   Solution: Answers will vary. You need to determine the amount of medication in the patient's body after each dose.

   (b) Consider one dose. The patient's body initially has 30 mg of the medication. How much medication is left after 4 hours?
   Solution: Since 50% of the medication eliminates from the patient's body in 4 hours, 15 mg of the 30 mg initial dosage will be eliminated. So what is left is 15 mg.

(c) At the 4th hour, the patient is administered another dose of 30 mg. How much medication is in the patient after this second dose (don't forget what is left from the first dose!)?

Solution: 50% of the first dosage, the 15 mg, remains in the patient's body after the 4th hour. Add to that the second dose, 30 mg, that makes what's in the patient's body at the 4th hour is $15 + 30 = 45$ mg.

(d) At the 8th hour, the patient is administered another dose of 30 mg. How much medication is in the patient after this third dose (don't forget what is left from the first dose!)?

Solution: 50% of the total from the last part, the 45 mg, remains in the patient's body after the 8th hour. Add to that the new dose, 30 mg, that makes what's in the patient's body at the 8th hour is $22.5 + 30 = 52.5$ mg.

(e) Continue the same approach as in (c) and (d) to complete the table below to determine the amount of medicine in the patient's bloodstream after each dose during the first 24 hours.

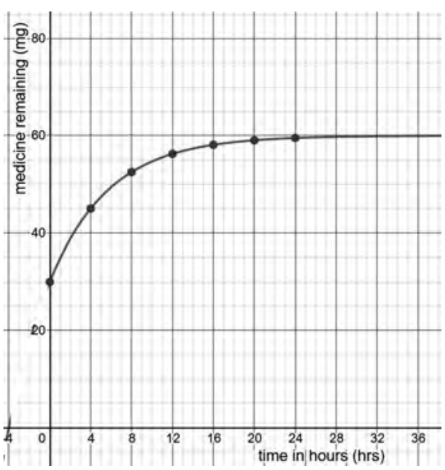
Hours since first dose	Amount of medicine (mg) in patient
0	30
4	45
8	52.5
12	56.25
16	58.125
20	59.0625
24	59.53125

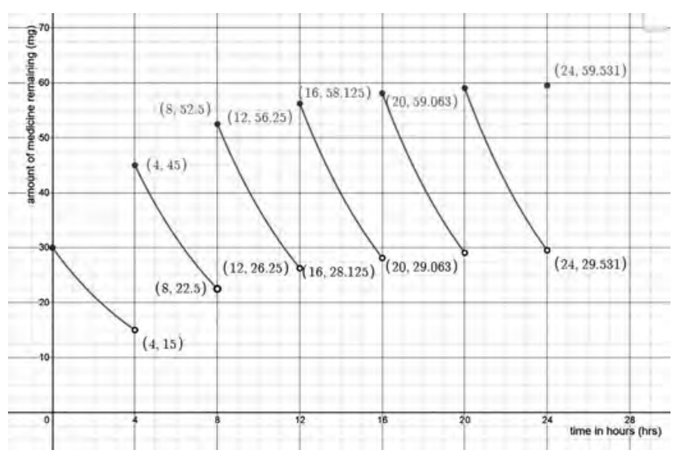
Solution: Solutions are in the second column of the table above.

4. Using the above table, make a graph that shows the relationship between the amount of morphine (mg) in the patient and the time since the initial dose. What kind of graph would you use to connect these points to reflect the amount of medicine in the patient's body?

Solution:

Most students will join the points with a smooth continuous curve as in the graph on the left. Some students will produce the graph on the right.





5. What trend do you see in the graph in question 4? Explain.

Solution:

If the student gives the graph on the left: There is an initial rapid increase in the medicine in the bloodstream, and then that amount in the bloodstream approaches 60 mg, or levels off.

If the student gives the graph on the right: The amount of medicine in the four-hour period between dose administration decreases. It increases at the administration of the next dose. The amount of medicine eventually approaches 60 mg.

6. Morphine may be harmful if more than 60 mg is in the patient. Based on the table and graph, do you think that it will ever become harmful to the patient at this dosage? Explain your reasoning.

   Solution: No, the amount cannot be harmful to the patient at this dosage since it appears that the graph levels off at 60 mg.

7. Did your answers to the previous questions agree with your expectations in question 3a? Were there any findings that surprised you?

   Solution: Responses will vary

3.9 Population Dynamics—Applying Rates of Change to Population Dynamics

John Hearn, Caroline Maher-Boulis, and Nicholas Nagle
Lee University and University of Tennessee–Knoxville
Contact: Contact: cmaherboulis@leeuniversity.edu

3.9.1 About the Module

- Course: Algebra for Calculus
- Partner Disciplines: Geography, Biology, Chemistry, and Health Science

3.9.2 Institutional and Course Contexts

- Type/size of institution: Liberal Arts University
- Size of Class: 30 or fewer
- Characteristics of Students: Undergraduate students who are seeking a degree which requires higher mathematics courses, such as Calculus I or trigonometry-based physics.
- Mathematical Content: Interpreting graphs.
- Purpose/Goal of the Module: This module challenges students to interpret graphs showing changing populations sizes and rates of those changes, and to connect the concept of rate with real-world challenges in estimating population change.
- After and Before: Before this module students need to be familiar with the algebraic definition of a slope. After this module, students should be able to recognize the differences between a variable, its changes, and its rate of change.
- Inspiration for the Module: Faculty want students to conceptualize the differences between variables and their derivatives without the rigor of calculus.

3.9.3 Partner Discipline Background and Discussion

Before starting the module, explain what an annual percent population growth rate is. An annual percent population growth is given by the following equation:

$$R = \frac{\Delta P}{P} \times 100\%$$

where ΔP is the annual change in the population size; that is,

$$\Delta P = \text{population size at the end of the year - population size at the beginning of the year.}$$

For example, if the population size falls from 100,000 to 90,000 over the course of a year, the annual percent population growth is:

$$R = \frac{90,000 - 100,000}{100,000} \times 100\% = -10\%$$

Point out that a negative population growth rate results from a decrease in population size, but the population size is always positive.

Depending on class interest, the instructor may engage the class in a number of topics from the partner disciplines of demography, sociology and geography. We give some ideas here, ranging from how these data and growth rates are used, to how data are collected, to how this knowledge is used, to other applications of population growth rates.

Discussions and background about the data: The data for these graphs were obtained from `www.macrotrends.net`, who in turn appears to have collected them from the United Nations. The United Nations, in turn, collected the data from the US Census Bureau. Population censuses in the United States are conducted every ten years, in years ending in zero. It appears that the United Nations has just drawn a straight line ("linearly interpolated") between the censuses. It is not clear what model has been applied to extrapolate the population trend after 2020. If the instructor has time, they might ask students, "How many data points are here?" If data are observations, then it is obvious that the values in census years represent data. But do the values outside of census years represent observations? Are they data?

The instructor might inquire about how else the data between census years might be calculated. For example, a smooth curve might be drawn connecting the census years, rather than straight lines. The US Census Bureau itself, gathers *vital statistics* about births and deaths, and estimates migration flows using tax records, to estimate how the population changes in non-census years.

The instructor might also discuss the question "What is Detroit?" The United Nations has not just used the population of the City of Detroit, but has also added the suburbs. They do this to enable international comparisons. Every nation defines cities differently. For example, the City of London is quite small, but the metropolitan area is one of the largest places in the world. By using a metropolitan area, the United Nations believes that it is easier to make these international comparisons. But you might even ask "What is a metropolitan area?" For instance, is the definition of a metropolitan area in the United States the same as the definition of a metropolitan area in India? Answering questions like these are part of the job for statisticians and other scientists who work with official statistics.

Discussions and background about visualization: Graphs are ubiquitous in scientific communication, and interpreting graphs is an important skill for students to develop. In the COVID-19 pandemic graphs are available to show cumulative cases, new daily cases, and active cases to just name a few. While these graphs are connected, those connections may not be obvious to everyone. The reader needs to be able to distinguish between the variable and the rate of change of that variable. This general interpretation arises in chemistry, biology, and health science. In chemistry, chemical concentrations and their rates of changes are central to the study of kinetics. In biology, population sizes and population growth rates are observed for various microorganisms. In health science, chemical concentrations or microorganism population sizes may be the subjects of study. In this module, students are asked to interpret population data for Detroit, Michigan. The subject of this module (population of a city) will be more familiar to students than examples from chemistry or microbiology, but the graphical interpretations are similar.

Discussions and background about real-world applications: Calculating future population is important for many important decisions. Many states and communities use population projections like this to forecast tax revenues, or to forecast changes in the demand for public services. For example, growing populations may require new schools and hospitals, which can take years of planning and construction. Similarly, declining populations may require communities to plan for hard decisions about closing existing ones. Communities are also interested in populations growth so that they can plan whether they have sufficient access to fresh water and to energy. In some places of the arid Western United States, cities are required to periodically forecast population change and to guarantee future water availability. In other places, population growth may be modeled to determine where land conservation efforts should be directed. Another important application is calculating political representation, such as the number of seats in congress, electoral votes, and setting boundaries for electoral constituencies.

Discussions and background about modeling population change: In order to estimate future population change, scientists usually think in terms of the growth rate chart rather than the population chart. For example scientists may look at the table of growth rate for Detroit, make an educated guess about the future growth rate, and then update the population graph using that growth rate. For example, experts might decide that the growth rate will be -0.1 % per year for the next decade, or they might decide that a trend is more reasonable, maybe starting at $-.5$ percent, and rising $.05$ percent each year: $-0.5 + .05t$. Experts would make these choices based on other data, such as trends in employment or the economy. In places where there are a lot of data, scientists do not usually make models of the population growth rate directly, but often make models of birth rates, death rates, and migration rates, which are then used to develop charts of births, deaths, and moves, and then used to update the population.

Similar population models are used elsewhere. For example, in the early stages of an epidemic, the growth rate of the disease is often a constant. In the early days of the COVID pandemic, many scientists were interested in estimating exactly what this growth rate was. In later stages of an epidemic, it becomes important to not only look at the population of infected people but also the population of people who have had the disease and are now immune, and the population of people who are still susceptible. These sorts of models are called *Susceptible - Infected - Recovered* (SIR) models.

In all of this, important things for students to understand are that: (1) it is possible to go back and forth between the charts of population count and the population growth rate, and (2) this is important because many scientists think in terms of the growth rate, rather than the count.

3.9.4 Implementation Plan

Formal Learning Objectives

- Interpret graphs by analyzing variables and their units.
- Distinguish between a change (slope) and a changing slope.
- Describe how rates are used in real-life analyses of population change

Time Required

- Introduction: 5 minutes.
- module: 15–20 minutes.

Implementation Recommendations The implementation of the module can be adjusted according to the instructor. It can be given as an in-class module or as a group module.

Students need to pay careful attention to what is being asked: a change in the population size or a change in the population growth rate. These concepts may be easily confused.

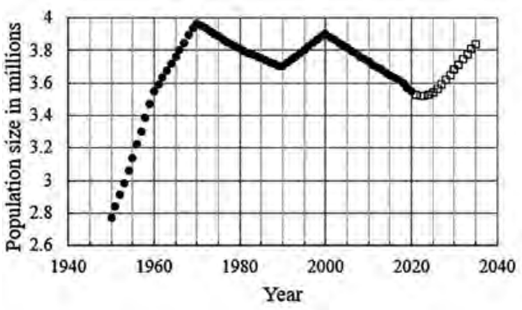
3.9.5 Additional Information

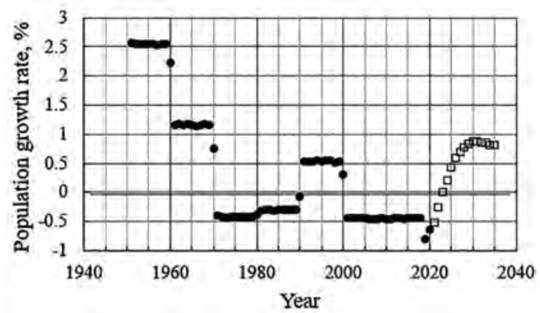
This module can be revisited in a calculus course once the derivative terminology has been introduced and defined.

Population Dynamics - Applying Rates of Change to Population Dynamics

Module with Solutions

Instructions to student: The following graphs show the annual population growth rate (top) and population size (bottom) for the Detroit metro area from 1950 to 2020. The graphs also show projections through the year 2035. The data for these graphs was obtained from `www.macrotrends.net`.





1. From the graph of the annual growth rate

   (a) Estimate the population growth rate in the year 1965.
   Solution: 1.1% (solutions may vary ranging between 1% and 1.5%).

   (b) Estimate the population growth rate in the year 1975.
   Solution: -0.4% (solutions may vary ranging between 0% and -0.5%).

   (c) Describe the annual population growth rate during years in the period 1960–1970, and explain what is happening to population size, from year to year. Compare that to the time period 1970–1980 and describe the annual population growth rate during that time, then explain what is happening to population size from year to year during that time period.
   Solution: The annual growth rates are positive in the period 1960–1970 and seem to be constant. The population size during that same period is increasing from year to year. While in the period 1970–1980 the annual growth rates are negative and the values seems to be close to each other. The population size during these years is decreasing from year to year.

2. How would you compare the population growth rate during the period 1970–1980 and that during the period 1980–1990?

   Solution: The growth rates in both periods are negative, which means the population size is decreasing in both periods. However, the rate during 1970–1980 is larger in absolute value than that of the period 1980–1990. This means the population size is decreasing faster in the first period than it is in the second period.

3. When does the growth rate first begin to decline?

   Solution: In 1960.

4. From the population graph

   (a) Estimate the population size in the year 1960.
   Solution: 3.5 million (solutions may vary ranging between 3.45 and 3.6 million).

   (b) Estimate the population size in the year 1990.
   Solution: 3.7 million (solutions may vary ranging between 3.6 and 3.8 million).

(c) Why does the population size graph not extend to include negative values on the vertical axis, as is the case in the annual growth rate graph?

Solution: Because while we can have negative growth rate (implying population is decreasing), we can not have a negative population size.

5. When does the population size first begin to decrease?

   Solution: 1970.

6. Why are your solutions to questions 3 and 5 different?

   Solution: A decline in the growth rate, while still being a positive number, means that the population size is still increasing but at a slower rate than where it was before 1960. On the other hand, a decline in the population will correspond to a negative growth rate.

7. If the population growth rate *increases*, does the population size necessarily *increase*? Explain by referencing specific time periods in the graphs. In other words, are there any time periods over which the population growth rate increases while the population size decreases?

   Solution: No, an increase in the growth rate does not necessarily imply an increase in the population size. An example is the period between 1970 and 1990. The growth rate shows an increasing graph (yet still negative), while the population graph shows a decreasing population size.

8. Using the data in the population size graph, calculate the percent population growth rate for a 5-year period during the 1950s. Read carefully from the graph - estimate to the nearest 0.01 million. How does your calculation compare with that from the population growth rate graph?

   Solution: solutions will vary, For example, for the 1950 to 1955 the difference in population size is approximately $3.15 - 2.75 = 0.4$ million. The population growth rate is

   $$\frac{0.4}{5} = 0.08 \text{ millions/year.}$$

   The **percentage** population growth rate is

   $$\frac{0.08}{2.75} = 2.91\%$$

   which is different from the information in the population growth rate graph (likely due to the averaging of the annual population growth).

9. Based on the class discussion, describe one application in which it is important to estimate population change, and describe one real-world challenge in actually estimating population change.

   Solution: Responses will vary.

3.10 Exploring Logarithm Rules with an Exploration of pH

Janet Bowers, Kathy Williams, and Diane Smith
San Diego State University
Contact: jbowers@sdsu.edu

3.10.1 About the Module

- Course: Precalculus
- Partner Disciplines: Biology and Chemistry
- Required Technology: Graphing Calculator or Desmos

3.10.2 Institutional and Course Contexts

- Type/size of institution: large, public, research institution
- Size of Class: 25-28 students
- Characteristics of Students: Undergraduates studying precalculus from a variety of backgrounds
- Mathematical Content: Logarithmic scales, laws of logarithms, using logs to compare small numbers of different magnitudes
- Purpose/Goal of the Module: This module is designed to engage students in a use of logarithms to solve a problem (how do I see distinctions between very small numbers on a graph?) It is also designed to engage students in a mathematical application that is "real life": the use of pH as a logarithmic measure of acidity.
- After and Before: Students need to know the general rules for manipulating logarithmic expressions
- Other Prerequisites: Familiarity with a graphing program such as Desmos or a graphing calculator
- Inspiration for the Module: Talking with biology and chemistry professors who have said that students need to understand logarithmic scales and how logarithms are used. They also note that more advanced classes use the rules for more complicated derivations involving differential equations.

3.10.3 Partner Discipline Background

The overall goal of the module is to study the mathematical model to describe pH. Instead of just telling students to use logs, we begin by creating and discussing the "problems" associated with graphing small numbers (concentrations of hydrogen ions). The module then guides them through the solution of using logs to graph the same information in a more effective way.

Knowing the acidity of various water-based solutions is critical for maintaining life on Earth. For example, human muscle proteins move best in environments with a pH between 5 and 7. Moreover, entire species are endangered when they encounter acid rain, which increases with pollution. In order to measure the acidity of a substance, scientists around 1909 developed the idea of pH to measure the amount of free hydrogen ions in a given solution. It is interesting to note that it is not clear what this "pH" originally referred to. Some suggest it represents the *power* of hydrogen while others believe it is the *percent* of hydronium. Others prefer to think of it as a measure of the *proportion* or the *probability* of choosing a free hydrogen ion in the substance. This lesson features part of a YouTube video in which Paul Anderson uses this latter description to introduce the idea of the pH of water by stating that the probability of finding a hydronium ion in distilled water is 1 in ten million, or .0000001. This can be contrasted, for example, by noting that the probability of finding a hydronium ion in lemon juice is much higher. In fact, there is a 1 in 100 or .01 chance of finding a hydronium ion lemon juice. Thus, lemon juice is 10^5 times more acidic than distilled water.

The goal of this lab is to generate the need to compare small numbers by thinking about them as probabilities. Once students have wrestled with this idea, the derivation of the equation for pH is introduced:

$$\mathrm{pH} = -\log[\mathrm{H}+]$$

For distilled water, this becomes

$$\begin{aligned} \text{pH} &= -\log[0.0000001] \\ \text{pH} &= -\log[1 \cdot 10^{-7}] \\ \text{pH} &= 7 \end{aligned} \tag{3.1}$$

For lemon juice, this becomes

$$\begin{aligned} \text{pH} &= -\log[1 \cdot 10^{-2}] \\ \text{pH} &= 2 \end{aligned} \tag{3.2}$$

Solutions that have a greater concentration of hydrogen ions such as lemon juice are considered acidic and have a pH less than 7. On the other hand, a solution of soapy water could have a concentration of 1 in $1{,}000{,}000{,}000{,}000$, so $[\text{H+}] = 10^{-12}$, which yields a pH value of 12.

3.10.4 Implementation Plan

Formal Learning Objectives

- Students will develop an intuitive idea of logarithms as numbers that can be used to organize a set of either extremely large numbers or extremely small numbers (i.e., sets of numbers with different magnitudes) so that they can all be placed on one scale and compared in relative terms.
- Students will develop a deeper understanding of the rules for manipulating logarithms by applying the rules to convert from concentrations to pH values and vice versa.
- Students will develop an understanding of how to compare quantities by comparing their relative magnitudes expressed as logarithms (applications of pH and Richter scale).

Materials and Supplementary Documents The following can be found on the volume website `https://bit.ly/SUMMITPNotesActivities`:

- The student worksheet.
- The slidedeck.
- The Desmos module.

Time Required

- 50 minutes (for advanced class)
- 50 minutes (for parts 1–3) and 50 minutes (for parts 4–5)

Implementation Recommendations Generally speaking, students can experiment with the graphing challenge, which is the heart of the module. We want them to use Desmos but do provide numbers to be entered. The hope is that students will experience cognitive dissonance because, usually when graphing with Desmos, patterns can be quickly discerned. In this case, they cannot. We encourage students to experiment by changing the scales of the x- and y-axes to see if they can better see a pattern. It is hoped that this drives home the realization that large differences in the size of the quantities being represented need an alternative representation method. We then suggest that they add a third column in Desmos to take the log of the numbers and discuss the resulting graph. Within Desmos, the group activities all have self-checks so that students can experiment with different solutions.

Active learning is threaded throughout the module. There are several "self check" questions so that students and teachers can get a sense of where the understanding is strong, and what areas may need to be reviewed further. In the Desmos lesson, these occur on slides 3, 8, 13, 14, 15). In the classroom module, the teacher could ask similar questions using either voting, or paper solutions, or simply a raise of hands and class discussions. The goal of the lesson is intended to be as interactive as possible.

- Part 1: Warm up Problems

  The first problem was designed to remind students of the link between the math they are learning in "lecture" and the math used in this lab. It is also designed to review mathematical thinking through the use of numbers and counterexamples. Most students find the use of $a = 10$ and $b = 1$ to be convincing, but other examples could be $a = b = 1$ or even using a different base such as 2.

  The second problem (ordering common substances) is designed to engage students in thinking about the acidity of everyday substances and to challenge them and build some interest in finding out how close they were to the actual solutions. The answers to this ordering are noted later in the lesson.

  As you transition for Part 1 to Part 2, note that in Part 1 question 1, students are asked to show that log(a+b) is not equal to log(a)+log(b). This was placed there because some of the more difficult transformations in Part 2 from pH to H+ concentration require this knowledge (see the example in Figure 4 where this is applied).It was also placed there because we are trying to encourage students to make sense of the algorithms they are learning and put them into context so that they don't make mistakes such as that common one of "distributing" the word 'log.' The second question in Part 1 involves having students rely on intuition to order common items in terms of acidity. This was done to get them ready to think about how acidity can be formally measured in Part 2. It was also a way of building some interest in the topic before the "big reveal."

- Part 2: Intuitive Introduction to Logs

  One of the basic assumptions of this lab is that logs can be understood through their use, rather than through their rules (which is often the only way they are discussed). In simple terms, scientists often want to compare things of very different magnitude where the big things are so big that we have trouble seeing their size relative to one another, or conversely, we may want to compare the size of things that are so small that they are hard to see. The teacher may want to elicit some examples of very "different" sized things to compare such as:

  - The speed of walking, riding a bike, and traveling in a rocket
  - The financial net worth of the students in the room and the CEO of Amazon
  - The number of years since: the Big Bang, the invention of the wheel, the invention of the car, the invention of the internet

  Once the general motivation is understood, ask students to plot the coordinate pairs given in question 1 of the worksheet. NOTE: It is best if no particular axis denominations are mentioned. The plotting question is designed to initiate cognitive conflict. Students will hopefully realize that the small quantities all land along the x-axis and cannot be distinguished (even when using a graphing program such as Desmos or Excel). [See Fig. 1]

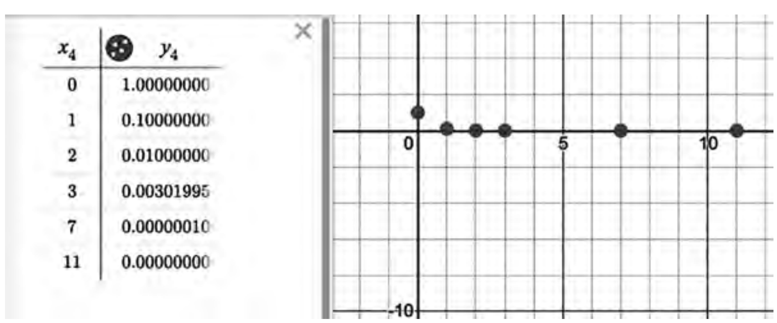


Figure 1: Initial plot of given data using Desmos.

To resolve this conflict, the instructor may first elicit answers to the question of "How are the numbers on the x-axis related to the values on the y-axis?" Hopefully an astute student notices that the x-values represent the number of decimal places to the right after the decimal point before the first nonzero digit is reached. This could be framed as the power of ten for each number expressed in scientific notation. The instructor can then relate this to the base ten logarithm and suggest taking the log of each number as a way to resolve the conflict of how to reveal differences among the y-values. This graph is shown in Fig. 2.

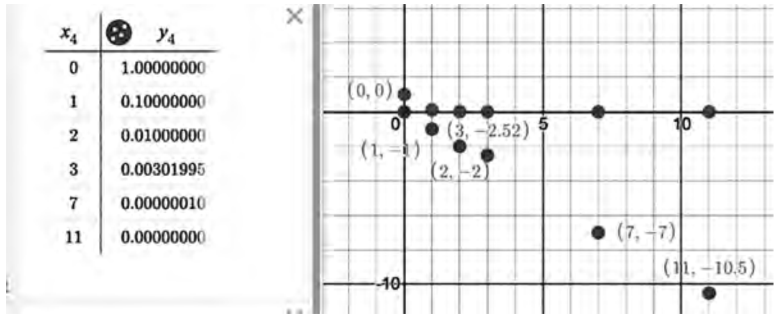


Figure 2: Desmos plot with $\log(y_4)$

- Part 3: A Brief Introduction to pH

  The next part of the lesson focuses on exploring what pH measures. The link between warm up question 2 and the graphing is revealed (See Figure 3).

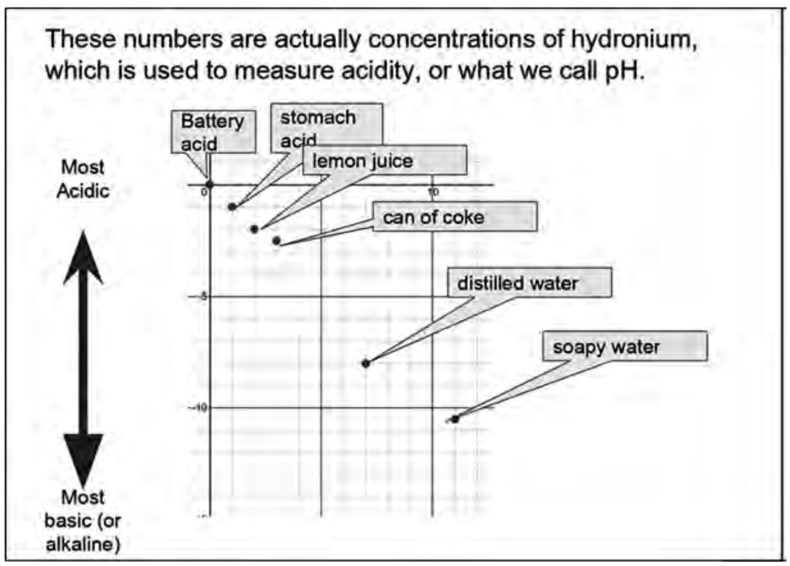


Figure 3. Graph showing the relation between the pH and the values plotted.

  Students are hopefully engaged as they see the order as it relates to their guesses. Other questions to consider:

  - Why are the y-values negative?
  - What does it mean to have a very small [highly negative] y-value?
  - What types of substances are most acidic? What are their pH values?

  In order to explain pH values, we introduce the YouTube video. The instructor helps students to visualize both pH as a molar concentration of hydronium and the relative magnitudes involved in powers of ten.

- Part 4: Using the Rules of Logs to Convert Between pH and Molar Concentrations

  The fourth section of the lesson involves laying out the steps to convert between concentrations of hydrogen ions and pH values. Here, the instructor is encouraged to allow students to solve the example on their own before revealing the correct solutions. It is also hoped that the instructor will be able to point out the utility of using laws of logarithms that are often learned in lecture classes but never applied to real life examples. The instructor can also reference back to the laws described in the first warm up problem. The examples used in this section can vary depending on the students' level of comfort with the material.

  To make the idea of converting between pH and concentrations more conceptual, the instructor may want to choose to use examples that are strictly powers of ten. A subsequent lesson may involve the more general conversion module that involves using the laws of logs. This slide has been used to scaffold students' work as shown in Figure 4.

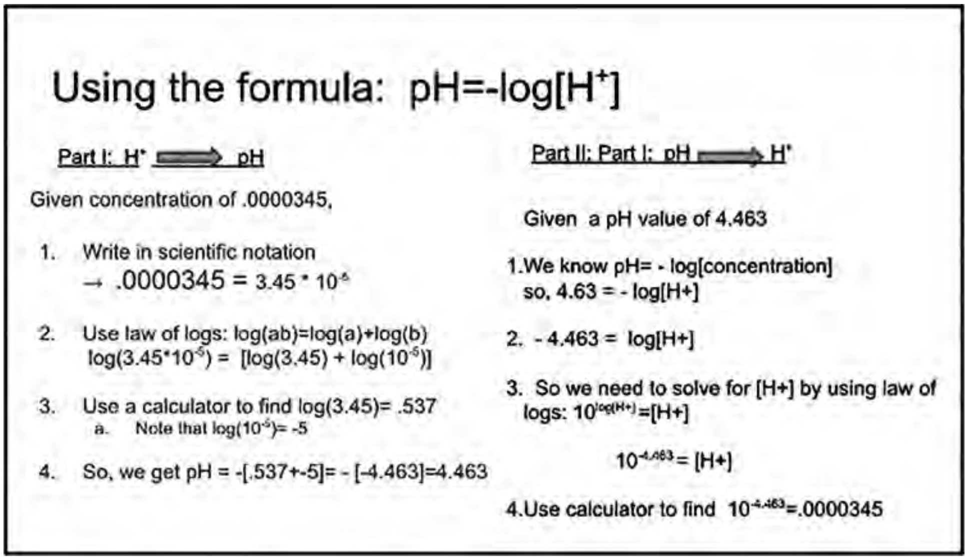


Figure 4: Step by step procedure for converting to/from pH and concentration numbers

- Part 5: Returning to the Utility of Logs as a Way to Compare Quantities

  The final section of the lesson involves comparing quantities using their logarithmic representations. Here, the goal is once again to elicit some cognitive conflict by using integral pH values that are multiples of each other (e.g., comparing a pH of 8 with a pH of 2). The teacher is encouraged to ask questions such as: does it make sense to say that stomach acid is only 4 times as acidic as soapy water? Go back to look at the concentrations of each or frame the question in terms of probabilities. Best practice is to refrain from introducing an algorithm here; allow students to determine which approach makes sense to them.

  A final example involves using the same idea of comparisons but using values on the Richter scale (See Rafferty, 2020).

  To conclude the lesson, we have often asked students to write one "Take away" message that they learned (encourage students to write either a math or science topic, as long as the insight is elaborated).

Common Errors and Questions

- Students may make the mistake of thinking they can "distribute" the word $\log$. This presents a good opportunity to point out that $\log(a)$ is a number, as is $\log(a+b)$. This is the intention of the first warm up problem.
- Students may compare the logs of numbers rather than the quantities that the pH values represent.
- Students may not realize that if numbers lie on the line $y = -x$, then the x and y coordinates have same magnitude, just different signs.

Adaptations to Online Learning The lesson was converted from a paper and pencil module worksheet to a Desmos module. The link to the teacher.desmos.com file is here:
`https://teacher.desmos.com/modulebuilder/custom/6043712338b7683da50c1f52`
Students are encouraged to work in pairs. In the Desmos module, students are assigned to breakout groups for slides 16-17. In the worksheet version, students work in groups of 2-3 to fill in the chart for question #2 and complete as many other questions as time allows.

3.10.5 Additional Information

This lab could be divided into two or three class periods, depending on the setting. Day 1 could involve the warm up problems and the initial graphing of very small numbers that are all powers of ten. This might support more students' recognizing the link between the x- and y-values as well a discussion of why some values are positive and some are negative.

Day 2 would be a deeper dive into the concept of pH in general, and the derivation of the formula. Students could then be introduced to the more complicated application of log rules.

Day 3 could involve the idea of comparing large and small numbers. Helpful to introduce the notion that comparing magnitudes of numbers does not equate linearly to comparing the actual numbers themselves. This is highlighted using the Richter scale, but could also include studies of decibels, particles of light, internet host counts (See Wikipedia for other examples).

This lesson can be extended to a Calculus I class to discuss applications of the derivative of a logarithmic function. It is also critical for students to understand logarithmic integration when determining actual pH values in a titration lab.

Students found that the "Where's Waldo?" analogy to explaining pH as the probability of finding a hydrogen ion in lemon juice versus water to be very useful. Also, students enjoyed the "Aha" moment on slide 12 when they were able to compare their guesses with the actual results. Some even asked if they could re-do their thinking.

Note that approximately half of the students who take this course are able to compute simple logs such as log(1000), but very few have any conceptual understanding of the concept or its utility in science. Similarly, about one quarter of the students have heard about (or even studied) pH in chemistry class but again, very few have deep conceptual knowledge beyond the understanding that it is a molar concentration measuring hydrogen ions. Therefore, almost all students (and most of the TAs as well) leave with a deeper understanding of what logs are (in terms of powers of ten or measures of magnitude), their utility to compare very large numbers by comparing their magnitudes, and how their calculational knowledge (that they might have come with including memorized algorithms such as "around the world") fits with the larger conceptual ideas.

3.10.6 References

Bradley, D. (2020). When it comes to caustic wit and an acid tongue, mind your Ps and Qs. *Materials Today*. `bit.ly/3LkTMXb`.

Rafferty, J. (2020). *Richter Scale*. Encyclopædia Britannica. `bit.ly/3vCZyMU`

Ritz, G.F. & Collins, J.A. (2008) *U.S. Geological Survey Techniques of Water-Resources Investigations*, book 9, chap. A6.4, `doi.org/10.3133/twri09A6.4`

Wald, Lisa. *The Science of Earthquakes*. `www.usgs.gov/natural-hazards/earthquake-hazards/science/science-earthquakes?`

Wikipedia contributors. *Logarithmic scale*. Wikipedia, The Free Encyclopedia, Accessed by author 18 Dec. 2020.

Using Logarithms to Model Acidity

Module with Solutions

Warm-Up

- Warm-up 1: Provide values for a and b to show that $\log(a+b) \neq \log(a) + \log(b)$.
  Solution: Solutions will vary. Some examples are $a = 10$, $b = 1$, or $a = 1$, $b = 1$.
- Warm-up 2: Just using your own experiences and intuition, order the following from least acidic to most acidic:
  - lemon juice
  - soapy water
  - battery acid
  - stomach acid
  - distilled water
  - can of Coke

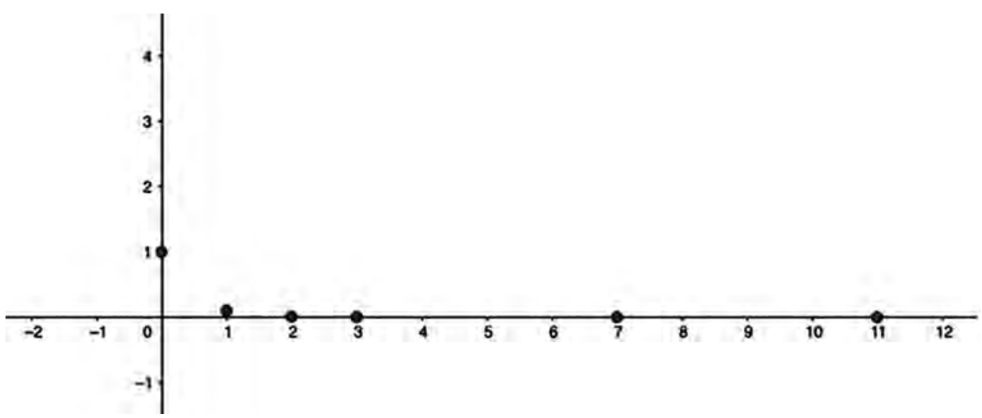
  Solutions: Solutions will vary. Correct order is: soapy water, distilled water, can of Coke, lemon juice, stomach acid, and battery acid.

Experiment

1. Plot the following table of points on a graph:

0	1.000000000000000000
1	0.100000000000000000
2	0.010000000000000000
3	0.003019950000000000
7	0.000000100000000000
11	0.000000000031622800

Solution:



Reflection Questions:

(a) Is your graph accurate?
Solution: No, the y-values are so small that they all appear to be sitting on the x-axis close to $y = 0$.

(b) How do the numbers on the x-axis appear to be related to the numbers on the y-axis?
Solution: The numbers on the x-axis appear to be related to the number of zeros to the right of the decimal point and before a non-zero digit appears.

2. Recall that we define pH$= -\log$(Molar concentration of H^+). Fill in the last three columns of the chart below (the first has been filled in as an example):

SUBSTANCE	MOLAR CONCENTRATION OF H^+, WRITTEN AS $[H^+]$	POWER OF 10	pH	Acidity compared to distilled water
Battery acid	1.000000000000000000	10^0	0	10^7 times greater
Stomach acid	0.10	10^{-1}	1	10^6 times greater
Lemon juice	0.010000000000000000	10^{-2}	2	10^5 times greater
Can of Coke	0.003019950000000000	$10^{-2.5}$	2.52	$10^{4.48}$ times greater
Distilled water	.0000001	10^{-7}	7	0 times greater
Soapy water	0.000000000031622800	$10^{-10.5}$	10.5	$10^{-3.5}$ as much

Solutions: Solutions are in the final 3 columns of the table above (the first row will be filled in for students).

3. How does the concentration of H^+ in lemon juice (pH=2) compare to that in soft water (pH=8)?

Solution: First, we calculate:

$$\frac{10^{-2}}{10^{-8}} = 10^6$$

This indicates that the concentration of hydronium in lemon juice is 10^6 or 1 million times as high as that of soft water.

4. How does the concentration of H^+ in a can of soda (pH=2) compare to that in soapy water?

Solution: First, we calculate:

$$\frac{10^{-2}}{10^{-10.5}} = 10^{8.5}$$

Next, we note that $10^{8.5} = 10^{0.5}10^8 \approx 3.16 \cdot 10^8$. Therefore the concentration of H^+ in a can of soda is about 316 million times that of soapy water.

According to Rafferty (2020), the Richter scale has been used to measure the amplitude (height) of the seismic wave produced by an earthquake. These numbers, which indicate the amount of energy released from the quake, are very large. Therefore, a logarithmic scale is used to compare amplitudes. Using this system, an increase of 1 unit (e.g., from $5.3 \to 6.3$) indicates a 10-fold increase in units. The units for this scale are called ML (magnitude local) which means that not all earthquakes from different places can be compared on one scale. Interestingly, the Richter scale has been replaced by other measures, but the word is sometimes still used in media reports because it is most recognized.

5. The amplitude of a seismic wave of a magnitude 5 is 100 times that of a magnitude 3 quake. Explain.

Solution: The amplitude of a magnitude 5 wave is defined to be 10^5 ML. Therefore, to compare a magnitude 5 with a magnitude 3, we compute the ratio:

$$\frac{10^5}{10^3} = 10^2$$

which is 100 times as great.

6. How does the amplitude of the largest wave measuring (6.2) earthquake in Japan compare with the magnitude 3.5 quake that struck in February?

Solution: Calculate the ratio:

$$\frac{10^{6.2}}{10^{3.5}} \approx 500$$

So the largest earthquake in Japan was approximately 500 times as great as the one that struck in February.

7. Juan said that he experienced an earthquake that was "a million times worse" than the one in Japan. What would the minimum size of Juan's earthquake have to have been?

   Solution: In order for the earthquake Juan experienced to be 1 million times more powerful than the largest one in Japan, calculate $10^{6.2} \cdot 10^{6} = 10^{12.2}$. According to the US Geological Society (see Spence, Sipkin, & Choy, 1989), the largest earthquake ever recorded measured 9.5. It occurred in Chile on May 22, 1960. So, Juan may have been exaggerating a bit!

References

En.wikipedia.org. (2021) Seismic magnitude scales - Wikipedia. [online] Available at: `https://en.wikipedia.org/wiki/Seismi_magnitude_scales`, Accessed by author 11 June 2021.

Rafferty, John P. *Richter Scale*. Encyclopedia Britannica, 8 May. 2020. `https://www.britannica.com/science/Richter-scale`. Accessed by author 12 June 2021.

Spence, W., Sipkin, S. A., & Choy, G. L. (1989). Measuring the size of an earthquake. *Earthquakes and Volcanoes 21* Retrieved by author 14 June 2021, from `https://earthquake.usgs.gov/learn/topics/measure.php`

3.11 Curve Fitting with Exponential Functions

Mike May and Debbie Pike
Saint Louis University
Contact: mike.may@slu.edu

3.11.1 About the Module

- Course: College Algebra: Business Flavored
- Partner Disciplines: Business
- Required Technology: Ideally, a spreadsheet. For access to associated data sets, see `https://mathstat.slu.edu/~may/ModelingWithExcel/`. Use the worksheet for exponential modeling.

3.11.2 Institutional and Course Contexts

- Type/size of institution: Comprehensive private university
- Size of Class: 40
- Characteristics of Students: This is a class of business students who are being prepared for business calculus.
- Mathematical Content: Fitting data to exponential curves.
- Purpose/Goal of the Module: The module aligns with a larger goal of preparing students to build a model that fits given data. This module looks at real data, with sizes too big to easily rely on calculators. One data set fits nicely, but the other data set has structural breaks.

  From a broader perspective, this module is part of a series of activities helping students learn to “think in Excel”—to take advantage of Excel’s functionality to solve complex problems, and to design spreadsheets that are user and reader friendly.
- After and Before: Students have recently been introduced to exponential functions given by formulas. They have used Excel for best fitting curves that are linear and quadratic.
- Other Prerequisites: In previous cases where the students have found best fitting curves, they have used the formulas to find projected values.
- Inspiration for the Module: The modeling approach to this course was inspired by a linear modeling worksheet developed at Virginia Commonwealth University.

3.11.3 Partner Discipline Background

While there is almost no specialized knowledge from business that one needs to know for this module, this module is based on the way business professionals “think in Excel”—how they design user and reader friendly spreadsheets usable for a given purpose. In terms of the content, it is helpful if students know that certain quantities, like the consumer price index (CPI), and the value of stock indices, like the Dow Jones Industrial Average, are expected over time to grow proportionally to their current size. Thus, they should model exponential growth. Of the stock indices, the Dow is the most famous.

The instructor should prepare by learning what expectations business faculty have for skills that will be useful for students in their business classes. The following notes provide context for such expectations at Saint Louis University:

- Business faculty want the instructions to be sparse, to simply be “build a model.” It is to be noted that this is a fifth worksheet in a series. The students have already been introduced to the mechanics of using Excel to find a best fitting curve and then using that formula to make predictions. If this were a first worksheet there would be more detailed procedural instructions.

- Business faculty want the students to grapple with the range over which the model is effective. The issue of an appropriate range shows up less with the CPI than with the Dow, where structural breaks occurred in 2001 and 2008. Our students are too young to remember why those years were different. For the CPI, the same issue would arise if the data went back to the 1970s.
- Business faculty are very happy with the idea that students are using real data that can be found online. They suggested the St. Louis fed site, `https://fred.stlouisfed.org/` as a good place for more data.

It is also worthwhile to note the following conventional differences between the disciplines:

- Mathematicians like to express exponential functions in the form Pe^{rt} with a growth rate r, because calculus formulas are simpler with a base of e. The commercial and business world like to express the functions in the form PR^t for a growth factor $R = e^r$.
- Math teachers rarely worry about base points since we work in exact numbers. The business discipline works with truncated decimals so the choice of base point is important. Exponential models should use a base point that makes sense in the model. The base for the CPI and Dow should be in the lifespan of the students.
- If the CPI is modeled by AR^t and an investment is modeled by PG^t with a growth factor of G, then the real income has a growth factor of G/R. (Cases in business course will routinely talk about inflation adjusted amounts.)

These differences will be nontrivial for our students. Instructors should be aware of notations and conventions used in partner disciplines.

3.11.4 Implementation Plan

Formal Learning Objectives

- Import a data set for use in Excel.
- Find a best fitting exponential curve to a data set.
- Be able to do transformations of exponential functions, changing both the base year and the exponential base of the function to improve the format of the formula.
- Use an exponential model to make projections.
- Explore issues of the reasonable limits in which an exponential model can be used.

Materials and Supplementary Documents You will need a spreadsheet with the CPI and Dow Jones average over a period of time, at least 30 years, which can be found on the volume's website.

Time Required The exercise was done in groups in a 50 minute class period. Groups finished outside of class with a one week time limit.

Implementation Recommendations I find it very useful to do any technology project in groups because the students do a certain amount of technical support for each other. The students need to be in a room where they all have computer access. Some technical tasks, adding documentation to a workbook with text boxes should have been experienced by students before this exercise.

Depending on the skill level of the class, how much the use of technology is incorporated into the course, and other course considerations, the instructor may want to provide the students with a template that has some of the Excel coding already filled in. Such templates are available at the website containing the Excel workbooks listed above.

Alternative Solutions The idea of a best fitting curve is a subtle issue with details beyond the scope of this course. The standard usage in Excel is to take a linear fit between x values and the log of y values. With some technologies, like Desmos, this is an optional approach and straight regression will give a slightly different solution. The project also explores an alternative approach where we use a starting point near the center of the data and use a growth rate that is the average of the annual growth rates.

Common Errors and Questions

- Some students will confuse the exponential and power models.
- Students will have trouble with the scientific notation used by Excel in the initial model.
- Students will have trouble translating the formula for the model given by Excel into one that Excel will accept.
- The final model fitting questions, which ask about what happened in 2001 and 2008 to cause structural breaks, are very unfamiliar types of questions for students to encounter in a math class.
- The computing of inflation adjusted investment rates will cause problems. If r is a growth rate, e^r is the growth factor. The use of trend lines finds growth rates. The inflation adjustment is the quotient of growth factors.
- The students may have issues with the idea that a model only works on a limited domain, and with the idea that we want a base point within the reasonable domain. This is similar to why they should like point-slope for mines rather than slope-intercept.

3.11.5 Additional Information

This project is part of a larger shift in a modeling-based college algebra course where the standard situation is to start with a data set, build a model, use the formula to make predictions, and give solutions in context of the original data. Most students have been trained that math is simply about manipulating formulas. The different viewpoint in this course makes the students feel better about math, but it needs to be made explicit.

Exponential Modeling

Module with Solutions

Put the names of all the group members on the first sheet of an Excel workbook. Save the file with the name Excel2Name1Name2Name3.xlsx where Name1, Name2, etc. are group member names.

Creating an Exponential Model in Excel

We have worked enough with building models for us to look at building models with real world data this time. In doing financial forecasting, we generally want to compute an average rate of inflation and an average rate of return for a generic investment. The inflation rate should be the growth rate of the price of a typical market basket of goods. For a generic rate of return on investments, we look at the price of a stock index.

We are going to start by building an exponential model for the Consumer Price Index (CPI). Download the ExponentialModels file from your learning management system. This contains data that you would otherwise have to type.

1. (The naïve model) Select the CPI data, insert a scatterplot and add a trendline, with the formula showing. The trendline should be an exponential model. Select the option to show the formula of the trendline on the scatterplot. Use that formula to add a column to your data with the predicted CPI.

   Solution: The formula for the naïve model is

   $$y = 9E - 19e^{0.0234x} = 9 \times 10^{-19}e^{0.0234x}.$$

2. (Shifting to get a better formula) The formula given is pretty ugly. The constant coefficient is given in scientific notation with only one digit of accuracy. It represents the value of the CPI in the year 0, assuming a steady rate of inflation through the centuries. This model is pretty useless for prediction. We can make things better by making our independent variable "years since 1990" rather than "years since 0." Insert a column between the column for year and the column for CPI. Fill it in with "years since 1990". Add a new scatterplot, produce a new equation, and add a new column of predicted values from your new equation. Your new predictions should be close to the actual values.

   Solution: The revised formula, with x measuring years since 1990, is

   $$y = 233.45e^{0.0234x}.$$

3. (More shifting to get a better formula) When Excel gives the equation of an exponential model, it uses the format

   $$f(t) = Pe^{rt}$$

   However, we would like to convert to

   $$f(t) = PR^t$$

   For your equation compute $R = e^r$. Add another column using the new formula. All of these methods should give the same prediction each year. Which year has the worst prediction, and how far is the prediction from the actual CPI that year?

   Solution: The revised formula is

   $$y = 133.45 \times 1.02368^x.$$

   The worst predictions are for 2018, the end year. It is off by nearly 42 in the naïve model and by 9.1 in the better model.

4. We want to explore some other ways to produce a model for the data. Instead of using Excel to find an exponential trendline, we can use the average of the annual rates of change on a curve that goes through the middle data point. For each year since 1991, find the ratio of the CPI in each year to the CPI in the previous year. Take the average of those rates of change to make a different estimate of R. Give a formula that uses that growth rate and the middle year as a base. What is the largest error using that method? Is this method more or less accurate than the trendline model?

   Solution: With this method
$$y = 185.2 \times 1.0241221^{x-2004}$$
   when x is the year. Once again, 2018 is the worst prediction and the error is 10.7. This method is less accurate than the trendline model.

5. Interpretation questions: What is the average rate of inflation for the given time period? (Use the trendline approximation.)

   Solution: The normal rate of inflation is 2.368% for this period.

6. Justify how we did the shift in years. Why does the following hold true?
$$e^{(\text{year})\text{rate}*\text{time}} = \left(e^{(\text{year}-\text{base})\text{rate}*\text{time}}\right)\left(e^{(\text{base})\text{rate}*\text{time}}\right)$$
   Solution: This is the normal rule for adding exponents when we multiply numbers with the same base.

7. Goal seek is in the "What-If analysis" menu of the data tab. Using Goal seek, find when the CPI is 100.

   Solution: Using Goal seek, the CPI should be about 2.3 through 1977.

Working with Another Data Set

We want to repeat the process with a different data set. The Excel workbook you downloaded also contains the opening values of the Dow Jones Industrial Average (Dow) for each year dating back to 1990. (The Dow is the best-known stock average. It thus models expected values of typical investments.)

1. Find the best fitting exponential model for the Dow Jones averages, convert the equation to a nice form.

   Solution: The basic formula is
$$f(x) = 5E - 55e^{0.067x}.$$
   Converting to a function in years since 1990 we get
$$f(x) = 3438.4e^{0.067x}.$$
   Converting the base gives
$$f(x) = 3438.4 \times 1.06929548^{0.067x}.$$

2. Explorations: What is the model's rate of return for passive investment in the Dow? Use your model to predict the value of the Dow in 2022. Use Goal seek to see when the Dow should have been 500. Check to see how accurate this value is.

   Solution: The rate of return is 6.9295% per year.

3. What is the inflation adjusted growth rate of the Dow? Explain how you got this value.

   Solution: In 2022, the model predicts the Dow will be 29341.

4. Find both the inflation rate and the investment return rate for the periods 1990–2000 and 2009–2018. Speculate why we get different solutions over different periods.

   Solution: The inflation rate was 2.71% for 1990–2000 and 1.65% for 2009–2018. The investment return rates were 16.26% and 11.04% respectively.

   Historically, 2001 and 2008 saw economic downturns that changed the return rates. The structure of the economy has a break from prior years. It is worth remembering that our current students will not remember 9-11 and the great recession. It is worth pointing out that the limits of models often depend on non-mathematical factors that change the structure of life.

3.12 “What If” Analysis, Parameters and Variables.

Mike May and Debbie Pike
Saint Louis University
Contact: mike.may@slu.edu

3.12.1 About the Module

- Course: College Algebra
- Partner Disciplines: Business
- Required Technology: Spreadsheets

3.12.2 Institutional and Course Contexts

- Type/size of institution: Comprehensive private university
- Size of Class: 40
- Characteristics of Students: This is a class of business students who are not ready for business calculus.
- Mathematical Content: Using technology to solve system for a variable based on context.
- Purpose/Goal of the Module: The module introduces the students to Goal Seek, a tool in Excel that does what if analysis. For basic problems, this is equivalent to finding the root with a graphing calculator. The use of a spreadsheet allows students to look at more realistic functions of several inputs, where on a given day one input is chosen to be a variable and the others are parameters. While the students are still learning to “solve a basic equation for a variable” they are given tools for motivating the process with realistic situations.
- After and Before: Students have done basic entry of formulas into Excel. They have seen absolute and relative cell references (see the Excel supplement in the Human Trafficking module). They will be asked to use numeric and symbolic approaches to solving systems of equations and means of checking their work.
- Inspiration for the module: Students in college algebra have great difficulty with functions of several variables. Faculty from the partner discipline complained that they have difficulty with the concept of parameter and variable.

3.12.3 Partner Discipline Background

All of the exercises look at the present value of a stream of payments stretching into the future. The basic concept is that a payment in the future is worth the amount you would need to invest today to get the future amount. If the interest rate is r as a decimal, then $GF = 1 + r$ is referred to as the growth factor, and a payment n periods in the future is discounted by multiplying by GF^{-n}.

The initial problem is designed to familiarize the student with Goal Seek. It asks about computing a payment when we assume no interest. It is a simple division problem and can be done without any technology. The second exercise looks at three payments with an interest rate of 10%. Algebraically, the principal is still linear in payment.

The second level of difficulty notes that the principal is really a function of payment amount, the growth factor, and the number of payments. Additionally, the function is complicated enough that the students will have difficulty correctly computing the function, let alone doing what if analysis. Furthermore, it is easy to envision scenarios where each variable has a turn as the one that counts. There are multiple lessons here. The obvious lesson is the what if analysis and finding the input that gives the desired output. Excel will solve problems numerically that are impossible to solve by algebraic means. The practical lesson is that good spreadsheet practice uses longer and more meaningful variable names, and works in small steps rather than trying to make one big equation. The modeled practice creates a template that can be reused and is easier to understand. The hidden lesson concerns parameters and variables. We can change what we are solving for at will. It is worth noting that loan payments are typically made on a monthly basis, but interest rates are quoted at an annual rate, compounded monthly. For computing a monthly rate, it is assumed that all months have the same length and are 1/12th of a year. The growth factor is 1 plus the growth rate.

The final level of difficulty is when the "equation" is defined by a process given as a stream of simple steps rather than by a formula. For this we use an amortization table, finding how much is still due on a load partway through the loan period in case someone wants to settle. (E.g., Someone with a 30-year mortgage wants to move after 8 years.) Intuitively it is easy to understand that for each period you want to add interest on the balance and subtract the payment on the loan. The payment should be set so the balance at the end is zero. For a 30-year mortgage, that is a lot of steps.

Business faculty want the instructions to be sparse. They want to simply state the problem and have the students give an solution. This module has added instructions in parentheses that were not in the original module. In the business college algebra class, this was the third worksheet in a series. The students have already been introduced to the mechanics of using Excel. To use this as an independent worksheet more detailed procedural instructions have been added.

In all these cases we want the student to understand that for a given function or process, we often want to find the input that produces a desired output.

It is also worthwhile to note some that good practice with a spreadsheet has a different format from the standard practice the students have experienced in math and what math teachers will expect.

- Mathematicians make symbol manipulation effective by using one letter function and variable names. The better spreadsheet practice is to build templates that can be reused over time for similar problems. Templates should use longer, easy to understand names or labels.
- Part of the task is creating a spreadsheet having the work neatly laid out, with easy to follow steps.
- When using a tool like Goal seek in teaching mathematics, it is often worthwhile to make a second copy of the cells that will change with goal seek, so the user can see before and after,
- Electrons are cheap. Mathematicians should resist the urge to put a complicated formula in one cell rather than repeating the step many times and using small steps that are easier to understand. The students need to be taught that templates should be designed where another student can understand the template a month or two later,

3.12.4 Implementation Plan

Formal Learning Objectives

- Use Goal Seek in Excel to do What If Analysis
- Make the connection between two graphs intersecting and the difference of the functions being zero.
- Work with multivariate functions where all but one of the inputs are fixed parameters for a given problem, but the role of being the variable changes for different problems with the same setup.
- Introduce students to problems with a many step process where solving for a variable symbolically is not practical.

Materials and Supplementary Documents There is an Excel file that gives fully worked problems available at `https://mathstat.slu.edu/~may/ModelingWithExcel/`. The file is titled Intro-Excel-III.

Time Required The exercise was done in groups in a 50-minute class period. Groups finished outside of class with a one week time limit.

Implementation Recommendations I find it very useful to do any technology project in groups because the students do a certain amount of technical support for each other. The students need to be in a room where they all have computer access. Technical issues, like adding documentation to a workbook with text boxes need to be addressed. If they are not handled before this module, then they need to be explicitly addressed in class.

Depending on the skill level of the class, how much the use of technology is incorporated into the course, and other course considerations, the instructor may want to provide the students with a template that has some of the Excel coding already filled in. Such templates are available at the companion site.

Alternative Solutions This project should work on Google Sheets. The financial examples in the second half of the project can be replaced with other functions of several inputs where several inputs could be the variable on any given day. The instructor may want to provide a pre-formatted spreadsheet with some of the work done. the instructor may also want to make the module part of writing a report with an Excel attachment.

Common Errors and Questions

- Students often have problems with using quick fill and using absolute and relative cell references.
- In part 3, where M is produced as a function of P, r, and t, students will have difficulty understanding that M is the quantity that needs to vary in Goal seek.

3.12.5 Additional Information

This project is part of a larger shift in a modeling based college algebra course where the standard situation is to start with a data set, build a model, use the formula to make predictions, and give solutions in context of the original data. Most students have been trained that math is simply about manipulating formulas. The change in viewpoint makes the students feel better about math, but it needs to be made explicit.

Students seem to pay more attention to breaking complicated functions down with explicit attention to order of operations. Also, this module elicits a newness to thinking about free variable, dependent variable, and parameter, and how those are artificial distinctions.

Present Value of Money and Goal Seek in Excel

Module with Solutions

Once we have of formula for a function, we often want to know what input produces a particular output. Given cost as a function of quantity, we want to know how many items we can afford to produce. This referred to as what if analysis. In Excel this is done with the tool Goal Seek. Goal Seek is part of what if analysis. In simple terms, it asks which x gives a specified y. As with any new technology, we work through a series of examples of increasing complexity. We start with an algebraic example more easily done by hand to understand how to set up problems. We then look at examples that could easily be done with a graphing calculator. Then we look at an example with several variables where the use of a spreadsheet helps keep things straight. Finally, we look at an example with a complicated construction where Goal Seek is the easiest way to do the problem.

Solving One Equation with Goal Seek

If we look at loan payments, the principal is a function of the payment. When we are given a function, prin(pay), we often want to find the payment-value that gives a specified principal-value. We want to solve the equation $\text{prin(pay)} = b$. In Excel this is called what-if analysis and is done with a tool called Goal Seek.

1. Following the normal pattern of a math class, we start by solving an equation that is easier to do by hand, without invoking Goal Seek. We start with the principal equation there are 4 equal payments and no interest. Then $\text{prin(pay)} = 4pay$. If we want to find the payment for a \$1000 loan, we want to solve $\text{prin(pay)} = 1000$. A simple computation tells us the solution is $\text{pay} = 250$. However, we want Excel to find the solution. We start by building a table of values for pay and prin(pay) that includes our guessed solution. I am going to start with a guess of $\text{pay} = 100$. I reproduce Table 3.1 in Excel:

   Table 3.1. Setting Up Goal Seek: Simple Example

	A	B
1	pay	prin(pay)
2	100	= 4*A2

   Now I invoke Goal Seek. I look in the data tab and find the what-if-analysis drop down menu. I select Goal Seek. I would like to change the value of B2 to 1000 by changing cell A2. Click OK and Excel solves the problem.

   Teaching Note: Students should do this themselves and get the solution of $\text{pay} = \$250$.

2. Now I use Goal Seek on a problem that I can't do by hand but could easily do with a graphing calculator. I am going to assume 10% interest per year and the payment over 3 years. I want to solve:

   $$\text{prin(pay)} = \text{pay}\left(1 + \frac{1}{1.1} + \frac{1}{1.1^2}\right) = 1000$$

   Looking at a graph or a table of the function, I have solutions near $\text{pay} = 300$. (Produce a table or a graph with pay going from -500 to 500 by 100 and see where the value of prin(pay) changes sign.) Once again, I produce a table. I have double rows so I will remember my starting points after doing the computation. With Goal Seek, change B3, to 1000 by changing cells A3.

   Teaching Note: Students should do this themselves and get the solution of $\text{pay} = \$365.58$.

Variables and Parameters

When looking at the relationship between monthly payments and the principal, we really have the principal as a function of three variables, the payment, the growth factor, and the number of payments. In the formula, we have the growth factor $GF = 1 + r$ where r is the interest rate, and n is the number of payments:

$$\text{prin} = \text{pay}\left(\frac{1 - \frac{1}{GF^n}}{1 - \frac{1}{GF}}\right)$$

Depending on the situation we may want to treat the payment, the growth factor, and the number of payments as something to be solved for.

Good spreadsheet practice is to use meaningful names and only do one step per cell. For example, if I want to find the principal given 5 payments of 500 and an interest rate of 10%, the calculation using good spreadsheet practice looks like Table 3.2.

Table 3.2. Goal Seek: More Complicated Example

	A	B
1	Payment	500
2	Growth Factor	1.10
3	1/GF	= 1/B2
4	n	5
5	top	= 1-B3 ∧ B4
6	bottom	= 1 - B3
7	fraction	= B5/B6
8	principal	= B7*B1

Using good spreadsheet practice and Goal Seek, solve the following:

1. If the principal =\$2000, $n = 5$, and the growth factor is 1.1, what is the payment?

   Solution: \$479.63

2. If the principal =\$2000, the growth factor is 1.1, and the payment is \$300, how many payments do I have to make?

   Solution: $n = 9.774$ payments.

3. If the principal =\$2000, the payment is \$400 and there are 7 payments, what is the growth factor?

   Solution: 1.1298, which corresponds to an interest rate of 12.98%. Note that this problem would be impossible to solve analytically since it would require solving a 7th degree polynomial equation.

Given a loan with principal P, growth factor $1+r$ (a monthly interest rate of r), and a term t in months, the monthly payment on a loan is:

$$M = \frac{Pr(1+r)^t}{(1+r)^t - 1}$$

We are typically given an annual (nominal) interest rate which has to be divided by 12 in order to obtain the monthly rate.

For an example in which we seek the monthly payment for a loan of \$200 to be repaid in 48 monthly payments at an annual interest rate of 8%, the Excel setup should look something Table 3.3 on the following page.

Use Goal Seek and good spreadsheet practice to solve the following:

4. I can afford a monthly payment of \$200 and the interest rate is 8% annually. How much can I borrow for a 4-year loan?

   Solution: I can afford to borrow \$8,192.38

5. I can afford a monthly payment of \$200 and the interest rate is 8% annually. I need to borrow \$5,000 How long will I be paying off the loan?

   Solution: I would pay off the loan in 27.44 months.

Table 3.3. Goal Seek: Monthly Payment on a Loan

	A	B
1	Annual Rate	8%
2	Monthly Rate	= B1/12
3	Growth Factor	= 1+B2
4	Principal	200
5	Months	48
6	factor ∧ months	= B3 ∧ B5
7	top	= B4*B2*B6
8	bottom	= B6 - 1
9	Monthly Payment	= B7/B8

6. I am offered a loan of $5,000 with a monthly payment of $200 and the loan is for 4 years. What is the annual interest rate?

   Solution: I would be paying 36.69% interest annually (!). Note that this problem would be impossible to solve analytically as it would require solving a degree-48 polynomial equation.

Goal Seek for a More Complicated Setup

The problems above can all be done with a graphing calculator because they have a straightforward formula with one input variable, one output variable, and several constant parameters. For the final example we look at a case where we have repeated steps and it is not clear how to enter a formula into a calculator.

1. Producing a loan table: There are formulas for finding loan payments and balances. They are complicated and hard to remember. It is relatively easy to make a loan table with Excel. The end of period balance is the beginning of period balance plus the interest minus the payment. The new beginning of period balance is the old end of period balance.

   Suppose we want to make a loan table (this is called **amortizing the loan**) for a loan of $200,000 to be repaid with monthly payments over 15 years (180 months) at an annual interest rate of 4%. We will start with a guess of a payment of $2000 and make a table in Excel. The table should look like Table 3.4 below. Note that the table also uses a mixed cell reference.

   When the formula =B8*B$4 in C8 is copied into C9 it becomes =B9*B$4. The B8 is a relative reference, In C8 it means the entry to the left. When copied to the ninth row it becomes B9, the entry to the left. In contrast B$4 means the item in the column on the left but in row 4. When copied down it still will refer to row 4.

Table 3.4. Goal Seek with a Loan Table

	A	B	C	D	E
1	Principal	$200,000		Final Balance	= OFFSET(E8,B2,0)
2	Months	180			
3	Annual Rate	= 0.04			
4	Rate	= B3/12			
5	Payment	$2000			
6					
7	Period	Begin Balance	Interest	Payment	End Balance
8	1	= B$1	= B8*B$4	= B$5	= B8+C8-D8
8	1	= E8	= B9*B$4	= B$5	= B9+C9-D9

We extend the table through 180 periods. Offset is a command that lets us move an entry to a nice place rather than search through pages of numbers. The entry in E1 says, go to cell E7, then go down B2 rows and over 0 columns. Put the value of that cell in E1. The initial payment was a guessed number. Now we want to make the Final Balance 0 by changing the payment with Goal Seek.

Do so, and record the payment

Solution: $1479.38

2. Reset the table and answer the following questions:

   (a) If I can afford a $2,000 monthly payment and 15-year mortgages charge an annual rate of 4%, how big a principal can I afford?

   Teaching Note: To let goal seek find the rate, it must start as a number without an equals sign.

   Solution: $270,384

   (b) If the loan is for $200,000 and my payment is $2,000 for 180 months, what is the interest rate?

   Solution: The rate is 0.73% monthly or 8.76% annually.

3.13 So Trendy! The Calculus of New Product Adoption

Jody Sorensen, Stella Hofrenning, and Suzanne Dorée
Augsburg University
Contact: sorensj1@augsburg.edu

3.13.1 About the Module

- Course: Calculus I
- Partner Disciplines: Business and Economics
- Required Technology: web browser, spreadsheet

3.13.2 Institutional and Course Contexts

- Type/size of institution: Small Comprehensive University.
- Size of Class: 28 or fewer
- Characteristics of Students:Undergraduates from diverse mathematical backgrounds. Some are first year students who took Precalculus or Calculus in high school, others came through Augsburg's Precalculus course. Most students are intending to major in STEM or Economics/Business.
- Mathematical Content: Rate of change in applied contexts, limit at infinity, derivative rules, and effects of parameters on function behavior.
- Purpose/Goal of the Module: This module is designed for students to see S-shaped models, including the logistic model and slightly more sophisticated Bass model; to see an application of Calculus in the social sciences; and to practice with derivative rules with parameters and in an applied context.
- After and Before: Students need to know how to take derivatives to complete this module, including exponential functions, the product/quotient rules, and the chain rule. They also need to be able to take limits at infinity. This project does not use the differential equation for the logistic model, so if you're going to cover that in your course it might be nice to do this beforehand.
- Other Prerequisites: Some familiarity with basic spreadsheet usage might be helpful.
- Inspiration for the Module: Working with an economist, and discovering the interactive graphic we use.

3.13.3 Partner Discipline Background

Every year tens of thousands of new products are introduced to the U.S. market, and most fail. (Depending on the definition of "fail", estimates range from 70–97% of new products fail.) Of those products that succeed, some are adopted quickly and others much more slowly over time. Recognizing early in the process which products are on the path to success and predicting adoption rate is especially challenging for innovative products and technology, where historical sales data of similar items is not available.

Innovators and Imitators

Consumer needs and tastes continually change. Producers respond by introducing new products to the market every year. For example, consumers are becoming more health conscious and want healthful alternatives to high-calorie or artificially-sweetened soda. In response, producers offer an increasing variety of flavored carbonated water drinks. Similarly, consumers want more options for accessing television programs, and so producers are offering a wider range of technological ways to access programming.

As new products (or technologies) hit the market, some consumers are quick to adopt a new product. These consumers are called "innovators". For example, early adopters of technology are considered innovators. More precisely, **innovators** are consumers who adopt a new product independent of the decisions made by others.

As time goes go, other consumers are influenced by the innovators through positive word-of-mouth, including communication over social media, or first hand observation (e.g., you see a friend's vacuuming robot and decide to buy one.) Those consumers who purchase a product (or technology) in this manner or are called "imitators." More precisely, **imitators** are consumers who adopt a new product because others consumers have.

The Bass Model of Diffusion

The Bass Model of Diffusion (Bass 1969) is the standard in Economics for forecasting the adoption patterns of products and then using that information to predict market success. Some products exhibit slow growth in the market, while other products experience rapid sales growth when first introduced in the market (for example, hot trends like the iPod©). Some products have huge sales growth and then quickly fade (fads such as Fidget Spinners or Pet Rocks), or have slow initial sales but eventually succeed. The Bass Model works well for a variety of products such as durable goods (such as refrigerators, air conditioners), computers and technology products, medical products and other consumer goods and services.

There are three important parameters that define a Bass Model for a product: the potential market, M; the coefficient of innovation, p; and the coefficient of imitation, q.

The **potential market** M is the number of customers who will eventually adopt the product. Determining a working value for M takes into account many factors, such as knowing which customers are likely to want (and be able to afford) the product, what the consumption rate of the product is (i.e. product lifetime), and usage information such as will this product be purchased per household or per capita. Different products usually have different values of M. The model assumes that all potential buyers eventually purchase the product and that the maximum number of potential buyers or adopters is fixed.

The **coefficient of innovation**, p, captures the impact of the external influences (primarily advertising). Technically, p is the proportion of consumers who adopt the product independent of other consumers. The model assumes that p does not change over time.

The **coefficient of imitation**, q, captures the influence of prior adopters. Technically, q is the proportion of consumers who adopt the product as a result of other consumers having adopted the product. The model also assumes that q does not change over time.

For a given product, write

$$f(t) = \text{the number of consumers who adopt at time} t.$$
$$F(t) = \text{the number of consumers who have adopted up to time} t.$$

Note that in a continuous version of this model, that $F(t) = \int_0^t f(s)ds$ and so $F'(t) = f(t)$. Also, no product is adopted at time $t = 0$, so $F(0) = 0$.

The **Bass model for diffusion** (Bass 1969) calculates the number of consumers who adopt at time t as the sum of the number of innovators who adopt at time t and the number of imitators who adopt at time t. Specifically:

$$f(t) = \underbrace{p(M - F(t))}_{\text{innovators}} + \underbrace{q\frac{F(t)}{M}(M - F(t))}_{\text{imitators}}$$

Note that the quantity $M - F(t)$ calculates the outstanding market, the number of consumers who have not yet adopted the product up to time t. The innovators component takes the proportion p of the outstanding market $M - F(t)$. The imitators component takes the proportion q of the outstanding market $M - F(t)$ scaled by the proportion of the market that has already adopted the product $F(t)/M$, since imitation is assumed to grow proportionally to the proportion of adoptees.

This equation often appears in the following format:

$$\frac{f(t)}{M} = p + (q - p)F(t) - \frac{q}{M}F(t)^2$$

Solutions and Logistic Approximation

Solving the differential equation (of Riccati type) yields the equation

$$F(t) = M\frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}$$

In practice, for trendy products p is likely to be much smaller than q. For example, the innovation coefficient is often less than one percent ($p = 0.01$) whereas the imitation coefficient averages around 38 percent ($q = 0.38$). (Majahan, et al. 1995)

We can rewrite the equation as

$$F(t) = \underbrace{\frac{M}{1 + \frac{q}{p}e^{-(p+q)t}}}_{\text{logistic}} - M\underbrace{\frac{e^{-(p+q)t}}{1 + \frac{q}{p}e^{-(p+q)t}}}_{\text{pertubation}}$$

Therefore, for trendy products (where $q >> p$), the equation can be approximated by the logistic equation

$$F(t) \approx \frac{M}{1 + \frac{q}{p}e^{-(p+q)t}} = \frac{M}{1 + Be^{-kt}}$$

where $B = \frac{q}{p}$ and $k = p + q$. The module focuses on this approximation.

3.13.4 Implementation Plan

Formal Learning Objectives

- Interpret the graph of a logistic function: identify points of inflection, identify that an increasing function has a positive derivative, estimate values of the derivative from a graph, tell the story of a function from its graph, and identify a graph that is not logistic
- Use the general formula of a logistic function: calculate the limit as t goes to infinity, apply derivative rules to calculate the derivative, and find parameter values for a logistic function through specific points
- Use a spreadsheet program: populate a column of a spreadsheet, create a scatterplot in a spreadsheet, graph a function in a spreadsheet, compare actual to model data graphically and numerically
- Use a graphing tool, such as the free online graphing calculator Desmos with sliders for parameter values to explore the impact of parameters on the model.

Materials and Supplementary Documents Access to website (`ourworldindata.org/technology-adoption`) and the Excel spreadsheet microwave.xls, available on the volume's webpage.

Time Required This module is designed to take about an hour, with students working in groups of 2 to 4. This should allow most students time to complete the module and be ready to turn it in.

Implementation Recommendations This module is done in a weekly calculus lab period where students apply the content they've been learning in new situations. No introduction is necessary, just let them dig in and get down to it. They are asked to access a website (`ourworldindata.org/technology-adoption`), and to analyze some data in a spreadsheet program such as Excel or Google Sheets. They could be given or shown a detailed color picture of the graph from the website if internet access is a challenge. If students don't have access to technology, the instructor could work through and display the spreadsheet steps at the appropriate times.

Students get to apply derivative rules in context on an example from the wider world. They are also exposed to the logistic model before coming across the differential equation, and have an intuitive example for when they do encounter that topic again.

Common Errors and Questions Students typically do well with this module. Questions 6 and 7 allow for quite varied responses so some students like a little bit of reassurance there. It can be a challenge for students to pull knowledge from different parts of the course and apply it in a new setting. This transference is a goal of our calculus labs.

3.13.5 Additional Information

This module could be modified for different level courses. Students in a Precalculus could work with the logistic equation and explore various logistic models by graphing. Alternatively, the module could also be modified to include the logistic differential equation (or the full Bass model for diffusion), for either a calculus or differential equations course, as the derivation of the differential equation in terms of the innovator vs. imitator affect is intuitive.

An extension would be to ask about the time of peak sales (largest derivative), which for the Bass model is given by the elegant formula

$$t* = \frac{\ln(q) - \ln(p)}{p+q}$$

While the adoption of microwave technology provides a good fit to our model, instructors could easily revise the module to feature another of the products fitting the model. This would allow instructors to create new versions easily, as needed, to avoid repeating the exact same module.

The data used to create the graphic is easily downloadable, and seems to be regularly updated as well.

References

Bass, F. (1969). A new product growth for model consumer durables. *Management Science 15*(5) 215–227. doi.org/10.1287/mnsc.15.5.215.

Bass diffusion model, Wikipedia en.wikipedia.org/wiki/Bass_diffusion_model

Mahajan, V., Muller, E., & Bass, F. (1995). Diffusion of new products: Empirical generalizations and managerial uses. *Marketing Science 14*(3) G79–G88. doi.org/10.1287/mksc.14.3.G79. [As cited by wikipedia]

Our World in Data website, /ourworldindata.org, based at Oxford University.

So Trendy!

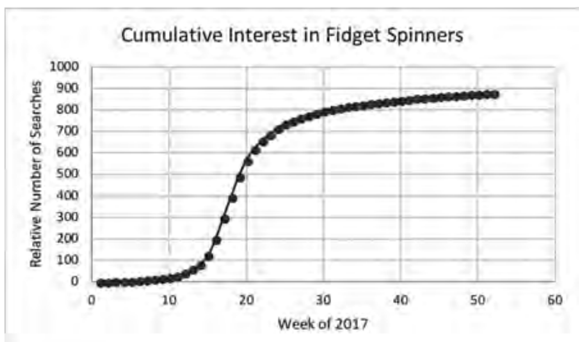
Module with Solutions

We are going to look at how fast trends are adopted. You can think about cultural fads (like the use of the word *basic*) or technology adoption (like smart phones). When a popular new trend starts, the growth often ramps up exponentially.

1. Consider a context like the sales of fidget spinners. Explain why the sales can't continue to grow exponentially in the long term.

   Solution: There are only a finite number of potential customers. At some point, everyone who wants a fidget spinner will have one.

2. One good place to find data for questions like this is at trends.google.com, which shares data on the number of searches for a particular topic. The data is relative, so it's not quite clear what the units are, but it gives a good idea of how interest changes over time. The chart below shows searches for *fidget spinner* over the course of 2017. Growth that begins as exponential but then turns an levels off is called "S-shape". The most famous type of S-shaped growth is called *logistic* growth. (Not logarithmic!)

   In what week does the interest seem to switch out of exponential growth?

   Solution: Approximately week 18, when the concavity changes

3. If $F(w)$ is the function that fits this data for $w > 0$, let's think about $F'(w)$. First of all, what can you say about the sign of the derivative (for $w > 0$)? Explain how your response makes sense in terms of the story.

   Solution: For $w > 0$, $F'(w) > 0$, since F is always increasing. Since F is cumulative, it can't go down.

4. What can you say about the value of $F'(w)$ for large values of w? Explain how your response makes sense in terms of the story.

   Solution: For large w, $F'(w)$ is small (close to 0). The rate of growth slows down eventually.

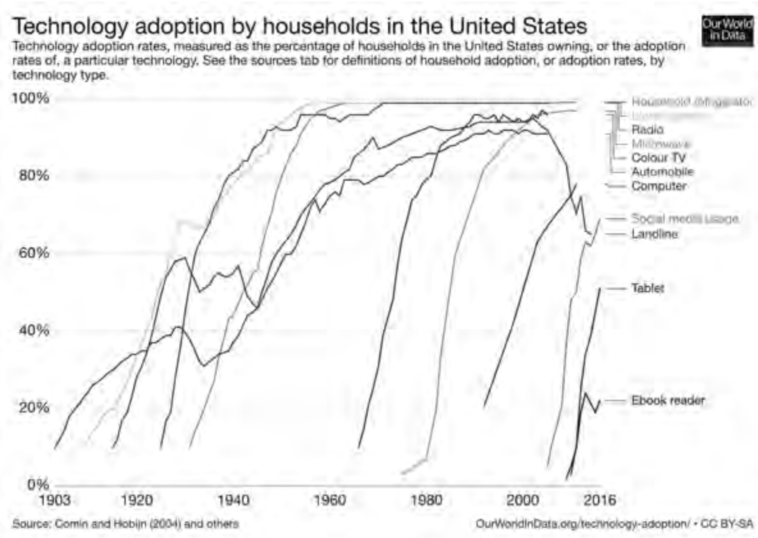
5. At what value of w do you expect $F'(w)$ to be the largest? How would you describe what happens in that week?

   Solution: $F'(w)$ would be largest at about week 18. In that week, interest in fidget spinners peaks. After that, it declines.

6. Let's look at a new context. The graphic below shows the adoption of various technologies in the US over the years. You should go to the interactive graph online by searching for "our world technology adoption." In this case the vertical axis is the percentage of households that have adopted the technology.

   Most of these technologies are adopted fairly quickly and then level off, so it is reasonable to consider a logistic growth model. Pick one example product and explain its growth over time. Like "Refrigerators first appeared in the 1930s. By 1950 . . . "

   Solution: Many possible solutions.



7. Find a product that doesn't fit this logistic model. Can you come up with a reason why?

   Solution: An obvious choice is the landline phone. After 2000, popularity decreases because people choose to only use cell phones.

8. The theory of trend adoption in Economics leads to the equation

$$F(t) = \underbrace{\frac{M}{1+\frac{q}{p}e^{-(p+q)t}}}_{\text{logistic}} - M\underbrace{\frac{e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}}_{\text{pertubation}}.$$

   However for trendy products it ends up that q is much larger than p and so the second term is very small. So we will assume that $F(t)$ has a logistic equation of the form:

$$F(t) = \frac{M}{1+Be^{-kt}}$$

   where M, B and k are all positive constants that depend on the product. Such constants are called parameters. Looking at the equation, can you see what happens to $F(t)$ in the long run? In other words, calculate $\lim_{t\to\infty} F(t)$ and interpret its meaning.

   Solution:

$$\lim_{t\to\infty} \frac{M}{1+Be^{-kt}} = M$$

9. We'll also want to know how F is changing, so calculate $F'(t)$ from this general formula (Chain rule, anyone?). Interpret the meaning of the derivative. Does this fit with your observations on F' from 3 and 4?

   Solution:

$$\begin{aligned} F(t) &= M\left(1+Be^{-kt}\right)^{-1} \\ F'(t) &= -M\left(1+Be^{-kt}\right)^{-2} \cdot \left(-Bke^{-kt}\right) \\ &= \frac{MBke^{-kt}}{\left(1+Be^{-kt}\right)^2} \end{aligned}$$

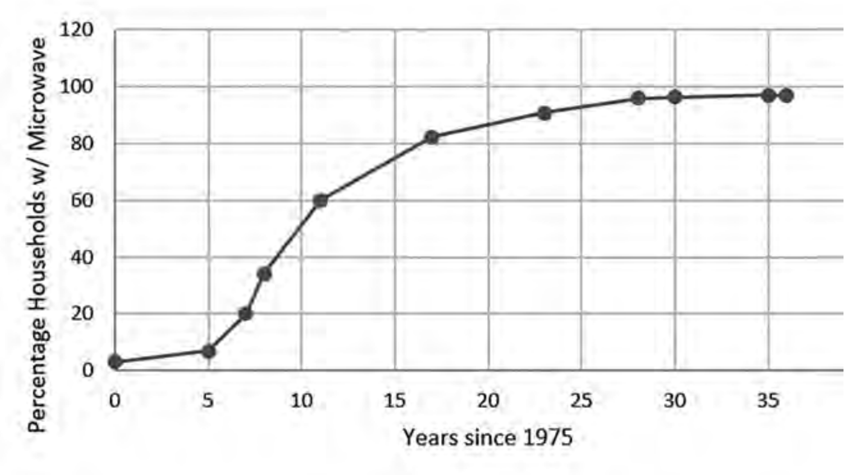
   #3 always positive

   #4 small for large t as top approaches 0 and bottom approaches 1

10. Let's focus on microwaves as a specific example. Access the microwave data file and open it in a spreadsheet. Our first goal is to fill in the 'Years since 1975' column. Click on the cell next to 1975 and enter '=A2-1975' . You should get a zero. Then click and drag that formula down to fill in the column. Your last entry should be 36.

11. Now make a connected scatter plot of columns 2 and 3. Sketch a little picture of that plot here.

Solution:



12. Now let's use the data we have to fit $F(t)$ for microwaves to a logistic model. It looks like the data levels off at about 98%. Which parameter in the logistic formula should be 98? Explain. Why do you think this happened?

Solution: M should be 98, since that represents where the function levels off.

13. To find the other two parameters, let's use two data points. The first point we have is $F(0) = 3$. Let's use that point along with $F(11)$ to calculate the two missing constants. Show your work. What does it mean in regards to the real world?

Solution: $F(0) = 3$ give $B = \frac{95}{3}$. Then $F(11) = 60$ give $k \approx .3556$.

14. We can add this function to our plot in our spreadsheet. For example, if I wanted to plot $F(t) = \dfrac{50}{1 + 12e^{-0.3t}}$, I would type '=50/(1+12*EXP(-0.3*B2))' and hit Enter. Then I would drag that formula down to fill in the column. Let's enter this formula using the values of M, k, and B you found in #12 and #13. Finally make a scatter plot showing the microwave data along with the function. Do they match up exactly? Pretty well? Terribly?

Solution: They should match pretty well, but it's not a perfect fit.

15. Use your formula from #9 to calculate $F'(15)$ for your microwave function. Interpret this solution, including units.

Solution:

$$F'(t) = \frac{98 * 31.67 * .3556e^{-.3556t}}{(1 + 31.67e^{-.3556t})^2}$$
$$F'(15) \approx 4$$

This means that the percentage of households with microwaves was growing by about 4% per year in 1990.

3.14 Titration

Pavel Bělík, Joan Kunz, Jody Sorensen, and John Zobitz
Augsburg University
Contact: sorensj1@augsburg.edu

3.14.1 About the Module

- Course: Calculus I
- Partner Disciplines: Chemistry
- Required Technology: web browser, spreadsheet

3.14.2 Institutional and Course Contexts

- Type/size of institution: Small liberal arts university
- Size of Class: 28 or fewer
- Characteristics of Students: Undergraduates from diverse mathematical backgrounds. Some are first year students who took Precalculus or Calculus in high school, others came through Augsburg's Precalculus course.
- Mathematical Content: Applications and numerical approximations of 1st and 2nd derivatives, concavity, inflection points, and working with data.
- Purpose/Goal of the Module: This module is designed for students to see a logistic-like curve arising naturally from data measured in one of our chemistry lab courses, apply calculus concepts to identify significant points on the curve, and connect these points to the goal of the chemistry lab – identifying an *equivalence point* in the process of titration.
- After and Before: This module comes, by design, later in the semester when unrelated topics are covered. It assumes the knowledge of the 1st and 2nd derivative and their interpretations. The goal is to recall and apply the concept of concavity some time after students first learned it.
- Other Prerequisites: Familiarity with basic spreadsheet functions; computation of (average) rates of change from data.
- Inspiration for the module: Working with a chemist; using data obtained in chemistry labs; using calculus to identify the acid used in the titration.

3.14.3 Partner Discipline Background

Titration is a method of chemical analysis in which a reactive substance is slowly added to another substance, and some property of the combined substance is measured. It is a foundational piece of analytical chemistry. Almost any fast reaction can be used to analyze an unknown chemical for quantitative information such as its concentration or molar mass. In performing a titration, the chemist holds a known solution (titrant) of known concentration in a buret. The buret is an analytical piece of glassware that dispenses precisely known volumes. The titrant is added into a flask containing the unknown solution (analyte). A visual indicator or a pH probe is used to determine the end of the titration (stoichiometric equivalence) and beyond. Each titration produces a characteristic S-shaped plot of some aspect of the analyte vs. volume of titrant added.

Acid-base titrations take advantage of the rapid and complete reaction between an acid and a base. This reaction can be represented as

$$\mathrm{HA} + \mathrm{OH}^- \rightarrow \mathrm{H_2O} + \mathrm{A}^-,$$

where HA is the acid being titrated, OH^- is the base being added, and H_2O is the water and A^- is the salt, products of the reaction. The ratio of acid to base is one molecule of acid to one molecule of base (or scaled by Avagadro's number, one mole of acid to one mole of base). Equivalence in the reaction occurs when the number of moles of base added equals the number of moles of acid originally present in the unknown. Solution concentration is typically reported in molarity (M), which is the number of moles of a chemical per liter of solution. The number of moles can be calculated from the solution concentration and the volume used:

$$\text{moles} = \text{molarity} \times \text{volume (in liters)}.$$

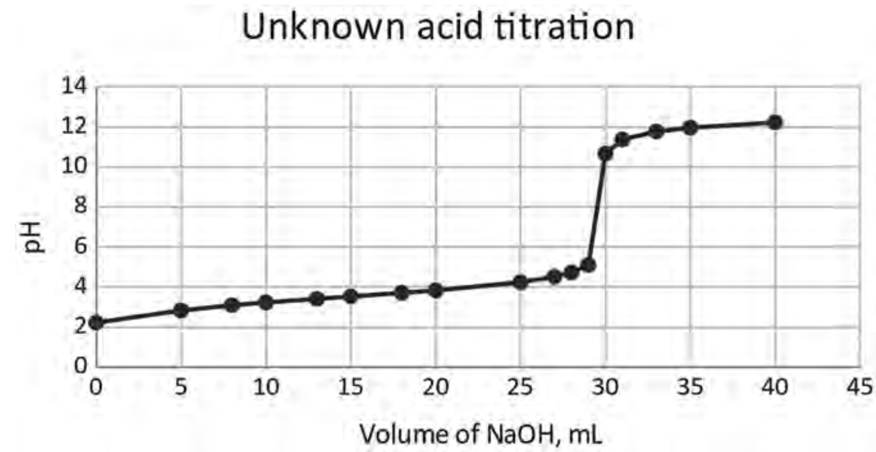
At the equivalence point in the reaction,

$$\text{moles of acid} = \text{moles of base}$$

or

$$\mathrm{M}_{\text{acid}} \times \mathrm{V}_{\text{acid}} = \mathrm{M}_{\text{base}} \times \mathrm{V}_{\text{base}}.$$

In a typical titration, the unknown acid is placed in a flask. A base of well-known concentration is placed in the buret. A pH meter is placed in the flask with the unknown acid. The pH is measured and plotted against the volume of base added to the acid. The result is an S-plot, as shown below.



The pH at the equivalence point can vary from the expected neutral 7.0 because of the hydrolysis of the salt, A^-, in water. The equivalence pH is best determined from the inflection point in the S-curve, which occurs in the near-vertical portion of the plot. By approximating the first derivative and looking for its maximum, the equivalence point can be calculated. When the equivalence point is known in terms of pH and volume of base, the concentration of the unknown acid can be calculated, as described above.

There is more information to be had from the titration curve! There is a second slight inflection point half-way to the equivalence point, in the nearly horizontal portion of the S-curve preceding the equivalence point. From this inflection point, the dissociation constant for a weak acid can be determined. Consider the equilibrium for a weak acid:

$$\mathrm{HA} \rightleftharpoons \mathrm{H}^+ + \mathrm{A}^-.$$

The extent of this equilibrium is represented by the acid dissociation constant, K_a, given by

$$K_\mathrm{a} = \frac{[\mathrm{H}^+][\mathrm{A}^-]}{[\mathrm{HA}]},$$

where [X] denotes the concentration of the quantity X. This K_a is valuable information about relative acid strength. At halfway to the equivalence point in the titration, half of the weak acid HA has been reacted to form its salt, A^-, and half of the weak acid remains. This means

$$[\mathrm{HA}] = [\mathrm{A}^-]$$

at half-way to the equivalence point. It also means that at half-way to the equivalence point we have $K_\mathrm{a} = [\mathrm{H}^+]$, or, in terms of base-10 logarithms,

$$\mathrm{p}K_\mathrm{a} = -\log_{10} K_\mathrm{a} = \mathrm{pH},$$

since, by definition, $\mathrm{pH} = -\log_{10}[\mathrm{H}^+]$.

3.14.4 Implementation Plan

Formal Learning Objectives

Mathematical: Identifying an inflection point from measured data, identifying that it corresponds to a point with largest derivative, estimating values of the 1st and 2nd derivative from the data.

Statistical: Using a spreadsheet: creating a scatter plot, creating a new column with average rates of change from the data, creating a column with approximations of the 2nd derivative, comparing and analyzing results from using 1st and 2nd derivative approximations.

Materials and Supplementary Documents Spreadsheet file `Titration.xlsx`; internet access for an instructor to play any short YouTube video on titration.

Time Required This module is designed to take about an hour, with students working in groups of 2 to 4. This should allow most students time to complete the module and be ready to turn it in.

Implementation Recommendations This module is done in a weekly calculus lab period where students apply in new situations the content they've been learning in class. No introduction is necessary, but it may be a good idea to show the class a video of the process of titration at the beginning (in the past we used `https://www.youtube.com/watch?v=g8jdCWC10vQ`). The students will analyze data in a spreadsheet program such as Excel or Google Sheets. If students don't have access to technology, the instructor could work through and display the spreadsheet steps at the appropriate times. Students get to apply derivatives in context, on an example from chemistry, one they perhaps experienced in an earlier class.

Alternative Solutions It is useful to connect the 3 different approaches to identifying the equivalence point: (1) identifying where the pH curve has largest slope; (2) identifying where the 1st derivative has the largest value; and (3) identifying where the 2nd derivative changes sign, or finding the inflection point.

Common Errors and Questions Students in our cohorts ask about how to correctly approximate the 1st derivative using the data, after which they usually have no problems with the 2nd derivative approximation. The last question, in which the students are going to identify the acid used in the titration, also elicits questions because it requires several steps and includes terminology from chemistry.

3.14.5 Additional Information

This lab reinforces students' working knowledge of 1st and 2nd derivatives and their meaning and applicability in the context of data. It provides additional exposure to working with spreadsheets which is one of the soft skills developed in the course. Finally, there is a discovery aspect at the end, when the acid used in the titration experiment is identified.

This module can easily be adapted down to even precalculus level by emphasizing average rates of change and focusing on the point/interval where the rate of change is at a maximum. With the same goal, it can also be used in a calculus course prior to introducing second derivatives and inflection points. Laterally, the module can be extended by providing data obtained from titrations with other acids and having groups of students work with different data sets. Although we haven't done so, one could also attempt to fit the data with various model functions (see, e.g., Eaker 2000) and use them to calculate the various derivatives and quantities of interest.

Finally, the inflection point half-way to the equivalence point could also be incorporated into the lab and a connection to the dissociation constant, K_a, could be made explicit for the interested students.

References

Eaker, C. W. (2000). Fitting and analyzing pH titration curves on a graphic calculator. *The Chemical Educator 5*(6) 329–334. `doi.org/10.1007/s00897000426a`

Titration

Module with Solutions

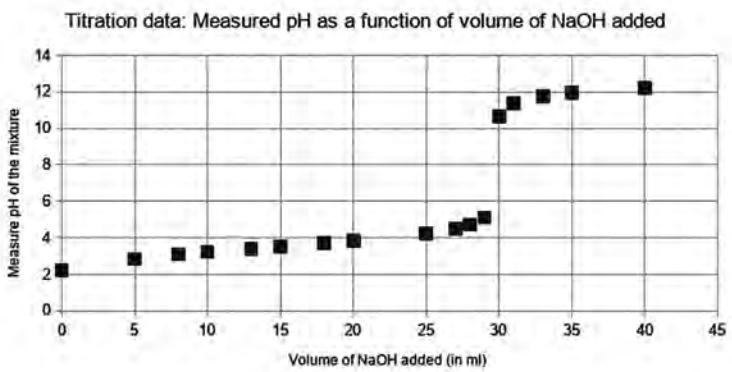
Titration is a method of chemical analysis in which a reactive substance is slowly added to another substance, and some property of the combined substance is measured. This procedure is typically taught in a General Chemistry course.

We will be looking at an example of acid-base titration. Specifically, 25 mL of an unknown monoprotic weak acid is titrated against 0.105M NaOH (which is a strong base). This means we are adding the base to the acid using a buret in a slow and precisely controlled manner. We measure the pH of the solution after each addition. (A solution with pH of less than 7 is acidic, and a solution with pH of more than 7 is a basic.) pH itself has no units.

1. Open the titration spreadsheet file. Explain in words what the point $(13, 3.41)$ means.

   Solution: This means that when 13 mL of NaOH are added to the acid, the pH is 3.41.

2. Create a scatter plot of the data. Do not connect the points. Sketch your plot below.



3. Imagine a smooth function $P(x)$ connecting the data points. What do the variables P and x represent, including units?

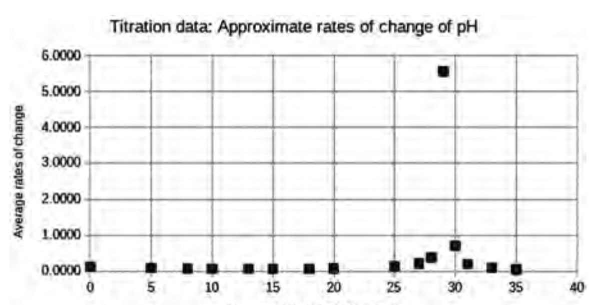
   Solution: P is the dependent variable, representing pH of the mixture (dimensionless, no units); x is the independent variable representing the volume of NaOH in milliliters.

4. How would you describe the function $P(x)$ in mathematical terms? (Consider terms like positive/negative, increasing/decreasing, concave up/concave down, maximum/minimum, etc.)

   Solution: On the shown interval, $P(x)$ is a positive function; it is increasing so doesn't have any local maxima or minima; it appears to change concavity twice: it starts off concave down, around $x = 15$ it changes to concave up, and then around $x = 30$ it changes to concave up again.

5. Write down a formula for the average rate of change of a function connecting points (x_1, y_1) and (x_2, y_2). Create a new column and calculate the rate of change between each pair of points. Put the first computed value into cell C4 and create a scatter plot of the computed values as a function of the values in column A. What observations can you make about your rate of change results? List a few things.

   Solution: AROC $= \dfrac{y_2 - y_1}{x_2 - x_1}$. All AROCs are positive. The values are initially quite small (below 0.1), then increase to the largest value of about 5.56, and decrease again below 0.1.



6. In titration we are interested in the *equivalence point*. **Chemically** this is when enough of the base has been added to completely neutralize the acid. **Mathematically** this is when the *rate of change is at its maximum*. How do we determine the equivalence point from P? How do we determine the equivalence point from P'? What does the data suggest? Estimate the equivalence point with these two approaches using your data and graph.

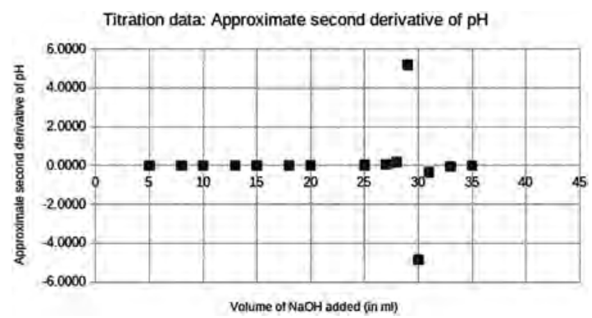
   Solution: Using P, we want to identify where it has the largest slope. Using P', we would be looking for its maximum. The data suggests this happens between $x = 29$ and $x = 30$ ml.

7. Suppose we have a formula for the titration function $P(x)$. What calculation would we want to do to find the equivalence point? What is our mathematical name for that type of point?

   Solution: To maximize $P'(x)$, we could solve the equation $P''(x) = 0$ for x and check that $P''(x)$ changes from positive to negative there. This is then an inflection point of $P(x)$.

8. Go back to your spreadsheet and calculate estimates of the second derivative. Put the first computed value into cell D5 and again create a scatter plot of the computed values. How many times does the second derivative change sign? What does that tell us about the concavity of $P(x)$? How would we determine the equivalence point from P''? What does the data suggest about the location of the equivalence point?

   Solution: The second derivative changes sign twice, starting off negative, then changing to positive, then to negative again. So, $P(x)$ changes from concave down to concave up to concave down. The equivalence point will be where the sign of the second derivative changes the second time. The data again suggests this is between $x = 29$ and $x = 30$ ml.



9. We are interested in when the second derivative is equal to zero. This actually happens a few times with our data set, but there is only one equivalence point. With our data there are a number of consecutive places early on where the second derivative is zero. To figure out why this happens, suppose we have a function whose second derivative is always zero. So we have $\dfrac{d^2y}{dx^2} = 0$. What is $\dfrac{dy}{dx}$ for this function?

   Solution: For such a function $\dfrac{dy}{dx} = c$, where c is any constant. That is, y has a constant slope.

10. What conclusion can be made from your observations in part 9.? Conclusion: If $f(x)$ is a ____________________ function on an interval, then $f''(x) =$ ______ on that interval.

    Solution: linear, 0

11. We are not interested in the places where the titration curve is linear, we are interested in where the concavity of the curve changes, specifically where it changes from concave up to concave down. Between what two data points does that happen? What is the concavity before and after the change?

    Between $x =$_______ and $x =$_______ the second derivative changes from ________ to ________.

    Solution: Between $x = 29$ and $x = 30$ the second derivative changes from *positive* to *negative*.

12. What is your best guess of the equivalence point for this titration? *(Your solution should be the mL of NaOH at the desired point.)*

    Solution: To provide a single solution, we can average the two x-values to get about 29.5 ml of NaOH.

13. The spreadsheet has a second page with more precise data near the equivalence point. Redo the scatter plot, and the first and second derivative calculations.

    Between $x =$_______ and $x =$_______ the second derivative changes from ________ to ________.

    What is your new estimate of the equivalence point?

Solution: Between $x = 29.7$ and $x = 29.75$ the second derivative changes from *positive* to *negative*. Averaging the two x-values, we get approximately 29.725 ml of NaOH.

Now solve for the value of K_a.

Solution: So, since the equivalence point is $x \approx 29.725$, half of that is 14.8625, and from our measured data the pH there is about 3.53 *(using $x = 15$ in the data)*.

Setting $3.53 = -\log_{10}(K_\mathrm{a})$, we obtain $K_\mathrm{a} = 10^{-3.53} \approx 2.95 \times 10^{-4}$.

14. Here's a list of common acids:

Acid	Use	K_a
benzoic acid	food preservative	6.5×10^{-5}
formic acid	ant bite sting	1.7×10^{-4}
acetylsalicylic acid	aspirin	3.0×10^{-4}
acetic acid	vinegar	1.8×10^{-5}
hydrofluoric acid	glass etching	7.1×10^{-4}

Which acid do you believe was used in our titration?

Solution: Based on our value $K_a \approx 2.95 \times 10^{-4}$, we conclude that the most likely acid was the acetylsalicylic acid (aspirin).

3.15 The Glucose Problem

Mary Beisiegel, Lori Kayes, Devon Quick, Steve Dobrioglo,
Michael Dickens, and Alena Arounpradith
Oregon State University
Contact: beisiegm@oregonstate.edu

3.15.1 About the Module

- Course: 10-week Calculus course (Differential Calculus)
- Partner Disciplines: Biology and Chemistry
- Required Technology: None

3.15.2 Institutional and Course Contexts

- Type/size of institution: Large, public research institution
- Size of Class:Lecture class has approximately 110 students; recitations have approximately 33-35 students each.
- Chararcteristics of Students: Predominately engineering-degree and STEM-degree intending undergraduate students; 60% have taken calculus prior to enrolling at Oregon State.
- Mathematical Content: Limits, rates of change, velocity, acceleration, algebraic rules of differential calculus and derivatives of polynomial, rational, and trigonometric functions, related rates, optimization.
- Purpose/Goal of the Module: The scenario is designed so that students can see how several concepts from calculus can be used to Solution many questions about a science-related problem over the course of the term. Students will use limits, first derivatives, and second derivatives to gain a deep understanding of glucose absorption.
- After and Before: Varies according to the week of the term; for the first set (week 2) of questions about glucose, students will need to understand exponential growth and decay as well as limits as a way for describing long-term behavior of a function; for the second and third sets of questions (weeks 5 and 9) students will need to know how to take derivatives.
- Other Prerequisites: None
- Inspiration for the Module: Working with biologists, we discovered that investigating function behavior through rates of change could deepen students' understanding of this applied problem.

3.15.3 Partner Discipline Background

Rationale

- Glucose, or sugar, is an important energy source that is required by all cells of the body in order to function properly.
- Glucose comes from the foods we eat, such as carbohydrates. When our diet does not contain enough glucose, our cells make it from fats and proteins.
- When we eat food, the food is broken down into glucose and absorbed into the bloodstream to be used by the cells

Teaching Notes

- What is glucose?
  - Glucose is sugar. The absorption of glucose is measured in milligrams per hour (mg/h). However, the function P is a percentage of the total amount of glucose that is absorbed, so the units for the function P is the percentage of glucose.

  - Glucose is a 6-carbon carbohydrate molecule that is used by almost all living organisms to provide energy for cellular processes.
  - In common language, when people talk about "blood sugar" they are referring to glucose dissolved in the blood.

- How do we obtain glucose?
  - Humans (and your dog, cat, etc.) obtain most glucose from their diets (we can convert other molecules into glucose if needed). Though we do not usually eat straight glucose (except in honey), we consume glucose connected to other molecules (for example, table sugar is glucose molecules connected to fructose molecules) and then use our digestive system to break it down to individual glucose molecules which then cross GI tract cells to enter the blood (absorption).
  - *Glucose cannot cross cell membranes without specific transport proteins embedded in cell membranes - this means that the absorption process has a limit to which it can work as determined by the rate of sugar breakdown and then saturation of these transporters.*

- What is diabetes mellitus?
  - People with diabetes mellitus have trouble regulating their blood sugar because they cannot move it out of their bloodstream into tissue cells. They either lack the hormone (insulin) needed for transport of glucose into tissue cells (cannot make it, Type I diabetes) or their cells are insensitive to the hormone (Type II diabetes).
  - For further information on diabetes, please see the website: `http://www.diabetes.org`

3.15.4 Implementation Plan

Formal Learning Objectives

For the first installment:

1. Investigate the impact of the sign of the exponent in the exponential function and the impact of the exponent on the model for glucose absorption
2. Use limits to understand the long-term behavior of glucose absorption

For the second installment:

1. Apply derivative rules to find the rate of change of glucose absorption.
2. Investigate the sign of the derivative and understand the meaning of the sign of the derivative for glucose absorption

For the third installment:

1. Apply derivative rules to find the second derivative of the glucose absorption function
2. Interpret the second derivative in the context of glucose absorption
3. Explore concavity of the glucose function using the second derivative and make sense of concavity in the context of glucose absorption

Time Required We expect each set of questions for the glucose problem to take students approximately 10-15 minutes. See implementation recommendations for suggestions on supplementing a recitation module or homework assignment with other problems.

Implementation Recommendations We recommend that you use this problem either as a recitation module or homework assignment in the way it is intended – that students see this applied problem multiple times over the course of a term as they learn new calculus concepts. This lowers the demand of reading and understanding new applied problems, increases familiarity with the problem, and this strategy illustrates that applied problems can be rich and complex and that calculus can be used to Solution many questions about a specific problem.

We recommend that this problem is presented with other problems in a recitation module or homework assignment. For example, in the second installment, once students have learned most/all rules for derivatives, we include several problems on taking derivatives. Typically, the recitation activities include about 5–10 more procedural problems and a few applied problems, depending on the section of the material and what problems are included in that section.

We also recommend that you use the introductory information to the problem each time it is used in a recitation module or homework assignment (e.g., from the sentence about glucose as a simple sugar through the definition of P). Students should be given time to process the information.

TAs or instructors should familiarize themselves enough with the issues of glucose and sugar absorption so that they can Solution students' questions should they arise.

Finally, for the first exploration of the value of a in the exponential function $P(t) = 1 - e^{at}$, instructors might consider having students use graphing software that allows them to easily experiment with different values of a.

Common Errors and Questions In terms of glucose, the misconception is that absorption is not limited: The more food you eat, the more glucose you absorb. This is not true because the cells have a limit to how much glucose they can absorb.

Another misconception is that all carbohydrates are glucose. This is not true. Carbohydrates are molecules of which there are many forms.

The Glucose Problem

Module with Solutions

The Glucose Problem: Installment One

Glucose is a simple sugar that serves as an energy source in organisms. Scientists have determined that glucose absorption from the gastrointestinal tract (GI tract) in rats and rabbits can be modeled by the exponential function $P(t) = 1 - e^{at}$.

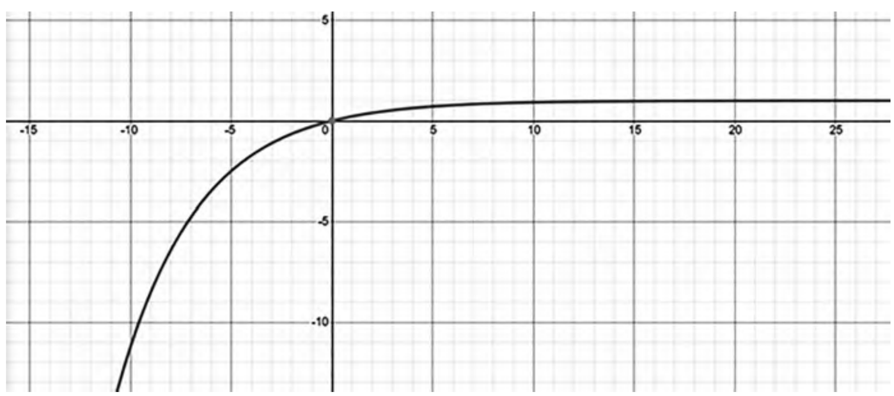
- t is time in hours
- a is a constant
- P is the *fraction* of glucose absorbed by the GI tract and in this case the range of P is $[0, 1]$. When $P = 1$, that means that all glucose has been completely absorbed, when $P = 1/2$, half of the glucose has been absorbed, and when $P = 0$ none of the glucose has been absorbed.

Complete the following:

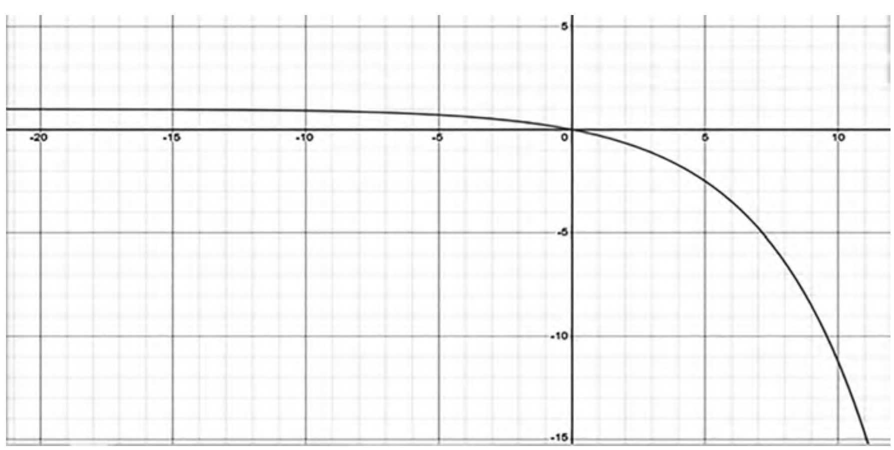
1. Create two graphs of the function $P(t) = 1 - e^{at}$: one graph where a is negative (try -0.25) and one graph where a is positive (try 0.25). Looking at these graphs and given how P was described, do you think a should be negative or positive?

   Solutions: The following graphs were generated in Desmos using $a = \pm 0.25$.

   $a < 0$:



   $a > 0$:



Since P is the portion of absorbed glucose, and because of the fact as time increases so should absorption until all glucose is absorbed, a should be negative.

2. Assuming $a < 0$, what is $\lim_{t\to\infty} P(t)$?

   Solution: Since $\lim_{t\to\infty} e^{at} = 0$ when $a < 0$, $\lim_{t\to\infty} P(t) = 1 = 100\%$.

3. What does this limit mean in the context of a rat or rabbit absorbing simple sugars?

   Solution: The function in this scenario, $P(t)$, shows the proportion of glucose absorbed over time, and as t goes to infinity (more hours pass in this system), we see that glucose will continue being absorbed until all of the glucose has been absorbed by the organism.

The Glucose Problem: Installment Two

Glucose is a simple sugar that serves as an energy source in organisms. Scientists have determined that glucose absorption from the gastrointestinal tract (GI tract) in rats and rabbits can be modeled by the exponential function $P(t) = 1 - e^{at}$.

- t is time in hours
- a is a constant
- P is the *fraction* of glucose absorbed by the GI tract and in this case the range of P is $[0, 1]$. When $P = 1$, that means that all glucose has been completely absorbed, when $P = 1/2$, half of the glucose has been absorbed, and when $P = 0$ none of the glucose has been absorbed.

Complete the following:

1. Find $\frac{dP}{dt}$

   Solution: $\frac{dP}{dt} = -ae^{-at}$

2. Using some algebra, show that you can rewrite the derivative entirely in terms of a and $P(t)$. Show your work.

   Solution: $\frac{dP}{dt} = -a\,(1 - P(t))$

   The tricky part here is observing that $1 - (1 - P(t)) = P(t)$

3. Using the formula you found in part 2, interpret $\frac{dP}{dt}$ in the context of rats and rabbits absorbing glucose. How does the function $P(t)$ behave in the long term? Why does it behave that way?

   Solution: As t approaches infinity, $P(t)$ approaches 1. Therefore:

   $$\lim_{t\to\infty} \frac{dP}{dt} = \lim_{t\to\infty} [-a\,(1 - P(t))] = 0$$

   This makes sense in the context of rats and rabbits because as time goes on less and less glucose is being absorbed per hour until reaching 0 glucose absorption per hour.

4. Is $\frac{dP}{dt}$ always positive, always negative, or sometimes positive/sometimes negative? What does that mean for the rate of change of glucose absorption?

   Solution: Always positive: we have $a < 0$ so $-a > 0$, therefore $-ae^{at} > -0$ for all t. This means that in this current system, glucose is always being absorbed but never is leaving the organism.

The Glucose Problem: Installment Three

Glucose is a simple sugar that serves as an energy source in organisms. Scientists have determined that glucose absorption from the gastrointestinal tract (GI tract) in rats and rabbits can be modeled by the exponential function $P(t) = 1 - e^{at}$.

- t is time in hours
- a is a constant

- P is the *fraction* of glucose absorbed by the GI tract and in this case the range of P is $[0, 1]$. When $P = 1$, that means that all glucose has been completely absorbed, when $P = 1/2$, half of the glucose has been absorbed, and when $P = 0$ none of the glucose has been absorbed.

Complete the following:

1. Find $\frac{d^2P}{dt^2}$

   Solution: Since $\frac{dP}{dt} = -ae^{at}$, then $\frac{d^2P}{dt^2} = -a^2e^{at}$.

2. What does $\frac{d^2P}{dt^2}$ mean in the context of rats and rabbits absorbing glucose?

   Solution: The second derivative is negative but increases towards zero over which means that the change in the rate of absorption of glucose decreases towards zero, and eventually levels off to zero when all the glucose is absorbed.

3. Does the concavity of the glucose function change at any point? Why or why not?

   Solution: The concavity does not change since $\frac{d^2P}{dt^2} = -a^2e^{at}$ never changes sign.

3.16 Graphical Analysis in Biomedical Engineering using ECG Signals

Maila Brucal-Hallare, Makarand Deo, and Shahrooz Moosavizadeh
Norfolk State University
Contact: smoosavizadeh@nsu.edu

3.16.1 About the Module

- Course: Calculus 1
- Partner Disciplines: Biomedical Engineering

3.16.2 Institutional and Course Contexts

- Type/size of institution: Historically Black College/University with about 5,000 students
- Size of Class: About 20 students
- Characteristics of Students: Students in engineering, mathematics, and physics programs who are mostly under-prepared mathematically. Such students appreciate seeing applications outside mathematics for motivation and further understanding.
- Mathematical Content: Graphical analysis skills in Calculus 1 to include continuity, differentiability, intervals of increase/decrease, intervals of concavity up/down; some mathematical modelling.
- Purpose/Goal of the Module: Emphasize importance and utility of graphical analysis outside typical Precalculus 1 and Calculus 1 syllabi
- In the Syllabus: Description of how electrical and mechanical activities of the heart are translated to graphs and visualizations can be introduced as early as the second week of the semester.
- Other Prerequisites: Graphical analysis concepts developed in Precalculus
- Inspiration for the Module: The Principal Investigator from the engineering department is an expert in modeling of the heart's electrical disorders and cardiac signal analysis.

3.16.3 Partner Discipline Background

In this module, we believe that we are able to send the message to the students that what we do in a Calculus class have many applications in real-life. An electrocardiogram (ECG or EKG) is a graphical visualization of a heart's electrical signals in a two-dimensional coordinate plane. The horizontal axis represents time while the vertical axis represents voltage recordings during heart beats. The discovery of electrocardiography is attributed to Willem Einthoven, a Dutch physician, who was awarded the 1924 Nobel Prize in Physiology and Medicine "for the discovery of the mechanism of the electrocardiogram" (see Figure 3.1). He proposed that if we connect electrodes on left arm, right arm and left leg, then these three limb points form an imaginary equilateral triangle (also called the Einthoven Triangle; illustrated in Figure 3.2), and the heart can be assumed to lie in the center of the triangle. These electrodes (usually plastic patches that stick to the skin) do not send electricity into the body and are connected to an ECG machine by lead wires.

Figure 3.2 shows various electrical leads derived from the three limb electrodes and a central reference terminal. One complete cardiac cycle consists of the following stages defining a normal pattern of propagation of electrical signals in the heart tissue. A heart beat originates in a region in the right upper chamber (atria) of the heart, called sinoatrial (SA) node. The SA node consists of specialized cells which generate an electrical stimulus regularly, 60 to 100 times per minute under normal conditions. These signals are spread in both the atria, causing them to contract; this pushes the blood from the atria chambers to the ventricles (bottom chambers). The electrical excitation of atria is represented by the first deflection in ECG, called P wave (see Figure 3.3). The electrical signal is then picked up by the junction node between the atria and ventricles, called atrioventricular (AV) node: this adds a slight delay to the signal. The electrical impulse is rapidly conducted by a specialized electrical network, called the Purkinje system, and is spread over the ventricular tissue almost synchronously. This causes the simultaneous contraction of both the ventricles that pumps

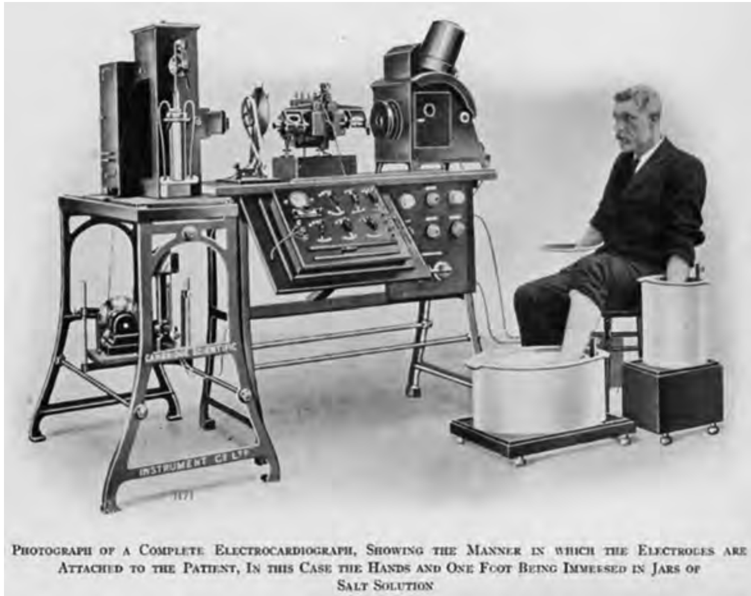


Figure 3.1. Willem Einthoven was honored with the 1924 Nobel Prize in Medicine due to his discovery of electrocardiography. Source: `https://en.wikipedia.org/wiki/Willem_Einthoven`

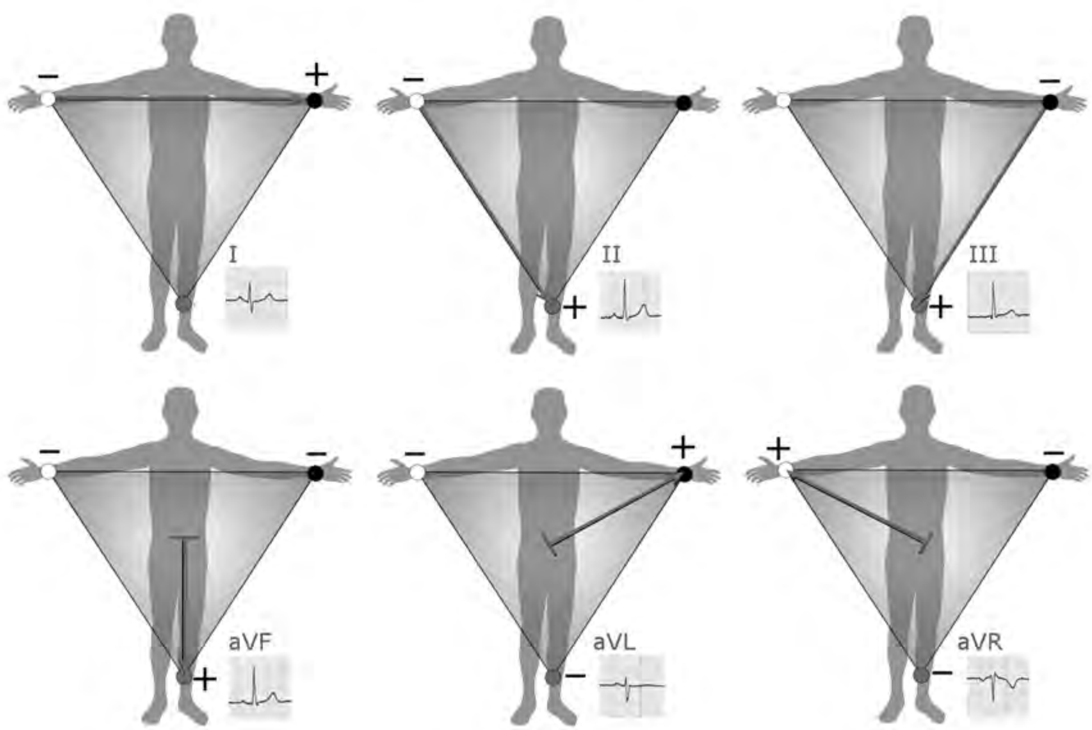


Figure 3.2. The Einthoven triangle is formed by taking three limb points with the heart in the approximate center. Leads I, II, III are usually called standard limb leads while leads aVF, aVL, aVR are called the augmented limb leads. There are six other leads. Although these are the 12 leads used in producing a complete ECG, one principle governs the conversion of electrical module data to time-voltage plots. Source: `https://en.wikipedia.org/wiki/Einthoven`. Image by Npatchett - Own work, CC BY-SA 4.0, `https://commons.wikimedia.org/w/index.php?curid=39235282`.

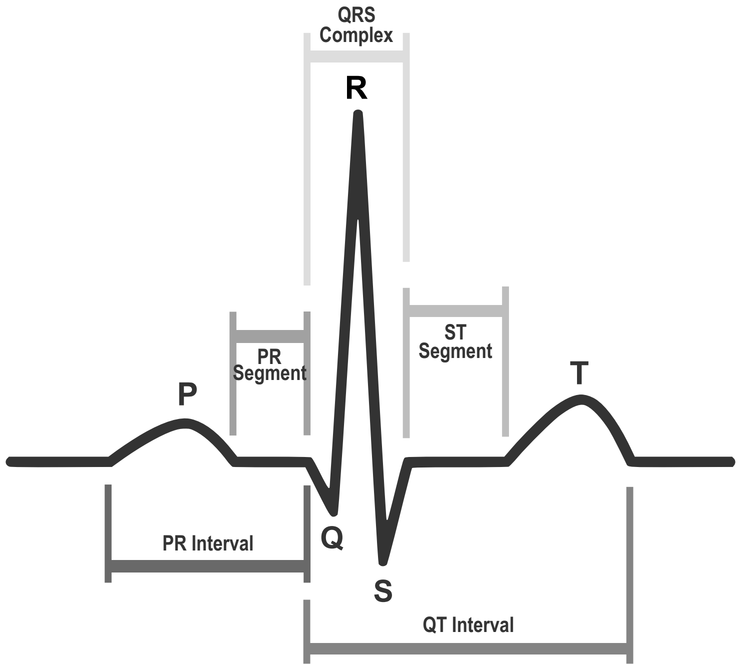


Figure 3.3. A schematic of a PQRST wave segment in an ECG. The P bump captures the atrial contraction, the QRS complex captures the ventricular contraction, while the T bump captures the re-polarization of the ventricles. Source: `https://en.wikipedia.org/wiki/QRS_complex`. Created by Agateller (Anthony Atkielski), converted to svg by atom. - SinusRhythmLabels.png, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1560893

blood from right chamber to the lungs for oxygenation and from the left chamber to the entire body. The synchronized contraction of ventricles is seen on ECG as a big QRS wave (Figure 3.3). The electrical signals gradually recede from the ventricular tissue making them relaxed again which is reflected in the ECG signals as another deflection, called T wave. Thus the normal ECG signals consists of a repeating pattern of P-QRS-T waves as shown in Figure 3.3.

The ECG signals captured by the limb electrodes can be explained by the vector projection of the cardiac vector onto the three sides of the Einthoven Triangle (i.e. primary ECG leads I, II, and III). The vector projection describes the movement of the electrical impulse during a cardiac cycle. When the movement of the electrical impulse is towards the positive electrode, then the corresponding segment of the ECG is above the baseline (positive deflection); when the movement of the impulse is towards the negative electrode, then the corresponding segment of the ECG is below the baseline (negative deflection); and when the movement of the impulse is perpendicular to the electrodes, then the corresponding segment of the ECG is parallel to the baseline (no deflection). Nurses, emergency medical technicians (EMTs), and physicians are trained to read and analyze the ECGs of patients in order to detect irregular heartbeats and heart's electrical rhythm disorders, called arrhythmia or dysrhythmia. In particular, when EMTs observe complete disruption of QRST waves, they must decide as soon as possible if use of a defibrillator is necessary. A defibrillator is a special equipment that aims to restore heart rhythm by applying a cross-field electrical shock.

3.16.4 Implementation Plan

Formal Learning Objectives

- Demonstrate the utility of graphing and graphical analysis in biomedical engineering.
- Apply graphical analysis techniques learned in Calculus I to analyze segments of ECGs.
- Use an online graphing calculator to analyze graphs.

Materials and Supplementary Documents

1. A short video explaining how ECGs are produced may be helpful in visualizing how the PQRST waves are created.
2. Worksheets can be developed for analyzing segments of ECGs.

Time Required It is suggested that the mathematical modeling and ECG fundamentals be introduced as early as possible during the semester, which may take about 15 to 20 minutes. Then, the ECG samples can be used to illustrate concepts and develop conceptual understanding whenever Calculus 1 graphical analysis techniques are introduced in the class.

Implementation Recommendations

1. This biomedical engineering application is a great way to demonstrate how mathematical modeling and graphical analysis are essential skills in the real world. There are several YouTube videos that explain how ECGs are plotted but there is a danger of being discouraged by the various biology and engineering terms. The key is to remember that an ECG is simply a time-versus-voltage graph and that the electrical module of a normal heart is an almost periodic natural phenomenon.
2. It is best to use an online graphing calculator while completing some questions in the module. Sketching the graphs without the use of an online graphing calculator will require more time. For example, we have found that students appreciate `http://desmos.com` and `http://geogebra.org`.
3. We also recommend creating questions where there are many possible solutions. See sample questions in the worksheet.
4. There is a website which goes into much more detail about the interpretation of features of ECG graphs: `https://bit.ly/ECGMath`

   Those using this module may consult this webpage for further interpretation of the derivative and convexity, or they may select portions for students to read in conjunction with the module, asking for them to interpret the derivatives and convexity that they find.
5. We have prepared a very short video that shows a simulation of how each heartbeat produces a P-QRS-T wave in the ECG. This video may be shown in class, if necessary.

Common Questions Students were hooked with the subject because the ECG is something that they see in real life or in the movies. Several questions that were put forward were: How is the ECG or EKG different from EEG? What do EMTs look for in the ECG to help them decide whether it is time to use the defibrillator pump? How do pacemakers work? How do ECGs of babies (due to their age) and athletes (due to their physical module) compare with other demographics? Why is it that certain segments of the P-QRS-T wave look like a line?

Adaptations to Online Learning We created a desmos module for the ECG signals project. This module was launched as a Calculus I virtual quiz and received equal weighting as the other required quizzes. Since the questions in the desmos module were not personalized and were not designed to be adaptive, some students copied other students' work. When the desmos module was discussed in virtual class the next day, it was apparent that many students had forgotten some topics from Precalculus. The desmos module was then used as a starting point for review of these prerequisite concepts. This is the link to the desmos.com module:
`https://teacher.desmos.com/modulebuilder/custom/60531b14d7b3e8481988ca51`

3.16.5 Additional Information

1. We have used a small portable ECG machine to gather actual ECG of members of the class, but be aware that some students may not want to share their ECG data results. The instructor may even want to arrange a short trip to the nursing laboratory (if any) in order to have access to actual ECG equipment. Gathering ECG data is a non-invasive and safe procedure.

2. Our university has a strong nursing program. One of the nursing professors was willing to share some ECG practice strips that they use in teaching their nursing classes. Student nurses are trained in reading ECGs to gather various information like heartbeats, shapes of the waves, lengths of the segments, etc. Unfortunately, nursing instructors do not use Calculus 1 graphical analysis methods to teach their nursing students. It may be true that employing Calculus language may help our nursing instructors and students in investigating and interpreting ECGs.

3. The internet has a plethora of non-copyrighted images of segments of ECGs that can be used to create various questions and worksheets for a Precalculus and a Calculus 1 course. When used in explaining graphical analysis concepts, students are reminded that graphs are not just abstract objects but they are used to represent real-life phenomenon. In this particular biomedical engineering example, "Every squiggle in the ECG graph is a heartbeat!"

3.16.6 Reference

Conover, M.B. (2003) *Understanding Electrocardiography*, 8th edition. Mosby.

Graphical Analysis in Biomedical Engineering using ECG Signals

Module

Pictured below in Figure 3.4 is a cartoon of a P-QRS-T wave on an ECG segment from a heart having a normal rhythm. Observe the shape of the QRS complex. Also, observe the general flatness of the portions of the curve labelled as PR segment and ST segment.

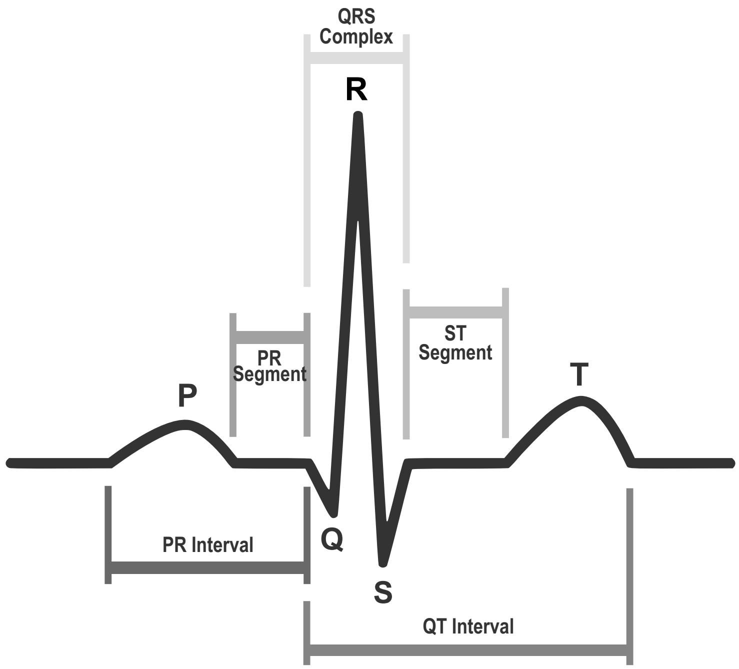


Figure 3.4. A P-QRS-T wave. Created by Agateller (Anthony Atkielski), converted to svg by atom. - SinusRhythm-Labels.png, Public Domain, https://commons.wikimedia.org/w/index.php?curid=1560893

Variations in the morphology of any parts of the P-QRS-T wave indicate possible irregular rhythms or heart disorders. An example of a variant of the QRS complex is what physicians call a Qr variant, where the first wave is large and negative (Q), the second wave is small and positive, and there is no S wave.

This may indicate an irregular rhythm in the heart. An example of a more grave scenario is when the PR segment becomes depressed and the ST segment assumes a saddle-shape. Such morphology may indicate existence of the heart disease called pericarditis, an inflammation of the lining of the heart called pericardium.

1. Use `http://desmos.com` or `http://geogebra.org` to sketch an approximation of the Qr wave (See Figure 3.5). Decide where you want the origin (0, 0) to be. Try your best to sketch graphs that preserve the proportions as close as possible.

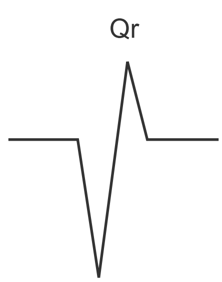


Figure 3.5. A variant of the QRS complex called a QR wave

(a) What is the domain of your graph? What is the range?

(b) The graph has four corners. At these points, we say that the graph is not differentiable. Use the definition of the derivative to show that the derivative does not exist at these points.

2. Figure 3.6 is a cartoon of an ECG segment of a person that suffers from pericarditis. To help you analyze the shape of the graph using the first and the second derivatives, several important points are labelled $P_i = (x_i, y_i)$ for $i = 1, 2, ..., 11$.

(a) Identify the critical points and the points of inflection.

(b) Complete the sign chart below for the first derivative of the picture in Figure 3.6. The first row indicates the interval that you are investigating. In the second row, determine whether the first derivative is positive, negative, or zero in each interval. In the third row, write a conclusion on whether the function is increasing, decreasing, or constant in each interval. Interpret what your Solutions mean in the real-life context.

Interval	(x_1, x_2)	(x_2, x_3)	(x_3, x_4)	(x_4, x_5)	(x_5, x_6)
First Derivative					
Conclusion					

Interval	(x_6, x_7)	(x_7, x_8)	(x_8, x_9)	(x_9, x_{10})	(x_{10}, x_{11})
First Derivative					
Conclusion					

(c) Complete the sign chart below for the second derivative of the picture in Figure 3.6. The first row indicates the interval that you are investigating. In the second row, determine whether the second derivative is positive, negative, or zero in each interval. In the third row, write a conclusion on whether the function is concave up or concave down. Interpret what your Solutions mean in the real-life context.

Interval	(x_1, x_2)	(x_2, x_3)	(x_3, x_4)	(x_4, x_5)	(x_5, x_6)
Second Derivative					
Conclusion					

Interval	(x_6, x_7)	(x_7, x_8)	(x_8, x_9)	(x_9, x_{10})	(x_{10}, x_{11})
Second Derivative					
Conclusion					

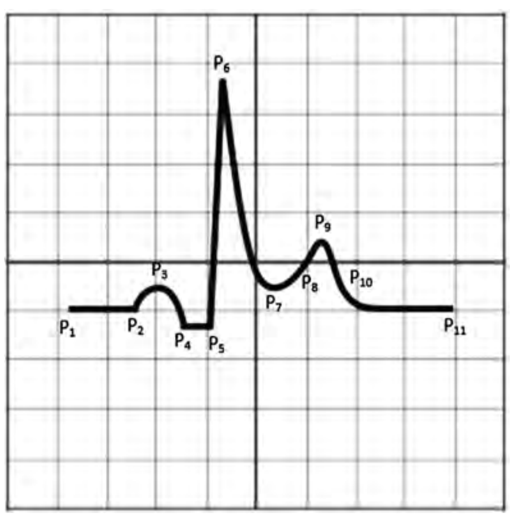
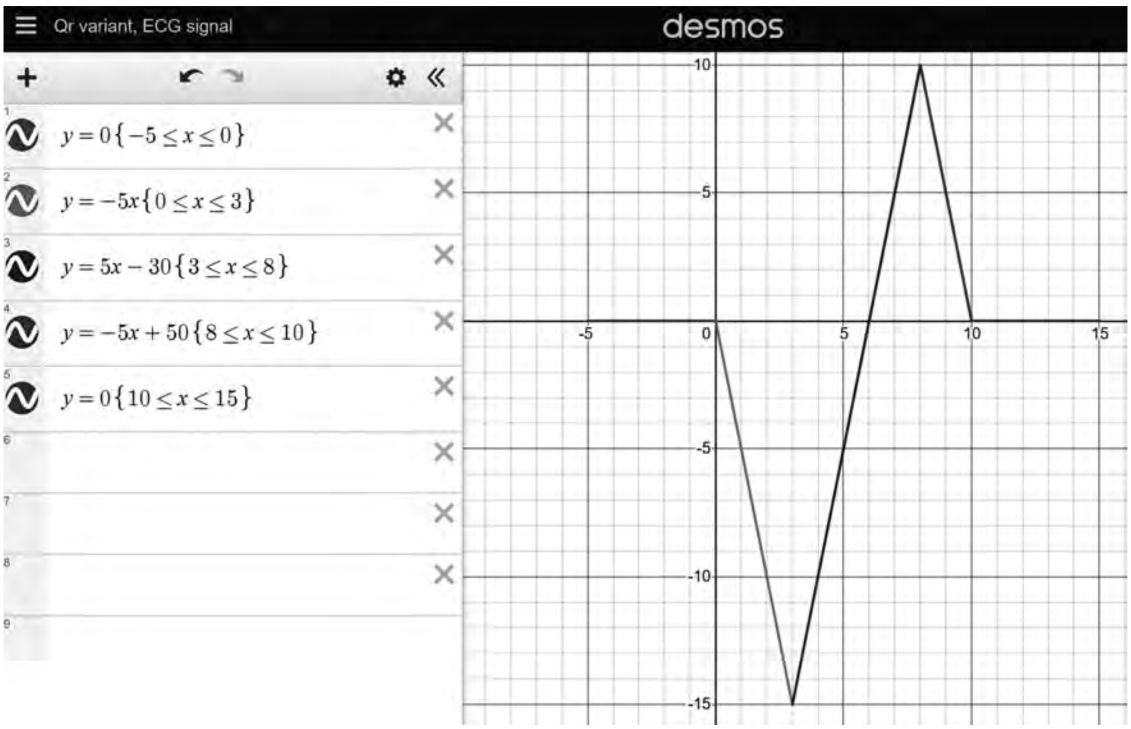


Figure 3.6. Pericarditis in the ECG

Graphical Analysis in Biomedical Engineering using ECG Signals

Module Solutions

1. There are many possible solutions. When presented as a group module, students will have a chance to learn from each other due to slight differences in their Solutions. Using `http://desmos.com`, a possible solution for the Qr variant is:



The domain is $[-5, 15]$ while the range is $[-15, 10]$. In this graph, the four points of non-differentiability are located at $(0, 0)$, $(3, -15)$, $(8, 10)$, $(10, 0)$. The graph is not differentiable at these points because:

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = 0 \qquad \text{while} \qquad \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = -5.$$

$$\lim_{x\to 3^-} \frac{f(x)-f(3)}{x-3} = -5 \qquad \text{while} \qquad \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = 5.$$

$$\lim_{x\to 8^-} \frac{f(x)-f(8)}{x-8} = 5 \qquad \text{while} \qquad \lim_{x\to 8^+} \frac{f(x)-f(8)}{x-8} = -5.$$

$$\lim_{x\to 10^-} \frac{f(x)-f(10)}{x-10} = -5, \qquad \text{while} \qquad \lim_{x\to 10^+} \frac{f(x)-f(10)}{x-10} = 0.$$

2. (a) Figure 3.6 has critical points at P_i where $i = 2, 3, 4, 5, 6, 7, 9$. The points of inflection are at P_8 and P_{10}.

   (b) Sign chart for the first derivative:

Interval	(x_1, x_2)	(x_2, x_3)	(x_3, x_4)	(x_4, x_5)	(x_5, x_6)
First Derivative	0	+	-	0	+
Conclusion	constant	increasing	decreasing	constant	increasing

Interval	(x_6, x_7)	(x_7, x_8)	(x_8, x_9)	(x_9, x_{10})	(x_{10}, x_{11})
First Derivative	-	+	+	-	-
Conclusion	decreasing	increasing	increasing	decreasing	decreasing

(c) Sign chart for the second derivative:

Interval	(x_1, x_2)	(x_2, x_3)	(x_3, x_4)	(x_4, x_5)	(x_5, x_6)
Second Derivative	0	-	-	0	0
Conclusion	none	down	down	none	none

Interval	(x_6, x_7)	(x_7, x_8)	(x_8, x_9)	(x_9, x_{10})	(x_{10}, x_{11})
Second Derivative	+	+	-	-	+
Conclusion	up	up	down	down	up

Every semester, there is at least one student who asks about the concavity of a linear function. This example is a good opportunity to discuss this question.

3.17 Finding Riemann Sums with a Spreadsheet

Mike May and Debbie Pike
Saint Louis University
Contact: mike.may@slu.edu

3.17.1 About the Module

- Course: Business Calculus with Excel
- Partner Disciplines: Business
- Required Technology: Ideally Excel, but almost any spreadsheet

3.17.2 Institutional and Course Contexts

- Type/size of institution: Comprehensive private university
- Size of Class: 35
- Characteristics of Students: This is a class of business students completing their math requirement with a one semester calculus class
- Mathematical Content: Numeric integration, approximating definite integrals with Riemann sums.
- Purpose/Goal of the Module: The course has very little time for techniques of integration. With a spreadsheet, Riemann sums are straightforward and quite effective for a broad variety of functions.
- After and Before: This is the beginning of integration, which is the last chapter of the course. This will be followed by antiderivatives. This course uses trend lines of accumulation functions produced by Riemann sums to produce antiderivative formulas.
- Other Prerequisites: Students have attained basic Excel skills and are familiar with quick fill and relative and absolute references. They know about the Show Formulas button.
- Inspiration for the Module: It is sad that a traditional one semester calculus class gives the students a lot of situations where a reasonable problem produces a definite integral the the student cannot solve by symbolic means.

3.17.3 Partner Discipline Background

This module is based on the way business professionals "think in Excel" - how they design user and reader friendly spreadsheets usable for a given purpose. In terms of the content, it is helpful if students know appreciate that accumulations are important.

While one might think that the use of a spreadsheet to do Riemann sums would be common in a calculus course aimed at business students, we are not aware of any commercially produced text that uses this approach and is currently in print. It is also worth noting that in business situations, the finite sum is often the desired value, and the integral is an approximation to that value. (Business functions are often defined in discrete time units, like days or quarters.). Thus, in business settings, the sum is often the exact solution and the definite integral is an approximation, reversing the standard math setup.

A typical one semester applied calculus course spends very little time on techniques of integration. Generally, the course only goes as far as basic formulas, with sometimes a day given to substitution. That makes Riemann sums with a spreadsheet even more effective, since it is quite robust with the kinds of functions used in a business setting. Part of a day is spent setting up a template for doing Riemann sums with easy functions where the area can be computed without any calculus. The template can then be used to do problems of interest beyond the scope of a typical one semester course.

The worksheet starts with integration of functions that can be done by elementary methods to show how to do sums on a spreadsheet, then moves to present value problems.

A present value problem asks for the current value of a stream of revenue. The promise of a dollar in the future is worth less than a dollar today, because the money could be invested. The standard approach is to set an investment rate and use an exponential function to discount the income over time. If the revenue function is $rev(t)$ and the investment rate is $r\%$ per year, then the value of the stream is:

$$\int_{\text{start time}}^{\text{end time}} rev(t)(1 + r/100)^{(-t)}\ dt$$

Places where present value shows up outside a math class include computing the cash payout value of a lottery, and setting the payment from a perpetuity. Many students will also know the commercials of companies that will buy out structured settlements. Unfortunately, if the revenue function is as complicated as a linear function, evaluation requires techniques beyond substitution. These problems show up in the last question of the worksheet.

Business faculty want the instructions to be sparse. Ideally, the instructions should simply be to find an accumulation. This worksheet has been added procedural instructions that would not be given in the actual class, since then the students would have acquired the Excel skills.

A followup module has the students start with a table of data, find a best fitting curve of an appropriate type, and then find the Riemann sum using that function.

It is also worthwhile to note some difference in conventions between the disciplines:

- Mathematicians make symbol manipulation effective by using one letter function and variable and function names. Business faculty want students to build templates that can be reused by several people for a prolonged period of time. Templates should use longer, easy to understand names or labels.

- In a business setting, having the work neatly laid out, with easy to follow steps is part of the task. The ideal spreadsheet has steps simple enough to understand months in the future.

- Electrons are cheap. Mathematicians should resist the urge to put a complicated formula in one cell rather than repeating the step many times and using small steps that are easier to understand. Ideally the steps in each cell should be small enough and the names clear enough, that a student could look at the worksheet a semester later and remember enough to modify the worksheet for a similar problem.

These differences in conventions will be nontrivial for our students.

3.17.4 Implementation Plan

Formal Learning Objectives

- Verify that Riemann sums can be used to find area, or accumulation from a function in cases where the results can be found by simpler methods.

- Learn to create a template to find midpoint rule Riemann sums for nice functions and understand that increasing the number of intervals cause the Solution to converge to the correct Solution.

- Use the technique to evaluate definite integrals from business applications where antidifferentiation techniques covered in the course are insufficient.

Materials and Supplementary Documents Sample Excel files are available at:
`https://mathstat.slu.edu/~may/ExcelCalculus/chap-7-Intergration.html`
You can also email the author.

Time Required The exercise was done in groups in a 50-minute class period. Groups finished outside of class with a one-week time limit.

Implementation Recommendations I find it very useful to do any technology project in groups because the students do a certain amount of technical support for each other. The students need to be in a room where they all have computer access. Technical issues, like adding documentation to a workbook with text boxes, should have been dealt with before this exercise.

Depending on the skill level of the class, how much the use of technology is incorporated into the course, and other course considerations, the instructor may want to provide the students with a template that has some of the Excel coding already filled in. Such templates are available at the companion site.

Students should note the robustness of the process—that it works with lots of different functions.

Alternative Solutions Depending on how this module fits in a course, the teacher may want to provide a stripped down version of the teacher's Excel file to the students. This module should work with other spreadsheets. In particular, this module works with Google Sheets and Apple Numbers.

Common Errors and Questions

- Some students will have trouble with quick fill. In particular, when extending the table to many rows, the column with n needs to include two rows to establish the progression, while the column with x_n has a first entry that does not follow the same pattern.
- If students are typing formulas into Excel, they may be confused by relative and absolute cell references.

Adaptations to Online Learning No significant modifications are necessary for online teaching modalities.

3.17.5 Additional Information

This project is part of a larger shift in a one semester calculus course for business students where more emphasis is given to a conceptual approach and less to symbol manipulation. Spreadsheets and numerical methods allow the core concepts of calculus to be used before a lot of symbolic skills have been developed.

Riemann Sums and Numeric Integration

Module with Solutions

In most calculus courses the definition of the definite integral as a limit of Riemann sums is given light treatment because it seems so theoretical and impractical. With a spreadsheet it becomes a practical means of evaluation, even for hard integrals. As is our standard practice we start with a case where the area under the curve is more easily done by geometry without any calculus, then an example more easily done symbolically, before moving to cases where using Riemann sums is the practical method.

The Left-Hand Rule for a Simple Function

For our initial exploration we look at $f(x) = 2x + 5$, from $x = 1$ to $x = 5$. The area is a 4 by 7 rectangle with a triangle of base 4 and height 8 on top, so the area is 44. We want to set up a spreadsheet to evaluate the Riemann sum. Following good Excel protocol, we want to use easy to understand labels with all the inputs and outputs in a single screen at top.

	A	B	C	D	E
1	Starting x	1		$f(x)$	$2x + 5$
2	Ending x	5		Area	OFFSET$(E6, B3, 0)$
3	Number Intervals	10			
4	Width Interval	=(B2-B1)/B3			
5					
6	n	x_n	$f(x_n)$	Area of Rectangle	Summed Area So Far
7	1	=$B$1	=2*B7+5	=C7*$B$4	=SUM(D$7:D7)
8	2	=B7+B$4	=2*B8+5	=C8*$B$4	=SUM(D$7:D8)
9	3	=B8+B$4	=2*B9+5	=C9*$B$4	=SUM(D$7:D9)

We then want to use quick fill to extend the pattern in rows 8 and 9, so that n is larger than our number of intervals. We will eventually look the case where n is 1000, so do at least 1100 rows.

Teacher note: There are three Excel features worth noting at this point, the Offset command, relative and absolute references, and the SUM command. OFFSET is used to find a value somewhere else and bring it to a nice place. Our command, =OFFSET(E6,B3,0) says to go to E6, the cell with the label "Summed Area so far", go down B3 rows, where B3 is the number of intervals, go over 0 columns, and bring that value back to cell E2. That brings our final Solution back to a convenient place. The $ in cell references makes them absolute references. Notice in rows 8 and 9 that the references with a $ are unchanged, while the references without a $ change. SUM(D$7:D9) gives the sum of cells D7 through D9.

1. Find the Riemann sum approximation of the area under $f(x) = 2x + 5$, from $x = 1$ to $x = 5$ with 10, 100, and 1000 intervals.

   Solution: The estimates for 10, 100, and 1000 intervals were 43.4, 42.84, and 42.984 respectively.

2. Compute the error in the Riemann sum approximation of the area under $f(x) = 2x + 5$, from $x = 1$ to $x = 5$ with 10, 100, and 1000 intervals. How does the error change if I increase the number of intervals by a factor of 10?

   Solution: The errors for 10, 100, and 1000 intervals were 1.6, 0.16, and 0.016 respectively. Increasing the number of intervals by a factor of 10 divides the error by a factor of 10.

3. Repeat the process with $f(x) = 4x + 7$, from $x = -3$ to $x = 7$. Start with a copy of what you did above. You need to change the formula in cell D1. You then need to change cell C7 to the new formula using cell references, then quick fill to replace the formulas in the rest of column C. This is easiest to do with "Show Formulas" enabled.

   Solution: The estimates for 10, 100, and 1000 intervals were 130, 148, and 149.8 respectively. Using geometry, and the area of triangles and rectangles, the area is 150. The errors for 10, 100, and 1000 intervals were 20, 2, and 0.2 respectively. Increasing the number of intervals by a factor of 10 divides the error by a factor of 10.

Switching to the Midpoint Rule

If our function is nice, we would expect to get better results if we used rectangles measured in the middle of the interval rather than at the beginning of the interval. We can do that with one simple adjustment to our Excel work, adding half an interval to our first evaluation point and moving on from there.

	A	B	C	D	E
1	Starting x	1		$f(x)$	$2x + 5$
2	Ending x	5		Area	OFFSET$(E6, B3, 0)$
3	Number Intervals	10			
4	Width Interval	=(B2-B1)/B3			
5					
6	n	x_n	$f(x_n)$	Area of Rectangle	Summed Area So Far
7	1	=$B$1+0.5*B$4	=2*B7+5	=C7*$B$4	=SUM(D$7:D7)
8	2	=B7+B$4	=2*B8+5	=C8*$B$4	=SUM(D$7:D8)
9	3	=B8+B$4	=2*B9+5	=C9*$B$4	=SUM(D$7:D9)

1. Find the midpoint rule Riemann sum approximation of the area under $f(x) = 2x + 5$, from $x = 1$ to $x = 5$ with 10, 100, and 1000 intervals. Notice, that with linear functions, the approximation is exact from the beginning.

2. Find the midpoint rule Riemann sum approximation of the area under $f(x) = x^2 + 5$, from $x = -3$ to $x = 6$ with 10, 100, and 1000 intervals. In the next section we will use symbolic techniques to show the exact area is 126. Compute the error in the Riemann sum approximation of the area under $f(x) = x^2 + 5$, from $x = -3$ to $x = 6$ with 10, 100, and 1000 intervals. How does the error change if I increase the number of intervals by a factor of 10?

   Solutions: The estimates for 10, 100, and 1000 intervals are 125.3925, 125.99393, and 125.99994 respectively, with respective errors of 0.6075, 0.006075, and 0.00006075 respectively. Increasing the number of intervals by a factor of 10 decreases the error by a factor of 100.

Cases where Riemann Sums are the Easy Way to Solve the Problem

In rough terms, Riemann sums work well to find the area under the curve as long as the curve does not wiggle back and forth too much. With a spreadsheet they give a good approximation for the accumulation of a broad collection of functions. We started with easy functions where other techniques are easier. We now want to look at some problems where the other techniques are much harder, but Riemann sums have the same level of difficulty. Estimate the following accumulations with 100 intervals and the midpoint rule.

1. You have a natural gas well. You have been told that as gas is extracted and the pressure in the well lessens, the rate of extraction also decreases. The weekly production is $10000e^{-0.01t}$ cubic feet per week.

   (a) Estimate the production in the first year. (Find the area under the curve as t goes from 0 to 52.)
   Solution: Our estimate is that we produce 405,479 cubic feet of gas in the first year.

   (b) Estimate the production in the third year. (Find the area under the curve as t goes from 104 to 156.)
   Solution: Our estimate is that we produce 143,318 cubic feet of gas in the third year.

2. Sales of a fad item are estimated to be $10000t^2(0.1)^{t/10}$ units per week.

   (a) Estimate the sales in the first year. (Find the area under the curve as t goes from 0 to 52.)
   Solution: Our estimate is that we sell 1,637,387 items in the first year.

   (b) Estimate the production in the second year. (Find the area under the curve as t goes from 52 to 104.)
   Solution: Our estimate is that we sell 875 items in the second year.

3. A revenue stream is worth $(10000 + 100t)(1.03)^{-t}$ dollars per year.

   (a) Estimate the value of the stream for the first 10 years. (Find the area under the curve as t goes from 0 to 10.)
   Solution: Our estimate is that the first 10 years of the revenue stream is worth $62,345.

   (b) Estimate the value of the stream for the next 10 years. (Find the area under the curve as t goes from 10 to 20.)
   Solution: Our estimate is that the first 10 years of the revenue stream is worth $49,471.

3.18 Calculating Solar Energy as an Application of the Integral

Liza Boyle, Sonja Manor, Bori Mazzag, and Ruth Saunders
California Polytechnic State University – Humboldt
(*formerly Humboldt State University*)
Contact: borbala.mazzag@humboldt.edu

3.18.1 About the Module

- Course: Calculus
- Partner Disciplines: Environmental Resources Engineering and Physics
- Required Technology: Spreadsheet software (Google Sheets or Microsoft Excel)

3.18.2 Institutional and Course Contexts

- Type/size of institution: Regional comprehensive state university with primarily undergraduate enrollment of approximately 6,500 students. It is a Hispanic-Serving Institution in a rural setting on the North Coast of California, with many strong science and natural resources programs.
- Size of Class: The class is capped at 35 students and a faculty member teaches all course meetings. The class meets four times a week for 50-minutes during the 15-week semester.
- Characteristics of Students: The course serves a broad range of STEM majors (chemistry, engineering, geology, mathematics and physics). Most students take a semester or two of prerequisite courses before taking Calculus, so for the typical student the class is taken in the first or second year of their undergraduate studies. Students in the course often need support with algebra.
- Mathematical Content: The project is an application of several concepts related to integration: (a) Riemann sums; (b) net change as the integral of a rate of change; (c) computing the integral of a polynomial using the Fundamental Theorem of Calculus.
- Purpose/Goal of the Module: Connect discrete Riemann sums of analytic integration using the Fundamental Theorem of Calculus. The module uses solar energy data collected at the Humboldt State University Library to illustrate how polynomials and Riemann sums can be used to estimate the total solar energy gain during a day. The project also asks students to relate the amount of energy collected by a solar panel to their own energy use, and to reflect on the impact of the solar array that was installed by a local Native American tribe.
- After and Before: The project comes after the course covers Riemann sums, the Net Change Theorem and the Fundamental Theorem of Calculus. This is near the end of the course, and the only topics covered afterward are a practice of integration and the u-substitution method. This module is a take-home project, but during one of the lectures on Riemann sums, students work on an in-class group module that shows them how to compute different types of Riemann sums using a spreadsheet (Google Sheets).
- Other Prerequisites: Students need to be able to use sums and simple algebraic operations in a spreadsheet. We intentionally cleaned and prepared the data set to be ready to use, and created the polynomial function fit.
- Inspiration for the Module: Similar activities exist in texts and online, but none were directly used to create the project. The projects are intentionally open-ended and thought provoking, requiring students to synthesize ideas.

3.18.3 Partner Discipline Background

Engineering students in the course may have taken an introduction to engineering course that would introduce them to concepts, units of measurements and broader social implications of solar energy. Most other students in the course, including physics students, have taken Science 100 which introduces them to gathering data and spreadsheets. Although this knowledge provides benefits, this is not necessary in order to complete the project. The ability to work with different units and perform unit conversion will give students an advantage in this project.

We should note that the partner discipline faculty were integral in developing the module and the connections between the mathematics and its meaning in the context.

3.18.4 Implementation Plan

Formal Learning Objectives

- Compute Riemann sums using data in a spreadsheet
- Set up and compute the integral that represents the total amount of solar energy received in a day at the site
- Interpret the total energy as the sum or integral over time of power (power is the rate of change of energy with respect to time) - direct application of the net change theorem
- Practice working with physical measurements and practice unit conversions

Materials and Supplementary Documents Solar data that can be obtained from the National Solar Radiation Database(`https://midcdmz.nrel.gov`) for a location of interest. We used data collected on campus. The data can be found in a spreadsheet on the volume's website. There are many other sites around the United States and the world that collect similar data. Some data cleaning and trendline fitting may be required for those who want to use a new dataset. A supplement following the module describes how to obtain a new dataset.

Time Required The module was given as a project outside of class. See Implementation Recommendations for more details on in-class time required.

Implementation Recommendations The module was given as one of three large projects during the course and it was completed outside of class. Students had three weeks to complete the project. In class, there were two days of activities that supported the project: one day (using Desmos) on Net Change; one day (using Google Sheets) on computing Riemann sums from data. Students also took a quiz on calculating energy cost savings for the local Tribe, the Blue Lake Rancheria, from a large (2 MW) solar array they have installed. This quiz was also completed outside of class

If necessary, the project can be broken into several parts and used as in-class activities. In this case, 2-3 days of class time is recommended for this module. One day for introducing the idea of solar energy and its measurements and units, and calculating Riemann sums in the spreadsheet. A second day (perhaps more) where the polynomial functions fit to the data set are used to compute the integrals and time to discuss the interpretation of the results and setting a broader context for them.

Our students were able to collaborate with each other on completing these projects, but each student had to submit their own individual project. The module intends to make use of active learning where students work collaboratively in teams discussing the problems. The questions in the specific module we highlighted integrates Calculus I topics and a more complex real world application from engineering.

Alternative Solutions In the original statement of the problem, the notation used for the Riemann sums was not specific. This has been corrected. When students are prompted to report their own personal energy use (question 9), multiple solutions are possible. This also makes solutions individualized for question 12.

Common Errors and Questions There were two particular challenging aspects of this project: writing the Riemann sums (questions 2 and 4) and working with the unit conversions in the problem (particularly questions 11 and 12).

The questions about the Riemann sums were updated to include more information on the notation to use in writing down the Riemann sums. The challenge students faced was in finding the correct formula for how the index of the term denoting the power at different times, P_i changes. It may benefit students to see similar examples prior to the project.

Working with units and measurements is another challenging aspect of the assignment. Asking students to report their units for each question helps with this. Practicing unit conversions is a valuable skill for all science students, and is useful practice in mathematics courses.

In questions 2 and 3 of the module (where students must find a formula for the Riemann sum), indexing was a very difficult task. It may warrant additional practice before the lab, or more scaffolding in the prompts.

The energy output conversion (question 11) was also quite difficult for students who often have not worked with units in the past. This is another part of the module where students may need more scaffolding.

Adaptations to Online Learning We designed these activities during the pandemic, so it was clear that we would be working with students online. In some ways, this made the course logistics easier because we could expect all of our students to have access to a computer or laptop during class. This was particularly helpful when we were showing students how to compute sums using Excel. We had used similar, Excel-based labs before the pandemic. In face-to-face modalities, instructors reserve one or two days in a computer lab, or ask students to bring devices to class (or both).

3.18.5 Additional Information

- Students reported positive feedback on learning about the local area and a local Tribe.
- The module built on the examples of other curricular materials that linked science content to mathematics for the first-year place-based learning communities.
- A new solar-panel module in the Trigonometry class (prerequisite some take before Calculus) provides additional context on solar energy for some students.
- The use of both Riemann sums with discrete data and the Fundamental Theorem of Calculus with explicit functions helps to connect the two approaches to problem solving. The discrete data mimics what engineers often see in practice, and works well with spreadsheet technology.
- The project allows students an in-depth opportunity to see how concepts in the course apply to complex and multifaceted problems, such as sustainable energy production through solar panels. Projects also involve data analysis and interpretation.
- This style of project has been used with success in other introductory mathematics courses at our institution, and the SUMMIT-P collaboration expanded the number of courses in which such projects were required.

Integration Application - Solar Panel Data

Module with Solutions

Background and Introduction

The National Renewable Energy Laboratory (NREL) is a national laboratory dedicated to the science and engineering of renewable energy systems, sustainable transportation and energy efficiency, and they publish a large amount of information on these systems including solar energy. Their webpage (NREL nd) writes:

> More energy from the sun falls on the earth in one hour than is used by everyone in the world in one year. A variety of technologies convert sunlight to usable energy for buildings. The most commonly used solar technologies for homes and businesses are solar photovoltaics for electricity, passive solar design for space heating and cooling, and solar water heating.
>
> Businesses and industry use solar technologies to diversify their energy sources, improve efficiency, and save money. Energy developers and utilities use solar photovoltaic and concentrating solar power technologies to produce electricity on a massive scale to power cities and small towns.

Photovoltaic (PV) panels, sometimes called solar panels, convert sunlight into electricity providing a renewable source of electricity. There are many factors that influence the amount of electricity that a PV panel produces, but the largest of these factors is the amount of solar radiation, or sunlight,that reaches the panel. This means that panels produce energy during the day, but not at night, and more energy on sunny days than on cloudy days.

Renewable sources of electricity are an important part of how we will reduce carbon emissions, decrease dependency on fossil fuels, and increase the redundancy and reliability in our electricity grid. The electricity generated by PV panels directly emits no carbon dioxide and can produce electricity with very low maintenance for decades. This makes these systems especially attractive both to produce electricity for communities (via rooftop or ground-mounted solar installations) and also for remote applications where connection to other generation sources is difficult (such as the international space station, remote research stations or dwellings, and traffic signals). While these systems are well tested and well understood, it is still important to understand the specific solar energy resource at a site in order to properly design a PV system.

In this project, you will estimate the amount of power collected by a PV panel at the HSU library during a day, and you will compare it to your own power consumption. The project will also give you a broader perspective on the impact of PV panels on communities through the example of the Blue Lake Rancheria's PV array and microgrid. These tools of estimation are very similar to the ones used by professionals who design solar energy systems.

Photovoltaic Panels and Solar Energy

The Solar Radiation Monitoring Station (SoRMS) at HSU is maintained by HSU students and in partnership with NREL the data is published on the NREL website as part of the National Solar Radiation Database (NSRDB) (Andreas & Wilcox 2007). The website provides data from the HSU Library for each day since May 2007, with a few interruptions for instrument maintenance and calibration. Calendars of data are also provided on the NREL website, and data from September 2020 is considered here. For each day, three curves are shown: global horizontal (red), direct (green) and diffuse (blue) radiation. Global horizontal radiation is the power from sunlight measured on a flat surface; direct radiation is the power coming directly from the sun measured on a surface facing directly at the sun, and diffuse radiation is the power from the sun that is bounced around (by reflection and refraction) in the atmosphere by airborne particulates, molecules, and clouds, that reaches a horizontal surface but not directly from the sun this is measured by shading the direct sun and measuring the horizontal radiation in the same method as the global horizontal s. Measuring all three of these parameters makes it possible to estimate PV production accounting for the tilt of the system. This project focuses on **global horizontal radiation**.

Global horizontal radiation is measured in Watts per square meter (W/m^2), where Watts is a measurement of power. Power itself is measured in units of energy per time (Joules per second, J/s), so power describes the amount of energy transferred in a unit of time. In other words, we can think of global radiation measuring how much energy (solar radiation) the solar panel can capture each second for each square meter - or the rate of change of energy with respect to time.

Our goal will be to calculate the total amount of energy a horizontal solar panel captures on a given day. We will:

- Approximate the amount of solar energy captured through a day using a Riemann sum
- Approximate the amount of solar energy captured through a day using an integral of a function that approximates our data
- Compare results from different days
- Compare the energy captured with our own energy use

Data Sets and Questions

We will work with data from two different days, September 7 and September 28. Figure 3.7 shows the two datasets:

1. Looking at the images of two days of data, why do you think making approximations may be difficult? Comment on the possible sources of noise in the Sep 28 data.

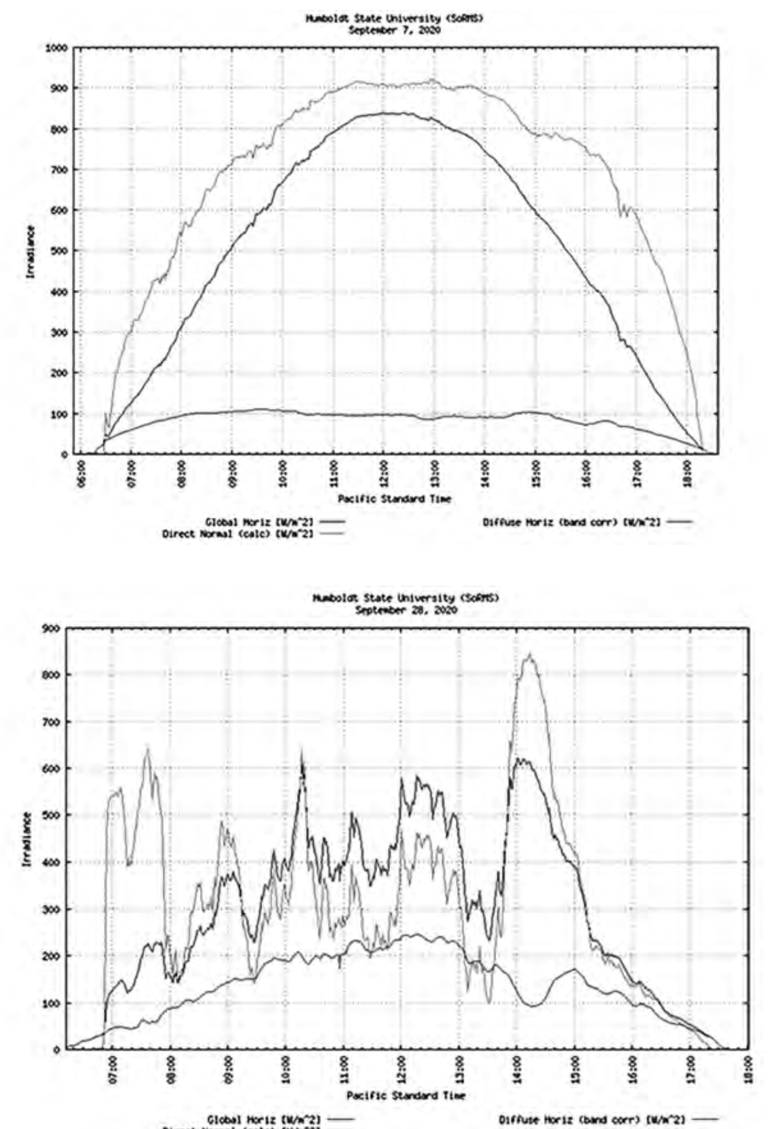


Figure 3.7. Solar panel data from the HSU Library on September 7 (top) and 28 (bottom), 2020

The spreadsheet contains data from these two days. Col A has the date, Col B has the time in "hours:minutes" format, Col C has the time in seconds, Col D has the global horizontal radiation data (W/m^2). The spreadsheet has two tabs, one tab for Sep 7, another tab for Sep 28. Copy this spreadsheet for yourself, and make the calculations in your own spreadsheet.

Solution: Solutions will vary here. Students should comment on the Sep 28 data set containing a lot of noise, most likely reflecting a day with a lot of cloud cover.

2. Write down a formula for the left Riemann sum that approximates the total solar energy per square meter for Sep 7. For this sum, assume that the length of your partition is 1 minute. Include the appropriate units. *Hints:* (a) Define a variable P_i to denote the power at the ith second. (b) Think about how many terms there will be in your sum. (c) Think about how the index, i needs to change if you want it to correspond to seconds.

   Solution: There are 763 terms in the Riemann sum, with the values of time running from $i = 21120$ (for the first term) and $i = 66900$ for the last term. The increment of time is $\Delta t = 60s$. This gives us the formula:

$$\sum_{i=1}^{764} P_{60i+21120}\Delta t = \sum_{i=1}^{763} P_{60i+21120}(60)$$

   This amount is in units W/m^2 times seconds or J/m^2.

3. Write down a formula for the left Riemann sum that approximates the total solar energy per square meter for Sep 28. Include the appropriate units. (See the hints in Question 2).

   Solution: There are 707 terms in the Riemann sum, with the values of time running from $i = 22320$ (for the first term) and $i = 64740$ for the last term.The increment of time is $\Delta t = 60s$. This gives us the formula:

$$\sum_{i=1}^{707} P_{60i+22320}\Delta t = \sum_{i=1}^{707} P_{60i+22320}(60)$$

   This amount is in units W/m^2 times seconds or J/m^2.

4. Use the left Riemann sum formula to approximate the amount of energy captured on Sep 7 per square meter. Include appropriate units.

   Solution: This question requires students to add values of Col D from cell 2 to cell 765 and multiply by 60. The solution we got was 22,032,865.66 J/m^2 or 2,203.29 kJ/m^2.

5. Use the left Riemann sum formula to approximate the amount of energy captured on Sep 28 per square meter. Include appropriate units.

   Solution: Similar method, using the spreadsheet. The solution is 12,210,111.99 J/m^2 or 1,221.01 kJ/m^2

   Next, we will approach estimating the total amount of energy on a given day by computing an integral. In order to calculate an integral, we need to have a function that *approximates* the trends in our data. Note that the functions we will obtain will not be exact - they will be approximations of the trend we see in the data.

   The two panels in Figure 3.8 show the global horizontal power as a function of time on Sep 7 and Sep 28, and a quadratic function that has been fit to each of these data (using built-in features of google sheets). The formulas for the quadratic functions are difficult to read on the figure. For Sep 7, the equation is (with x denoting time in seconds and f denoting energy in W/m^2):

$$f(x) = -2723 + 0.159x - 0.0000018x^2$$

   And, for Sep 28, the equation is (using the same units as above):

$$g(x) = -1755 + 0.103x - 0.00000119x^2$$

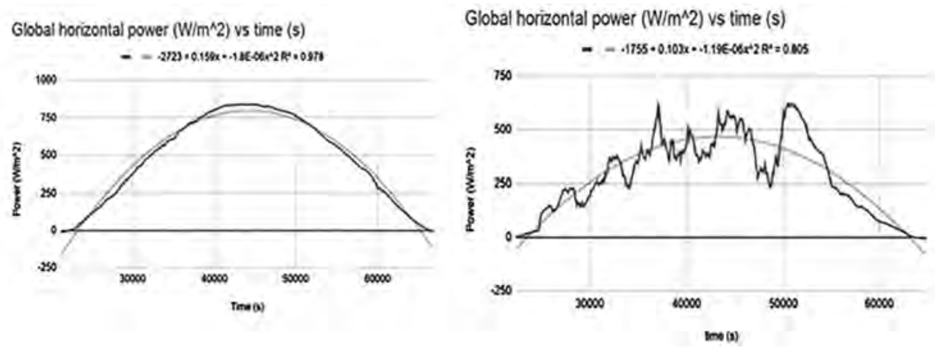


Figure 3.8. Fitting quadratic functions to the global horizontal power data from the HSU Library on September 7 (left) and 28 (above), 2020. Data is shown darker, and the fitted curve is shown lighter.

6. Compute the total amount of energy (per square meter) on Sep 7 using an integral of the function fitted to the data. Include appropriate units for your solution.

   Solution: We integrate:

$$\int_{21120}^{66960} \left(-2723 + 0.159t - 0.0000018t^2\right) dt = 2,168.35kJ/m^2$$

7. Compute the total amount of energy (per square meter) on Sep 28 using an integral of the function fitted to the data. Include appropriate units for your solution.

   Solution: We integrate:

$$\int_{22320}^{64800} \left(-1755 + 0.103x - 0.00000119x^2\right) dt = 1,252.03kJ/m^2$$

8. Compare your approximations in questions 4 and 5 to the approximations in questions 6 and 7. Think about the following questions. What are your observations? What do you think is the more accurate estimate? What are some pros and cons of the different methods?

   Solution: The Riemann sum was fairly close to the integral for both Sep. 7 and for Sep 28. We expected the values on Sep 7 to be closer, because it was clear from the figures that the polynomial curve we fit to the data did not match well on Sep. 28.

We had talked about integrals representing the true area under the curve. However, in this case, we are fitting a curve to a data set, and we clearly have a potential for a lot of error in this curve fit (depending on the amount of noise). So, in this case, we expect the Riemann sums to be better approximations of the true value.

Broader Implications

Now you have an idea of how much energy you are able to collect using a solar panel. Usually, energy use is measured in kWh (or kilowatt-hours) where, 1 kWh = 3,6000,000 J (Joules). And 1 W = 1 J/s This final portion of the project will help you put the numbers you got in questions 4-7 in some context.

Use The United Solar website to calculate your own energy use.

9. Approximately how many kilowatt hour (kWh) of power do you use each day?

   Solution: Solutions varied widely for individual students.

   The next set of calculations will allow you to estimate how many square meters of solar panel you would need to meet your daily energy needs.

10. You calculated the energy output per square meter of the solar panels on two different days, using two different methods. Average these four results.

    Solution:

$$\frac{2,203.29 + 1,221.01 + 2,168.35 + 1,252.03}{4} = 1,711.17kJ/m^2$$

11. The average energy output estimate is in Joules per square meter. Convert this to kilowatt hour per square meter. The conversion factor is that 1 kilowatt hour = 3.6 x 106 Joules.

    Solution:
    1 kilowatt hour = 3.6 x 106 Joules
    1 kilowatt hour = 3.6 x 103 kJoules
    0.4753 kilowatt hour = 1,711.17 kJoules

    The average energy estimate (per m2) of the solar panel is 0.4753 kilowatt hour.

12. Estimate how many square meters of solar panels you would need to meet your daily energy needs assuming that the panels you use are 100% efficient. (In reality, solar panels cannot convert all of the solar energy to electrical energy.)

    Solution: Here, solutions vary for students again. To carry out the calculations, students will have to divide their energy use (in Question 9) by 0.4573 to get the number of square meters of solar panels.

    Now, we want to connect these calculations with a broader question about the impact of solar panels in a community. The Blue Lake Rancheria is a federally recognized tribe in Humboldt County. Read about the Blue Lake Rancheria microgrid (Schatz Energy Research Center nd) and watch a short video about it from the Sierra Club (nd).

13. Write a short paragraph that summarizes in your own words how the solar microgrid may have been able to help Blue Lake Rancheria. Also, think critically about any potential negative issues the solar microgrid could have caused, and address those too.

    Solution: Solutions will vary. Local students might also comment on how BLR was able to provide power during the recent California power outages. The video describes energy independence and the website highlights the collaboration between the BLR tribe and an HSU research institute, the Schatz Energy Research Center (SERC).

References

Andreas, A. & Wilcox, S. (2007). *Solar Radiation Monitoring Station (SoRMS)*: Humboldt State University, Arcata, California (Data). NREL Report No. DA-5500-56515. `http://dx.doi.org/10.5439/1052559`. Accessed Nov. 1, 2020.

Energy calculator. The United Solar Website `https://www.theunitedsolar.com/energy-usage-calculator/`. Accessed Nov. 1, 2020.

NREL, *Solar Energy Basics*. US Department of Energy, National Renewable Energy Laboratory, `https://www.nrel.gov/research/re-solar.html`. Accessed Nov. 1, 2020.

Schatz Energy Research Center, *Blue Lake Rancheria Microgrid* `https://schatzcenter.org/blrmicrogrid/`. Accessed, Nov. 1, 2020.

Sierra Club, *Faces of Clean Energy: Jana Ganion*. `https://schatzcenter.org/blrmicrogrid/`. Accessed, Nov. 1, 2020.

Supplement: Obtaining Solar Energy Data in the United States from the National Solar Radiation Database, National Renewable Energy Laboratory

There are two strategies for downloading data from this site. First is to obtain data from an active monitoring station (this is what we did at HSU as we have an active monitoring site on our campus). These sites are scattered across the United States. To obtain this data go to: `https://midcdmz.nrel.gov/`

Use the map or dropdown menu to select the site of interest, then select ‘Daily Plots and Data Files.’

The data for this module is Global Horizontal, select this box, the start and end date you would like, and the time interval desired. For this project, we chose the data given for each minute. Select ‘Submit,’ and the requested data will be supplied.

We used two different days, one day where the data looked very smooth (so the polynomial approximation would fit closely) and one that was noisy. Comparing the approximations for the different data sets is not an essential learning outcome for the project, so the project can be modified to use one data set, and the questions streamlined to reflect this. (If one data set is used, questions 3, 5, 7 and 8 can be cut from the project.)

The second method is using the National Solar Radiation Database Viewer. This is modeled and extrapolated data from a larger set of monitoring stations, and can be obtained for anywhere in the world. Instructions for downloading this data are here: `https://nsrdb.nrel.gov/data-sets/download-instructions.html` Be sure to select ‘GHI’ as the data to download.

Converting Time to Seconds Once you have downloaded the data set, you should have a data set with three columns: Date MM/DD/YYYY, PST (HH:MM), Global Horiz [W/m^2]. The measurement of power, “Watts” is given as Joules/seconds, so the time must be converted to seconds. To do this conversion, insert a new column after PST (Col B) and use the following formula to compute the elements of Col C (Time, seconds):

(example for cell 2 in Col C) :$C$2 = HOUR(B2)*3600 + MINUTE(B2)*60.

Now you will have the following columns: Col A: Date (MM/DD/YY), Col B: PST (HH:MM), Col C, Time (Sec), Col D: Global Horiz [W/m^2].

Creating Figures and Polynomial Data Fit Highlight Col C and D in the spreadsheet and insert a chart showing Global Horiz [W/m^2] vs Time (seconds). Add the trendline by right clicking the chart, choosing “Series (Global Horiz [W/m^2])”, then, in the menu that appears, marking “Trendline” and choosing to add a polynomial trendline with degree 2. One can also add the R2 value and the “Use Eqn” as the label. The equation of the trendline is necessary to use for questions 6 and 7 where student compute the integral of these functions.

3.19 Studying the Response of Second-Order Electrical Circuits

Makarand Deo, Maila Brucal-Hallare, and Shahrooz Moosavizadeh
Norfolk State Univesity
Contact: smoosavizadeh@nsu.edu

3.19.1 About the Module

- Course: Differential Equations
- Partner Disciplines: Electrical Engineering

3.19.2 Institutional and Course Contexts

- Type/size of institution: Historically Black College/University with about 5,000 students
- Size of Class: 20-25 students
- Characteristics of Students: Sophomore or junior students who major in mathematics, engineering, or physics
- Mathematical Content: Second-order differential equations
- Purpose/Goal of the Module: To introduce students to a typical application of second-order differential equations in electrical/electronics engineering
- After and Before: This module is implemented after completing a unit on second order differential equations and their solutions, particularly linear, homogeneous second order differential equations with constant coefficients.
- Other Prerequisites: Knowledge of basic electronic components (e.g. resistors, capacitors and inductors) and basic circuit analysis laws (e.g. Ohm's law, Kirchhoff's voltage and current laws). This can be introduced in the class by the instructor (see partner discipline content below) or by a guest speaker. These basic principles are also found in several standard differential equations textbooks.
- Inspiration for the Module: One outcome of our fish-bowl conversation with the engineering faculty is a suggestion on emphasizing graphs of elementary functions. We believe that this module demonstrates the importance of graphs in analyzing responses to an electric circuit expressed as differential equations.

3.19.3 Partner Discipline Background

A second-order electrical circuit has two distinct energy storage devices such as a capacitor (C) or inductor (L). As a reminder, capacitors store energy in the form of an electric field between their charged plates, inductors store energy in the form of a magnetic field, whereas resistors (R) limit the flow of electrons (current) in a circuit. Voltage across an inductor (v_L) is proportional to the rate of change of current flowing through the inductor (i_L), i.e.:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

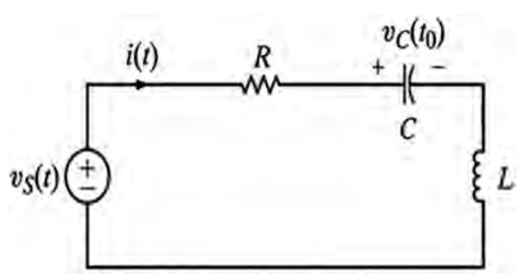


Figure 3.9. An RLC Circuit

On the other hand, the current through a capacitor (i_C) is proportional to the rate of change of voltage across the capacitor (v_C), i.e.:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

Thus the voltage across the capacitor is given by:

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(x)dx$$

Voltage across a resistor (v_R) is linearly proportional to the current flowing through it (i_R), i.e. $v_R(t) = Ri_R(t)$. A typical second-order electrical circuit (series RLC) is shown in Figure 3.9 (Irwin & Nelms 2011). Applying basic electrical circuit analysis methods (i.e. Kirchhoff's Voltage Law) produces the following second- order differential equation by summing the voltages across L, R and C:

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0 \tag{3.3}$$

where $i(t)$ is the current flowing through the circuit. The total response of the system in equation (3.3) is a sum of natural response (circuit response to initial conditions with all external forces set to zero) and forced response (circuit response to an external stimulus with zero initial conditions).

To categorize, identify, and analyze system behaviors, the concept of damping is used. Damping is an influence on or within an oscillatory system that restricts, limits, or prevents oscillations. In case of electrical circuits, the resistance, R, provides damping. A very little or no resistance in the circuit will produce excessive oscillations in voltages and currents, whereas large resistance will make the circuit response very sluggish.

To describe the concept of damping, consider an LC circuit (that is, there are no resistors). In this case, since the capacitor has an initial charge, as it discharges, it will build a magnetic field. If it is resonant (C and L values coupled exactly), when the capacitor is fully discharged, the magnetic field collapses and recharges the capacitor in the opposite polarity. This oscillation process will continue indefinitely. This scenario is the undamped case.

Now, if we add a resistor in the circuit (series RLC), the oscillations will gradually get smaller, until they stop. This results into a damped system. In other words, an RLC-series circuit such as the one in Figure 3.9, exhibits a damped scenario. The damping ratio is given by:

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \tag{3.4}$$

As such, there are three types of system behavior based on the value of the damping ratio: underdamped ($\zeta < 1$), producing oscillatory response; overdamped ($\zeta > 1$), producing sluggish response; and critically damped ($\zeta = 1$), producing just about the perfect response (see Figure 3.10). Table 3.5 summarizes the three responses and corresponding solutions to the second-order differential equation 3.3.

Table 3.5. Summary of Circuit Responses for Series RLC Circuit

Type of roots of characteristic equation	Damping ratio behavior	General solution	System behavior
Real and distinct, m_1, m_2	$\zeta^2 > 1$	$i(t) = Ae^{m_1 t} + Be^{m_2 t}$	Overdamped
Real and equal, $m_1 = m_2$	$\zeta^2 = 1$	$i(t) = Ae^{m_1 t} + Bte^{m_1 t} = e^{m_1 t}(A + Bt)$	Critically damped
Complex roots $m_1 = -\zeta\omega_0 + j\omega_0\sqrt{1-\zeta^2}$ $m_2 = -\zeta\omega_0 - j\omega_0\sqrt{1-\zeta^2}$	$\zeta^2 < 1$	$i(t) = e^{-\sigma t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right)$, where $\sigma = \zeta\omega_0$, the damping factor $\omega_d = \omega_0\sqrt{1-\zeta^2}$, the damped oscillation frequency, and ω_0 is the natural (undamped) frequency	Underdamped

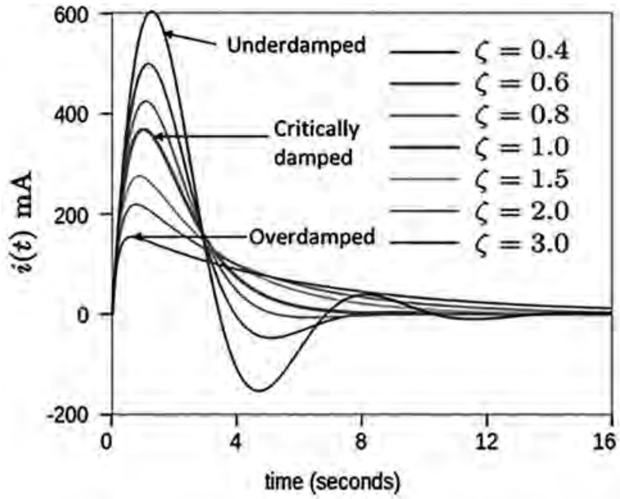


Figure 3.10. Underdamped, overdamped, and critically damped circuit responses for varying values of ζ

- In engineering applications, the imaginary number i is written as j, that is $j = \sqrt{-1}$. This can be seen in Table 3.5 in the description of the complex roots of the characteristic equation. The symbol i is reserved for the current function $\iota(t)$.
- In Equation 3.4 and Table 3.5, the condition $\zeta^2 = 1$ is equivalent to a zero discriminant of the characteristic equation, that is $R^2 = \frac{4L}{C} = -0$. Analogous statements for $\zeta^2 > 1$ and $\zeta^2 < 1$ hold. In engineering applications, the magnitude of the square of the damping ratio is emphasized, not the sign of the discriminant of the characteristic equation.
- It can be shown that the natural frequency (referred to in the description of the general solution to Equation 3.3 in the complex roots case in Table 3.5) is $\omega_0 = \sqrt{1/LC}$.
- The units for R, L, C are Ω (ohm), H (Henry), and F (Farad), respectively. The notations for values in Table 3.6 (in the implementation recommendations below) are:
  - $1000\Omega = 1k\Omega$ (one kilo-ohm)
  - $2.2 \times 10^{-3}H = 2.2mH$ (2.2 milli-Henry)
  - $0.22 \times 10^{-6}F = 0.22\mu F$ (0.22 micro-Farad).

3.19.4 Implementation Plan

Formal Learning Objectives Apply second order differential equations theory to the analysis of RLC circuits.

Materials and Supplementary Documents Students will need access to website `https://www.partsim.com`. A PowerPoint introduction is available on the volume's website.

Time Required This module is designed to take 45 minutes. The first 15 minutes may be used to review the solutions of second-order differential equations and to discuss the concept of damping. Note the transition from the mathematics language and notation to the engineering language and notation. Thirty minutes should be devoted for group module, completing the worksheet by the end of the class.

Implementation Recommendations An RLC circuit as shown in Figure 3.9 will be implemented in an open-source online circuit simulator (`https://www.partsim.com`). Overdamped, underdamped, and critically damped responses will be generated by changing the value of resistor R, keeping the inductor L and capacitor C values unchanged, see equation 3.4. In this case, the resistor value changes in direct proportion to the damping ratio ζ: as the resistor value increases, the damping ratio also increases and the system is driven towards an overdamped response.

Table 3.6. Values for Circuit Simulations

Circuit I	Circuit II	Circuit III
$R = 3.9\Omega$	$R = 200\Omega$	$R = 1k\Omega$
$L = 2.2mH$	$L = 2.2mH$	$L = 2.2mH$
$C = 0.22\mu F$	$C = 0.22\mu F$	$C = 0.22\mu F$

The effect of damping ratio and corresponding responses will be investigated using circuit simulations (transient response). Table 3.6 lists the component values used to generate the responses. For effective learning, the module should be done in groups of 2 students (3 students in larger classes). Uninterrupted access of broadband internet should be ensured as the circuit simulations are run online through a web browser. Students should complete a worksheet and submit for grading/feedback.

In addition, find time to discuss the role of initial values in finding the solutions to second-order differential equations and how they are used in simulating circuits.

Alternative Solutions Some alternative approaches the instructor can implement include:

1. Instead of providing component values for three circuits, keep values of L and C constant as given in Table 2, but in place of discrete resistors, use a potentiometer (variable resistor) of value $10k\Omega$. Let the students vary the potentiometer and obtain the overdamped, underdamped and critically damped responses. When a response is obtained, remove the potentiometer and measure the R value using a multimeter. Substitute the measured value in Equation 3.4 and calculate the damping ratio to confirm whether its value is in agreement with the damping response.
2. The circuit implementation and simulation can also be done in more advanced circuit simulators such as MATLAB/Simulink or Multisim, if the university has access to these licensed software.
3. This lab could be preceded by a study of responses of first order circuits (RC or RL series circuits) which follow first order differential equations.

Common Errors and Questions Some students who have not taken an electrical circuits course may find the transference from mathematics to engineering challenging. Help them transition by making a listing of translations.

Adaptations to Online Learning This module has been used as a springboard for an undergraduate research project, and students created this video, which could be useful in online teaching modalities.
`https://photos.app.goo.gl/WFNQXGh1LokJaLZi9`

3.19.5 Additional Information

We hope the students realize that the answer to the first question is obtained from the background material. This is an opportunity to reinforce the background and connect it with the circuit shown in the simulator.

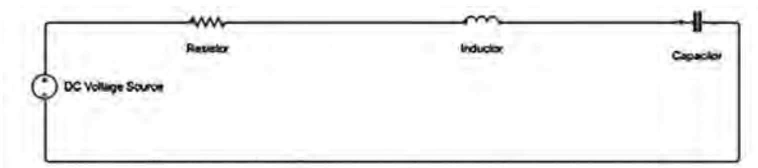
3.19.6 References

Irwin, J.D. & Nelms, R.M. (2011) *Basic Engineering Circuit Analysis*, 11th Edition, John Wiley & Sons, Inc.

RLC Circuits

Module with Solutions

Exercise: Implement second-order series RLC circuit using a circuit simulator and study the effects of damping ratio.



Implement the circuit shown in online simulator (PartSim or Simulink) and observe the voltage across capacitor in simulations.

1. Write the differential equation for the current $i(t)$ in the circuit.

   Solution:

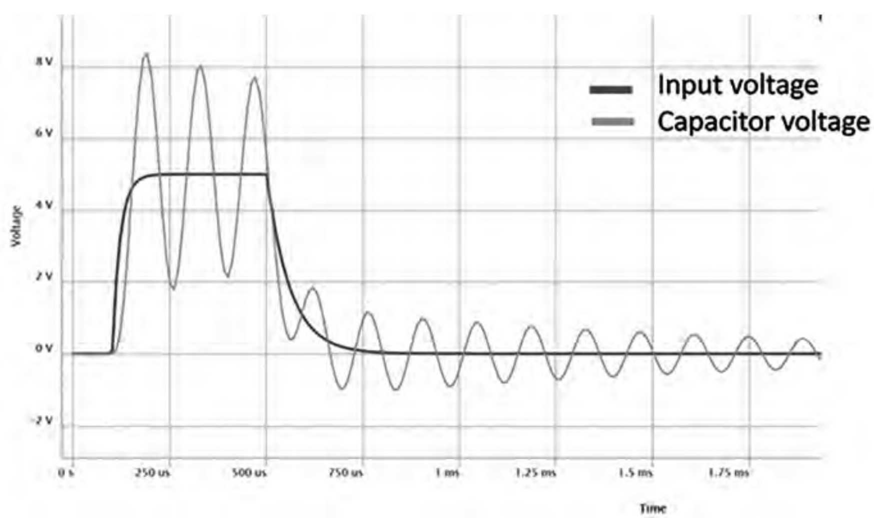
$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

2. **Circuit 1**: Use $R = 3.9\Omega$, $L = 2.2mH$, and $0.22\mu F$.

   (a) Compute ζ^2 and determine the behavior of the system.

   (b) Approximately sketch the capacitor voltage waveform as observed on the scope. Comment on the type of response based on damping ratio.

   Solutions: $\zeta^2 = 0.00038$, underdamped

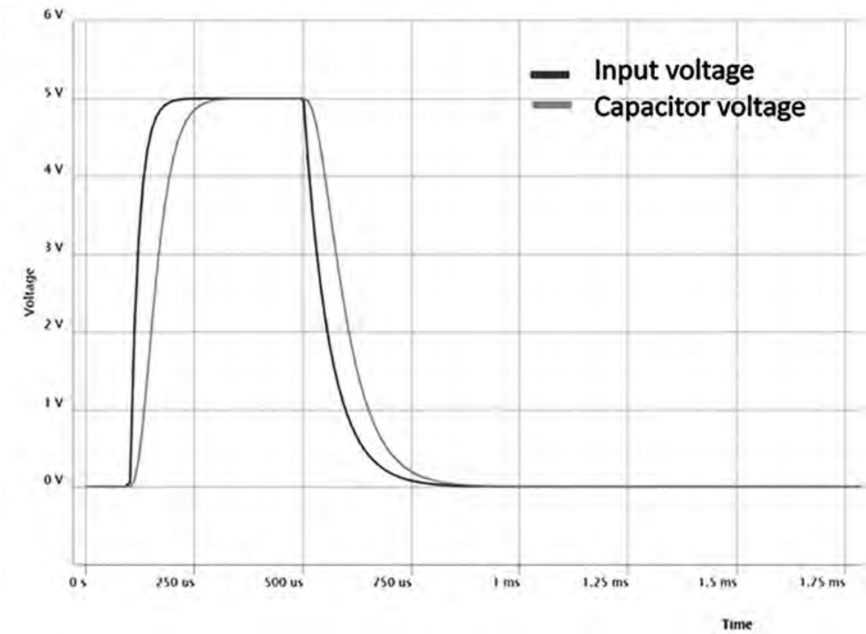


3. **Circuit 2**: Use $R = 200\Omega$, $L = 2.2mH$, and $0.22\mu F$.

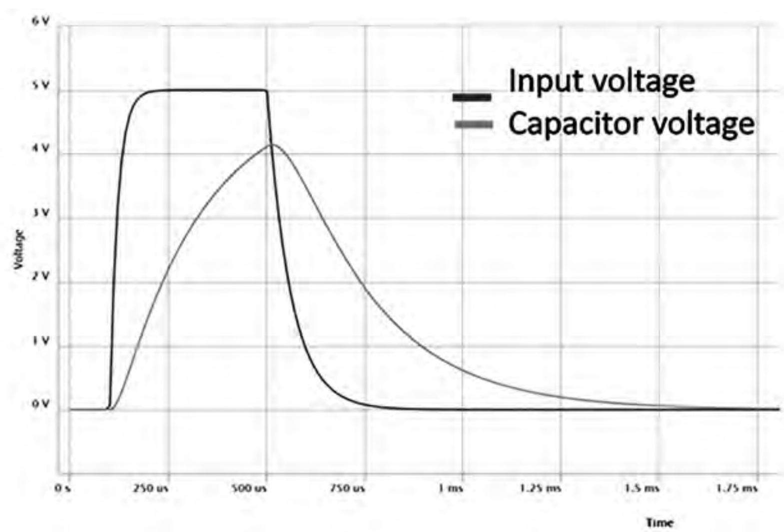
   (a) Compute ζ^2 and determine the behavior of the system.

   (b) Approximately sketch the capacitor voltage waveform as observed on the scope. Comment on the type of response based on damping ratio.

   Solutions: $\zeta^2 = 1$, critically damped



4. **Circuit 3**: Use $R = 1k\Omega$, $L = 2.2mH$, and $0.22\mu F$.
   (a) Compute ζ^2 and determine the behavior of the system.
   (b) Approximately sketch the capacitor voltage waveform as observed on the scope. Comment on the type of response based on damping ratio.
   Solutions: $\zeta^2 = 25$, overdamped



3.20 Modelling Wireless Power Transfer

Maila Brucal-Hallare, Makarand Deo, Hongzhi Guo,
Shahrooz Moosavizadeh, and Ana Vivas-Barber
Norfolk State University
Contact: smoosavizadeh@nsu.edu

3.20.1 About the Module

- Course: Differential Equations
- Partner Discipline: Electrical Engineering

3.20.2 Institutional and Course Contexts

- Type/size of institution: Historically Black College/University with about 5,000 students
- Size of Class: About 20 students
- Characteristics of Students: Students who are required to take a differential equations course are mostly underprepared mathematically. They appreciate seeing applications outside mathematics for motivation and further understanding.
- Mathematical Content: Mathematical modelling, system of ordinary differential equations, Laplace method
- Purpose/Goal of the Module: Demonstrate mathematical modelling of electrical circuits, solve the resulting system of differential equations, and use an online circuit simulator to analyze the circuit.
- In the Syllabus: The first part, mathematical modelling, may be introduced alongside the other modelling applications in the middle of the semester. The second part, solutions method via Laplace, must be introduced after introducing properties of Laplace operator.
- Other Prerequisites: A working knowledge of basic electricity components (see the module on RLC circuits in Section 3.19 of this volume), Laplace transforms of elementary functions.
- Inspiration for the Module: One of the the engineering faculty consultants is an expert on wireless technology.

3.20.3 Partner Discipline Background

In the last decade, wireless power transfer (WPT) has significantly changed our daily life by enabling a large variety of wireless charging applications, including water-proof toothbrush, cell phones, smart watch, and electric vehicles (Kurs, et al 2007). Since WPT transfers power without the use of cables or wires, this engineering breakthrough creates a lot of advantages. Large-scale applications of this technology is still in research stage.

Although the implementation of a WPT system is complex, the underlying principle is simple. Briefly, a WPT system consists of two electric circuits that are joined by magnetic resonance. When electric current flows in the first circuit, it creates a magnetic field around it; this is called Ampère's law (Irwin and Nelms 2015). If another circuit comes in close proximity to this circuit, the magnetic field gets linked with it and produces a voltage in the second circuit; this is called Faraday's law (Irwin and Nelms 2015). Although this occurs to some extent in all circuits, the effect is magnified in coils because the circuit geometry amplifies the linkage effect. Thus, if current is passed through a primary coil (embedded in a charging circuit), it can produce adequate magnetic flux to create voltage in a secondary coil (embedded in a device to be charged) which can be used to charge the device without any electrical connection with the charging circuit. Note that Faraday's Law also requires that the primary circuit must be connected to a time-varying voltage as a source of power.

In this module, we will model a WPT system using an equivalent circuit that consists of two RLC circuits coupled by a magnetic resonance. An RLC circuit gets its name from its electrical components to include resistor (R), inductor (L), and a capacitor (C). See Deo et al (this volume) for more information.

This learning module has two parts. Part 1 describes how to use Kirchhoff's Voltage Law to create a mathematical model on a WPT system. Part 2 shows some ideas on how the resulting system of differential equations can be used as a rich source of practice problems on Laplace methods in solving initial value problems. As with most electric circuit teaching applications, it is best that students use online circuit simulator to supplement learning.

Part 1. Modelling the Wireless Power Transfer System

A schematic diagram of a WPT system is given in Figure 3.11. For example, the transmitting coil is part of the charging base of an electric toothbrush or cellphone, which is connected to a power source, while the receiving coil is embedded in the electric toothbrush or cellphone.

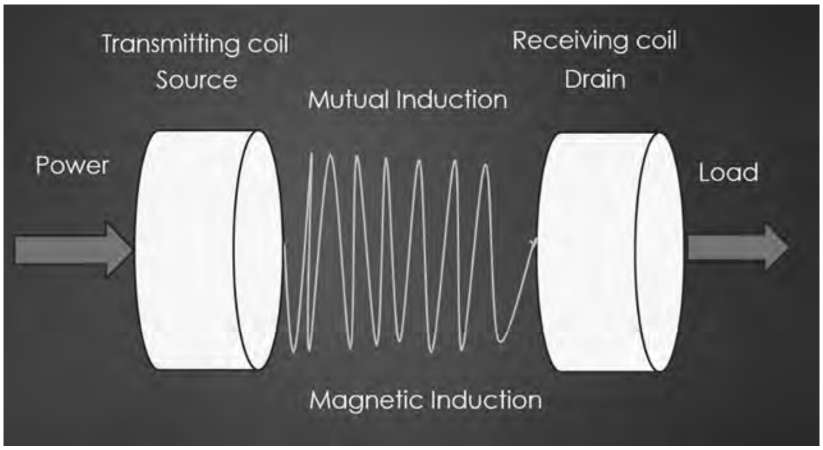


Figure 3.11. A schematic diagram of a WPT system

Figure 3.12 is an equivalent circuit diagram of a simple WPT system, where V_1 is the time-varying voltage source, V_A is the voltage that corresponds to $A = R, L, C$, and M is the mutual inductance between the two circuits. According to Kirchhoff's Law applied to the first loop in this circuit, we have

$$-V_1(t) + V_{R_1}(t) + V_{L_1}(t) + V_{C_1}(t) + V_M(t) = 0, \tag{3.5}$$

where V_M is the voltage that corresponds to the mutual inductance M. Analogously, the Kirchhoff's Law applied to the second loop gives us

$$V_{R_2}(t) + V_{R_3}(t) + V_{L_2}(t) + V_{C_2}(t) + V_M(t) = 0. \tag{3.6}$$

Now, let i_1, i_2 be the currents in the first and second loops, respectively. Recall that the relationship between voltage and current differs across a resistor R, an inductor L, and a capacitor C, namely,

$$V_R(t) = Ri(t), \;\; V_L(t) = L\frac{di(t)}{dt}, \;\; i_c(t) = C\frac{dV_c(t)}{dt}. \tag{3.7}$$

The first equation in (3.7) is also known as Ohm's Law and it is one of the most important results in electricity.

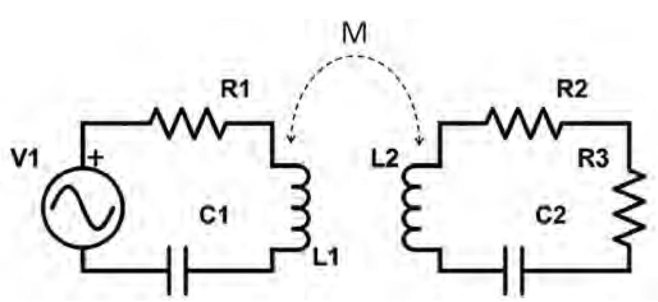


Figure 3.12. A WPT system consists of two circuits coupled by a mutual inductance

Using these equations, we can now write equations (3.5) and (3.6) as

$$\begin{cases} -V_1(t) + R_1 i_1(t) + L_1 \dfrac{di_1(t)}{dt} + V_{C_1}(t) + M\dfrac{di_2(t)}{dt} &= 0 \\ (R_2+R_3)i_2(t) + L_2\dfrac{di_2(t)}{dt} + V_{C_2}(t) + M\dfrac{di_1(t)}{dt} &= 0 \end{cases} \tag{3.8}$$

Note that the mutual inductance M is governed by Faraday's Law and Ampere's Law. Ampere's Law predicts that the flow of electric current will create a magnetic field through the coil. Since this magnetic field links the two circuits, Faraday's Law predicts the creation of a voltage within the linked circuit. Hence, for the second loop, we have $V_M = M\dfrac{di_1(t)}{dt}$ and for the first loop, we have $V_M = M\dfrac{di_2(t)}{dt}$. These two terms couple the two equations.

The final step is to solve for V_C in both equations (3.8) and use the third equation in (3.7), as follows:

$$\begin{aligned} i_1(t) &= C_1\frac{d}{dt}\left(-V_1(t) + R_1 i_1(t) + L_1\frac{di_1(t)}{dt} + M\frac{di_2(t)}{dt}\right) \\ i_2(t) &= -C_2\frac{d}{dt}\left((R_2+R_3)i_2(t) + L_2\frac{di_2(t)}{dt} + M\frac{di_1(t)}{dt}\right) \end{aligned}$$

Differentiating and re-arranging, we now have our system of equations that govern the WPT:

$$\begin{cases} L_1\dfrac{d^2 i_1(t)}{dt^2} + M\dfrac{d^2 i_2(t)}{dt^2} + R_1\dfrac{di_1(t)}{dt} + \frac{1}{C_1}i_1(t) = \dfrac{dV_1(t)}{dt} \\ L_2\dfrac{d^2 i_2(t)}{dt^2} + M\dfrac{d^2 i_1(t)}{dt^2} + (R_2+R_3)\dfrac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) = 0 \end{cases} \tag{3.9}$$

subject to the initial conditions $i_1(0), i_2(0), i_1'(0), i_2'(0)$. It is important to mention again that without the mutual inductance M, we obtain two separate second-order RLC circuits. Moreover, in (3.9), one equation is non-homogeneous because there is an input voltage source while the other equation is homogeneous.

Part 2. Solving the System of Differential Equations by Laplace Methods and MultiSim Live

The system of equations (3.9) consists of two second-order linear ordinary differential equations with unknown functions $i_1(t), i_2(t)$. In a typical undergraduate differential equations course, the function $V_1(t)$ may be taken to be a polynomial, a trigonometric, an exponential, or a combination of these functions. However, in the engineering application sense, $V_1(t)$ is usually taken as a trigonometric function of the form $A\cos(mx) + B\sin(nx)$ to account for the AC voltage source (connecting to a power outlet); or as a constant function $V_1(t)$ to account for the DC voltage source (connecting to a battery).

Three Cases

In this section, we consider three cases that depend on the existence of capacitors in the full WPT system (3.9). In the first case, we assume that there are no capacitors on either side; in the second case, we assume that there is a capacitor on the primary circuit; and in the third case, we assume that both primary and secondary circuits have one capacitor each. From a mathematics teaching perspective, these three examples showcase a system of two first-order differential equations, a system of one first order and one second-order differential equation, and a system of two second-order differential equations (respectively).

In commercial WPT systems, capacitors and inductors create LC resonant circuits in the primary and secondary sides which are used to strengthen the inductive coupling and create a higher efficiency power transfer. However, a basic WPT circuit still works without capacitors in both coupled circuits or with only one capacitor in the primary circuit, albeit at a reduced efficiency.

We present the solutions to specific initial-value problems (IVPs) that arise from these three cases. In each of these IVPs, all initial values will be set to zero to simplify computations, that is,

$$i_1(0) = i_2(0) = i_1'(0) = i_2'(0) = 0.$$

We present a sample module that can be used in a Differential Equations course that aims to provide practice on Laplace methods. There is one question for each of the three scenarios.

No Capacitors

Assume a system of two electrical circuits that does not have a capacitor on either side (see Figure 3.13). In this case, by setting $V_C = 0$ in (3.8), we obtain:

$$\begin{cases} R_1 i_1(t) + L_1 \dfrac{di_1(t)}{dt} + M\dfrac{di_2(t)}{dt} & = & V_1(t) \\ (R_2+R_3)i_2(t) + L_2\dfrac{di_2(t)}{dt} + M\dfrac{di_1(t)}{dt} & = & 0 \end{cases} \tag{3.10}$$

with initial conditions $i_1(0) = i_2(0) = 0$. Upon taking Laplace transforms, the resulting linear system that solves for $I_k(s) = \mathcal{L}\{i_k(t)\}$ for $k = 1, 2$ and $s > 0$ is

$$\begin{pmatrix} L_1 s + R_1 & Ms \\ Ms & L_2 s + R_2 + R_3 \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \mathcal{L}\{V_1 t\} \\ 0 \end{pmatrix}$$

Using Cramer's Rule, we have

$$\begin{aligned} I_1(s) &= \frac{-Ms\mathcal{L}\{V_1(t)\}}{(L_1 s + R_1)(L_2 s + R_2 + R_3) - M^2 s^2}, \\ I_2(s) &= \frac{(L_2 s + R_2 + R_3)\mathcal{L}\{V_1(t)\}}{(L_1 s + R_1)(L_2 s + R_2 + R_3) - M^2 s^2}. \end{aligned} \tag{3.11}$$

Finally, the solutions are given by the inverse Laplace transforms of these functions.

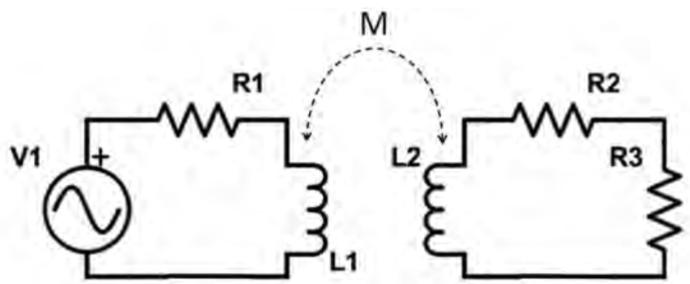


Figure 3.13. In this WPT system, both circuits do not have capacitors. It is modelled by a system of two first-order differential equations.

One Capacitor

Another helpful example to explore in class is a WPT system where the primary circuit has a capacitor while the secondary circuit does not have one. Refer to circuit diagram in Figure 3.14. In this case, the resulting system of differential equations is

$$\begin{cases} L_1 \dfrac{d^2 i_1(t)}{dt^2} + M\dfrac{d^2 i_2(t)}{dt^2} + R_1\dfrac{di_1(t)}{dt} + \frac{1}{C_1} i_1(t) & = & \dfrac{dV_1(t)}{dt} \\ (R_2+R_3)i_2(t) + L_2\dfrac{di_2(t)}{dt} + M\dfrac{di_1(t)}{dt} & = & 0 \end{cases} \tag{3.12}$$

with initial conditions $i_1(0) = i_2(0) = i_1'(0) = i_2'(0) = 0$. Note that we still need four initial conditions because both i_1, i_2 involve second derivatives. Upon taking Laplace transforms, the resulting linear system that solves for $I_k(s) = \mathcal{L}\{i_k(t)\}$ for $k = 1, 2$ and $s > 0$ is

$$\begin{pmatrix} L_1 s^2 + R_1 s + \frac{1}{C_1} & Ms^2 \\ Ms & L_2 s + R_2 + R_3 \end{pmatrix} \begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix} = \begin{pmatrix} \mathcal{L}\{V_1' t\} \\ 0 \end{pmatrix}$$

Using Cramer's Rule, we have

$$\begin{aligned} I_1(s) &= \frac{-Ms\mathcal{L}\{V_1'(t)\}}{(L_1s^2+R_1s+\frac{1}{C_1})(L_2s+R_2+R_3)-M^2s^3}, \\ I_2(s) &= \frac{(L_2s+R_2+R_3)\mathcal{L}\{V_1'(t)\}}{(L_1s^2+R_1s+\frac{1}{C_1})(L_2s+R_2+R_3)-M^2s^3}. \end{aligned} \tag{3.13}$$

Finally, the solutions are given by the inverse Laplace transforms of these functions.

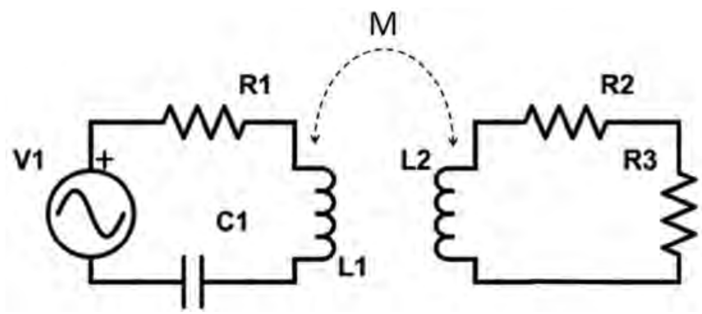


Figure 3.14. In this WPT system, the left circuit has a capacitor while the right circuit does not have one. It is modeled by a system consisting of one second-order and one first-order differential equation.

Two Capacitors

A third example is a WPT system where both circuits have capacitors, refer to circuit diagram in Figure 3.12. In this case, we are looking at the full-system (3.9):

$$\begin{cases} L_1\dfrac{d^2i_1(t)}{dt^2}+M\dfrac{d^2i_2(t)}{dt^2}+R_1\dfrac{di_1(t)}{dt}+\frac{1}{C_1}i_1(t)=\dfrac{dV_1(t)}{dt} \\ L_2\dfrac{d^2i_2(t)}{dt^2}+M\dfrac{d^2i_1(t)}{dt^2}+(R_2+R_3)\dfrac{di_2(t)}{dt}+\frac{1}{C_2}i_2(t)=0 \end{cases}$$

with initial conditions $i_1(0)=i_2(0)=i_1'(0)=i_2'(0)=0$. Upon taking Laplace transforms, the resulting linear system that solves for $I_k(s)=\mathcal{L}\{i_k(t)\}$ for $k=1,2$ and $s>0$ is

$$\begin{pmatrix} L_1s^2+R_1s+\frac{1}{C_1} & Ms^2 \\ Ms^2 & L_2s^2+(R_2+R_3)s+\frac{1}{C_2} \end{pmatrix}\begin{pmatrix} I_1(s) \\ I_2(s) \end{pmatrix}=\begin{pmatrix} \mathcal{L}\{V_1't\} \\ 0 \end{pmatrix}$$

Using Cramer's Rule, we have

$$\begin{aligned} I_1(s) &= \frac{-Ms^2\mathcal{L}\{V_1'(t)\}}{(L_1s^2+R_1s+\frac{1}{C_1})(L_2s^2+(R_2+R_3)s+\frac{1}{C_2})-M^2s^4}, \\ I_2(s) &= \frac{(L_2s^2+(R_2+R_3)s+\frac{1}{C_2})\mathcal{L}\{V_1'(t)\}}{(L_1s^2+R_1s+\frac{1}{C_1})(L_2s^2+(R_2+R_3)s+\frac{1}{C_2})-M^2s^4} \end{aligned} \tag{3.14}$$

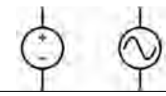
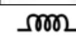
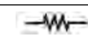
Finally, the solutions are given by the inverse Laplace transforms of these functions.

Armed with a table of Laplace and inverse Laplace transforms, solving these initial-value problems becomes mainly an algebraic problem of solving the transformed equations in the s-domain. In engineering, this $s-$domain is sometimes called the frequency domain. Note that the website `https://www.symbolab.com/` is a free online calculator that can compute the Laplace and inverse Laplace transforms of elementary functions.

MultiSim Live (`https://multisim.com`) is a free online circuit simulator where students can create their own electric circuits, simulate it, and see the resulting graphical solutions. It is also a great tool to learn about basic electric components in an online hands-on manner. Table 3.7 is a summary of such components.

In conclusion, we have presented here three possible systems of differential equations and their solutions. The mathematics instructor may want to use these systems and their corresponding solutions, choosing the voltage power source

Table 3.7. A Summary of Basic Electric Components

Component and Notation	Unit and Notation	Electrical Symbol in the Circuit Diagram
Current i	Ampere A	
Capacitance C	Farad F	
Voltage V	Volt V	
Inductance L	Henry H	
Resistance R	Ohm Ω	

(constant and trigonometric functions are preferred) and coefficients L_k, R_k, C_k and M. There is a very important result to remember: the value of the mutual inductance M cannot exceed the geometric mean of the inductors L_1, L_2, that is:

$$0 \leq M \leq \sqrt{L_1 L_2}.$$

The proof of this requires some power and energy analysis (Irwin and Nelms 2015). Furthermore, in mathematics, the coefficient M is called a coupling coefficient between the two equations; in electrical engineering, however, the coupling coefficient between two RLC circuits joined by a mutual inductance is given by

$$K = \frac{M}{\sqrt{L_1 L_2}}, \tag{3.15}$$

and hence, the coupling coefficient is a quantity between 0 and 1.

3.20.4 Implementation Plan

Formal Learning Objectives

- Apply electricity principles to create a mathematical model of a wireless power transfer system.
- Solve resulting system of differential equations using the Laplace method.
- Use an online circuit simulator to investigate circuits and visualize solutions to the system of equations.

Materials and Supplementary Documents You can find the following by searching online:

1. A short video that demonstrates how to create a wireless power transfer system in an engineering laboratory
2. Worksheets on solving systems of differential equations using the Laplace method plus the Multisim Live software.
3. A table of Laplace and inverse Laplace transforms.

Time Required Part 1 requires about 25 minutes starting from showing examples of real-life applications, viewing the video of the laboratory setup, up to the process of writing the system of equations using Kirchhoff's Laws. Part 2 can be done in 50 minutes because the students will supplement their handwritten computations with electrical circuit simulations via MultiSim Live.

Implementation Recommendations We take a mathematical modeling approach, meaning that we use a differential equations framework to explore a physical system by making as few assumptions as possible. We refer so some worksheets as class projects, implying that students have permission to discuss the solution to longer problems with classmates and the professor while working in real time.

We recommend introducing a prior example of RLC circuits prior to introducing the wireless power transfer system. The module by Deo et al in this volume will provide such an introduction.

Students sometimes get caught up in the application details if they are not familiar with them. Other students may be overly excited by the application and go off on a tangent. In both cases, refocusing them on the mathematical approach can be helpful.

Alternative Solutions

1. Students should be given an opportunity to explore the MultiSim Live software before they come to class. This can be accomplished by assigning any of the several YouTube videos that demonstrate how to create electrical circuits and how to generate graphs of solutions.
2. The instructor who wishes to emphasize the application/solution in class may skip the first part on modeling and proceed directly to the solutions method via Laplace and investigation via the online circuit simulator. However, we do not recommend skipping the first part because the mathematical modelling provides invaluable insight between engineering applications and mathematics.

Common Questions Some of the natural questions that may arise are: What should be the distance between the transmitting and receiving coils in order to ensure power transfer? How much power and energy is transferred? How can we optimize the power transfer? How does the coil geometry affect the power transfer? Do the individual inductors affect the mutual inductance between the circuits?

Adaptations to Online Learning We used this module in an online class with breakout rooms. However, we only asked students to do half the worksheet because everything seemed to take longer when doing it online. Once we were back to teaching the class in a face-to-face setting, we used the full worksheet again.

3.20.5 Additional Information

This engineering application may be used to develop some undergraduate research projects. Also, if time permits and access to an electrical/electronics laboratory is possible, the students may choose to create their own wireless power transfer system using the materials in the laboratory. In this way, the hands-on component of the module will keep their hands busy while grappling with the theory presented by the equations!

The benefit of introducing WPT as an engineering application of a system of differential equations helps students visualize the essential coupling between two differential equations. Students are able to see that they are not just dealing with abstract functions but they are actually manipulating electricity components; that they are not just looking at a differential equation but they are actually investigating an electric circuit; and that they are not just solving two related equations in a system but they are actually solving two circuits that are joined by a mutual inductance.

In general, we find that students engage with WPT as an example because they see it in everyday applications. We emphasize that the transmitting circuit or primary loop in a WPT is a linear non-homogeneous differential equation with constant coefficients while the receiving circuit of secondary loop in a WPT is a linear homogeneous differential equation with constant coefficients. This module led to an undergraduate research poster and a student-created video, further evidence that the students find this topic engaging.

Creating mathematical models of electrical circuits is a task that electronic simulation software, like MultiSim, cannot do. Although MultiSim is a powerful simulation tool that can be used in analyzing electrical circuits when access to an engineering laboratory is not possible, analysis of such circuits through mathematical modelling presents critical and important perspectives that go deep into the properties of electrical components. For engineering students, in particular, the synergy of theoretical approaches and computer or laboratory simulations allow them to appreciate the theories and methods that they learned in their mathematics courses.

3.20.6 References

Deo, M.,Brucal-Hallare, M., & Moosavizadeh, S. (2022) *Studying the response of second-order electrical circuits*. This volume.

Irwin, J.D. and Nelms, R.M. (2015) *Basic Engineering Circuit Analysis*, 11th edition, Wiley.

Kurs, A., Karalis, A., Moffatt, R., Joannopoulos, J.D., Fisher, P., & Soljacic, M. (2007) Wireless power transfer via strongly coupled magnetic resonances. *Science 317*(5834) 83–86.

Modelling Wireless Power Transfer

Module

Analyze each of the three wireless power transfer systems.

1. Compute the coupling coefficient K using (3.15).
2. Sketch a circuit diagram that corresponds to this WPT system. Write the corresponding system of differential equations.
3. Write the solutions of the system. Use `https://symbolab.com` in computing inverse Laplace transforms.
4. Sketch the solutions to the system. Use `https://desmos.com`.

A. Consider the WPT system that has no capacitors, whose voltage source is a sine function and whose values for $L_1, R_1, L_2, R_2, R_3, M$ are given below.

Voltage Source $V_1(t)$	$\sin(2t)V$
Inductor L_1	$2H$
Resistor R_1	1Ω
Inductor L_2	$2H$
Resistor R_2	0.5Ω
Resistor R_3	0.5Ω
Mutual Inductor M	$1H$

B. Consider the WPT system that has one capacitor, whose voltage source is cosine, and values for $L_1, R_1, L_2, R_2, R_3, C_1, M$ are given below:

Voltage Source $V_1(t)$	$2\cos(t)V$
Inductor L_1	$1H$
Resistor R_1	2Ω
Capacitor C_1	$1F$
Inductor L_2	$1H$
Resistor R_2	0.5Ω
Resistor R_3	0.5Ω
Mutual Inductor M	$1H$

C. Consider the WPT system that has two capacitors, whose voltage source is sine, and values for $L_1, R_1, L_2, R_2, R_3, C_1, C_2 M$ are given below:

Voltage Source $V_1(t)$	$\sin(t)V$
Inductor L_1	$1H$
Resistor R_1	2Ω
Capacitor C_1	$1F$
Inductor L_2	$1H$
Resistor R_2	1Ω
Resistor R_3	1Ω
Capacitor C_2	$1F$
Mutual Inductor M	$1H$

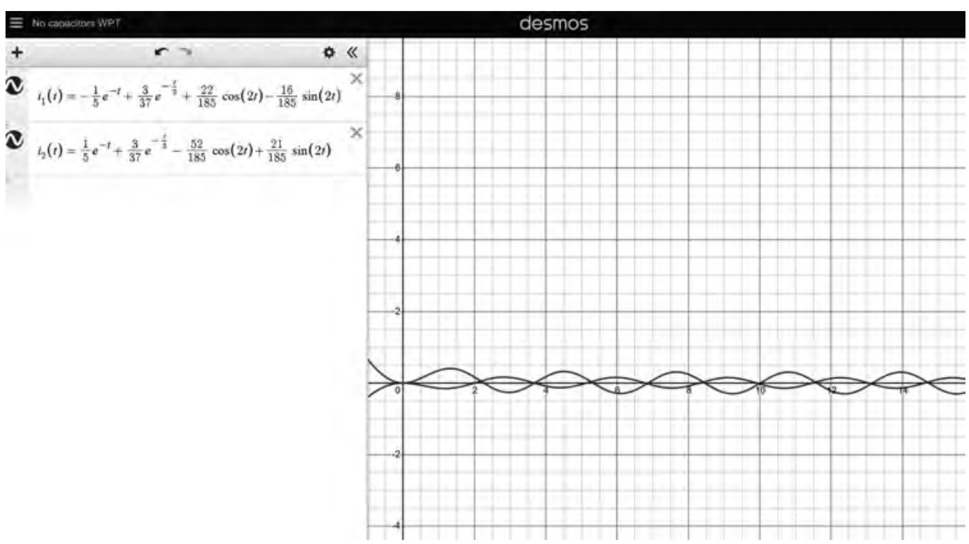
Modelling Wireless Power Transfer

Module Solutions

In this module, we look at three WPT systems (3.10), (3.12), and (3.9). Use the solutions for each of these three initial-value problems as summarized in (3.11), (3.13), and (3.14), respectively. Students should have a copy of a table of Laplace and inverse Laplace transforms and access to the internet (to use both `https://symbolab.com` and `https://desmos.com`).

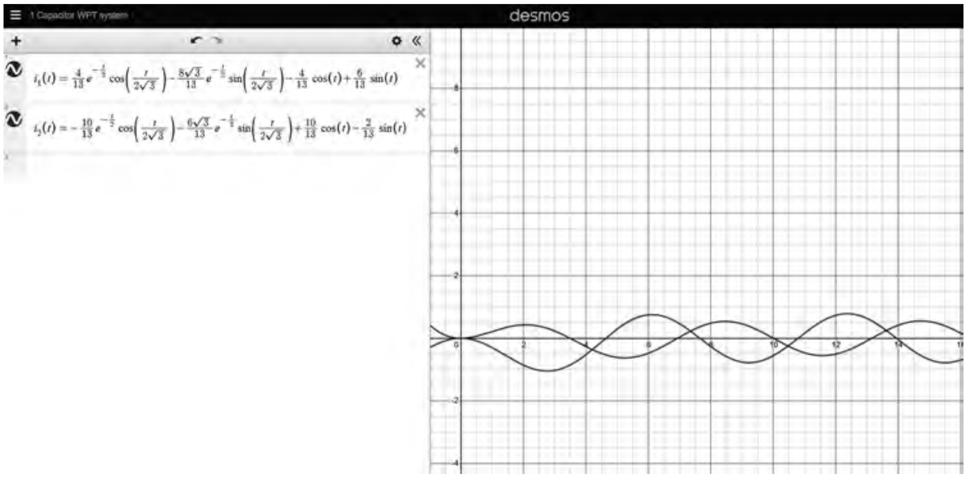
For system [A], the coupling coefficient is $K = \frac{M}{\sqrt{L_1 L_2}} = \frac{1}{2}$. The solutions to the system are

$$\begin{aligned} i_1(t) &= \mathcal{L}^{-1}\left(\frac{-2s}{(s^2+4)((2s+1)(2s+1)-s^2)}\right) \\ &= -\tfrac{1}{5}e^{-t} + \tfrac{3}{37}e^{-\frac{t}{3}} + \tfrac{22}{185}\cos(2t) - \tfrac{16}{185}\sin(2t) \\ i_2(t) &= \mathcal{L}^{-1}\left(\frac{2(2s+1)}{(s^2+4)((2s+1)(2s+1)-s^2)}\right) \\ &= \tfrac{1}{5}e^{-t} + \tfrac{3}{37}e^{-\frac{t}{3}} - \tfrac{52}{185}\cos(2t) + \tfrac{21}{185}\sin(2t) \end{aligned}$$



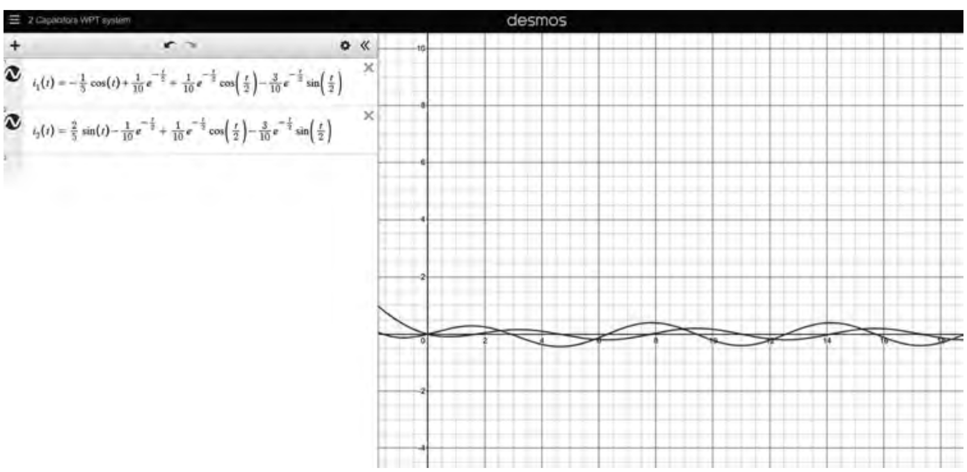
For system [B], the coupling coefficient is $K = \frac{M}{\sqrt{L_1 L_2}} = 1$. In this case, we say we have a perfect coupling. The solutions to the system are

$$\begin{aligned} i_1(t) &= \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)((s^2+2s+1)(s+1)-s^3)}\right) \\ &= \tfrac{4}{13}e^{-\frac{t}{2}}\cos\left(\tfrac{t}{2\sqrt{3}}\right) - \tfrac{8\sqrt{3}}{13}e^{-\frac{t}{2}}\sin\left(\tfrac{t}{2\sqrt{3}}\right) - \tfrac{4}{13}\cos(t) + \tfrac{6}{13}\sin(t) \\ i_2(t) &= \mathcal{L}^{-1}\left(\frac{(4s+1)}{(s-1)((s^2+2s+1)(4s+1)-2s^3)}\right) \\ &= -\tfrac{10}{13}e^{-\frac{t}{2}}\cos\left(\tfrac{t}{2\sqrt{3}}\right) - \tfrac{6\sqrt{3}}{13}e^{-\frac{t}{2}}\sin\left(\tfrac{t}{2\sqrt{3}}\right) + \tfrac{10}{13}\cos(t) - \tfrac{2}{13}\sin(t) \end{aligned}$$



For system [C], the coupling coefficient is $K = \frac{M}{\sqrt{L_1 L_2}} = 1$. The solutions to the system are

$$\begin{aligned}
i_1(t) &= \mathcal{L}^{-1}\left(\frac{-s^3}{(s^2+1)((s^2+2s+1)(s^2+2s+1)-s^4)}\right) \\
&= -\tfrac{1}{5}\cos(t) + \tfrac{1}{10}e^{-\frac{t}{2}} + \tfrac{1}{10}e^{-\frac{t}{2}}\cos\left(\tfrac{t}{2}\right) - \tfrac{3}{10}e^{-\frac{t}{2}}\sin\left(\tfrac{t}{2}\right) \\
i_2(t) &= \mathcal{L}^{-1}\left(\frac{s(s^2+2s+1)}{(s^2+1)((s^2+2s+1)(s^2+2s+1)-s^4)}\right) \\
&= \tfrac{2}{5}\sin(t) - \tfrac{1}{10}e^{-\frac{t}{2}} + \tfrac{1}{10}e^{-\frac{t}{2}}\cos\left(\tfrac{t}{2}\right) - \tfrac{3}{10}e^{-\frac{t}{2}}\sin\left(\tfrac{t}{2}\right)
\end{aligned}$$



3.21 Motivating Differential Equations with Nuclear Engineering

Afroditi Filippas, Supathorn Phongikaroon, and Rebecca Segal
Virginia Commonwealth University
Contact: rasegal@vcu.edu

3.21.1 About the Module

- Course: Introduction to Differential Equations
- Partner Disciplines: Nuclear and Mechanical Engineering

3.21.2 Institutional and Course Contexts

- Type/size of institution: Large public university with over 30,000 students.
- Size of Class: 30-35 students with an average of 8 sections per semester.
- Characteristics of Students: This module was used in a class that is composed of engineering students from various areas, and a few mathematics majors and sciences majors. Most of the students are second or third year undergraduate students. All the students in the class are required to take the class for their major.
- Mathematical Content: First order linear DE. The students will set up and solve the equation. Secondary review occurs with algebraic manipulation of the solution.
- Purpose/Goal of the Module: This module is designed to give the students exposure to engineering applications that use differential equations content. Students will have just seen how to solve first order equations. The module also requires students to practice algebra skills.
- After and Before: Students know how to solve first order equations using separation of variables and using an integrating factor. After this topic, the course moves to nonlinear first order equations.
- Other Prerequisites: Students need to recognize that the solution to a differential equation is a function. These problems also require students to understand the difference between a parameter and a variable.
- Inspiration for the Module: These problem ideas were supplied by our engineering faculty.

3.21.3 Partner Discipline Background

Mechanical engineers develop heating systems, such as the furnace and pipes that bring hot water to all the taps in your house, as well as power generation systems, such as nuclear power plants, which use a nuclear power source to heat water that generates steam that powers turbines that generate electricity. It is necessary to transport these fluids with minimal heat loss; in order to achieve this, mechanical engineers develop and solve models that describe the heat loss from cylindrical pipes.

The relevant variables in study of these systems include:

1. The length of the pipe,
2. The inner and outer radius of the pipe along its length,
3. The type of liquid flowing in the pipe,
4. The material out of which the pipe is constructed.

Simplifying assumptions include:

1. Assuming the pipe is straight,
2. Assuming the pipe profile is circular,
3. Assuming the pipe's inner and outer radius are constant along the length of the pipe,

4. Assuming the liquid completely fills the pipe.

Experimentation has also shown that the following assumptions are reasonable and will also simplify our model:

1. Assuming the pipe is long enough so that the heat transfer is radial, that is, flows out perpendicular to the outer surface of the pipe,
2. As the heat flows through the length of the pipe, the amount of heat that transfers from the liquid out through the pipe walls is the same per unit length. This flow is modeled, therefore, as a gradient with respect to the length of the pipe, i.e., in the direction perpendicular to the cylindrical radius of the pipe.

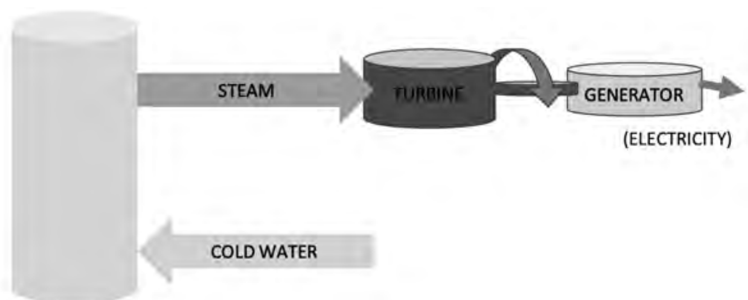


Figure 3.15. Block diagram of a nuclear power plant. One of the things mechanical engineers have to do is design the system so that the steam retains its heat before going to the turbine. This improves the efficiency of the system.

The model of how heat dissipates through the wall of the pipe (which is typically constructed out of heat insulating material) is developed through experimental testing of a simplified system, such as the heat loss of a heated liquid flowing through a straight pipe. This loss is measured for various lengths and radii of the inner and outer cylinder.

An example of such a cylinder, which is a simplified model of an actual pipe, can be seen in Figure 3.16, which shows a 3-D view of a straight section of pipe with a uniform cross-section. On this view, you can see the inner radius, r_i, the outer radius, r_o, and the length of the pipe, l. Of course, this view provides a distorted view of these parameters.

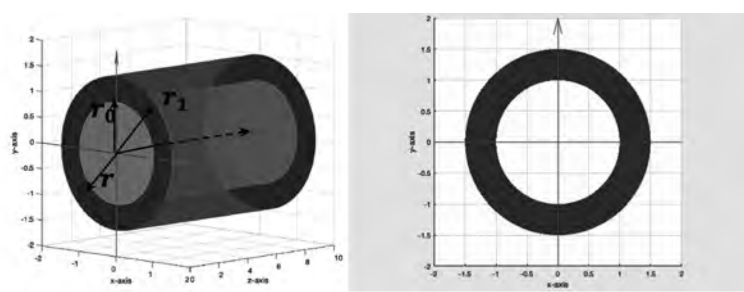


Figure 3.16. (left) General 3-D view of an idealized pipe used to carry heated fluid from point A to point B. (right) Cross-sectional view.

As we discuss the solution, we should also discuss the law that helped us develop the differential equation we will use to develop the equation for heat flow – Fourier's Law, the law of heat conduction, which describes the rate at which heat is transferred through a material. This law states that the rate at which heat transfers through the material is proportional to the negative gradient in the temperature and to the area through which the heat flows and that is perpendicular to the direction of heat flow.

Note that the above law is not obvious. The rate describes the time rate at which the heat transfer through the material (expressed as d/dt). The gradient is the change of heat in 3-D space (expressed as $\nabla \mathbf{q}$.) In our case, the gradient is expressed as d/dr due to the fact that we assumed all the heat flow is radial (i.e., one-dimensional).

The parameter that describes this heat flow is called the heat flux density and is denoted by the vector $\mathbf{q}$. The unit of $\mathbf{q}$ is $W \cdot m^{-2}$, where W is Watts, the SI unit of power. The material through which heat is transferred is characterized through the parameter k, which is the material thermal conductivity, and is a value that characterizes the material's ability to conduct heat. The unit of k is $W \cdot m^{-1} \cdot K^{-1}$, where K denotes kelvin, which is the SI base unit of temperature.

A second problem is presented, pertaining specifically to the process by which heat is generated in the containment structure of a boiling water reactor, such as the type shown in Figure 3.15. The configuration of the containment structure is shown in more detail in Figure 3.17. The heat for the steam generation is coming from the radioactive rods that are inserted into the water. The rods are activated so they go through a process called "nuclear fission," which splits the nucleus of the atom into two lighter, smaller nuclei and thus produces a significant amount of heat. This is enough to create the steam that then escapes out through the steam line and is used to rotate a turbine that powers a generator to generate electricity.

A very significant calculation engineers have to make is how rapidly the nuclei split; this is called the radioactivity or decay rate. If we symbolize the number of radioactive nuclei by $N(t)$, the decay rate A will be given by:

$$A = -dN/dt$$

Experimentation has shown that the decay rate is a statistical calculation, which depends on the probability of decay per unit time, $A = \lambda N(t)$, where λ represents the probability of decay per nucleus per unit time. Thus, the final equation for A is:

$$A = -dN/dt = \lambda N$$

We will see how engineers use this equation to calculate important parameters that control the behavior of the nuclear reactor. A failure to control this reaction appropriately leads to catastrophes such as Three Mile Island (large amounts of coolant were allowed to escape which could have led to an explosion) and Chernobyl (caused by an uncontrolled nuclear chain reaction triggered by a training exercise and causing a core meltdown leading to radioactive

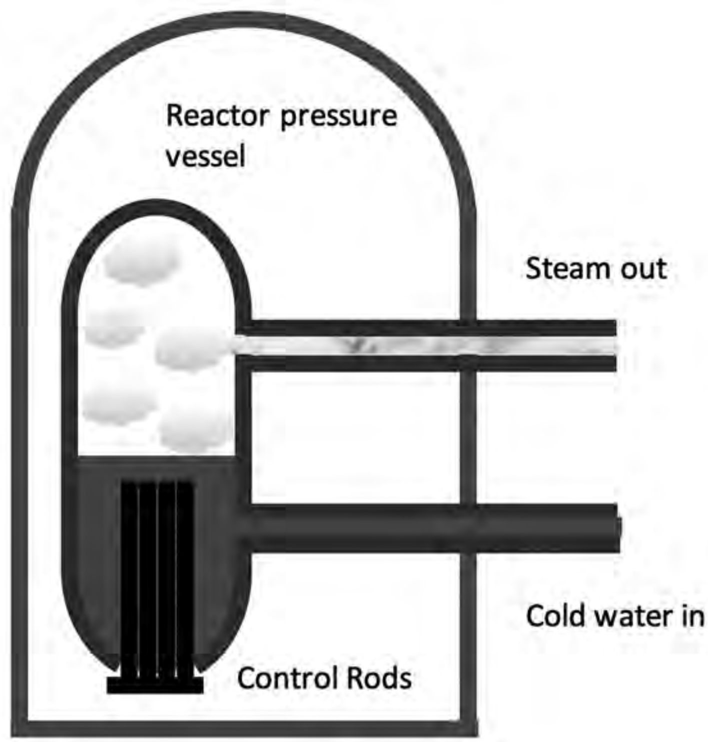


Figure 3.17. In the "Boiling Water Reactor," the water is heated through a process called nuclear fission, which is generated through a radioactive decay process. This is the process by which the nucleus of an atom splits into two or more smaller, lighter nuclei. This is a reaction which in the Reactor Pressure Vessel is controlled through the positioning of the rods in the water. The farther into the water they are inserted, the higher the reaction rate is. The heat of the water needs to be adjusted to a specific temperature, so tracking the rate at which fission occurs is of paramount importance.

contamination). The most recent disaster is the Fukushima Daiichi nuclear disaster, which was caused by the 2011 Töhoku earthquake and resultant tsunami. In all of these cases, the primary cause of the disasters was human error but the ultimate cause of reactor malfunction was overheating of the coolant resulting in a large amount of energy being released.

3.21.4 Implementation Plan

Formal Learning Objectives

- Use standard techniques for solving first order linear differential equations in an application problem.
- Demonstrate the relationship between descriptive language about the system and the form of the differential equation.

Time Required This module is designed to take about 20 minutes. The two problems can be split over different class meetings, if desired.

Implementation Recommendations Students often need prompting to begin the problem. Sometime that involves the instructor prompting the student or possibly the class as a whole. In particular, some students need to be directed to look at the form of the equation and to think about what solution technique is needed. Because the equation is given in the problem description in nonstandard form, it takes pointing them to that equation and leading them to rearrange the equation into a standard form. Often, students also need help talking through which letters are variables and which ones are parameters that are specified in the problem.

3.21.5 Additional Information

The students engaged well with this module. The presentation of the problem with the language and notation common to engineering allowed them to recognize the relevance of the problems immediately. Even if students complain about going beyond just finding an equation's solution and solving for particular states of the system, encourage them to practice making meaning out of the mathematics, and it will help motivate them to practice with algebra proficiency.

Nuclear Engineering Problems

Module with Solutions

In this module, we look at two problems in nuclear engineering: the heat flow problem and the radioactivity problem.

The Heat Flow Problem

Mechanical engineers develop power generation systems, such as nuclear power plants, which use a nuclear power source to heat water that generates steam that powers turbines that generate electricity. This is a block diagram of a nuclear power plant, showing the main stages of the process. To improve the efficiency of the system, engineers need to minimize the heat loss in transporting the steam to the turbine. This allows for the production of maximum electricity. One approach is to develop and solve models that describe the heat loss from cylindrical pipes. Here, we are looking

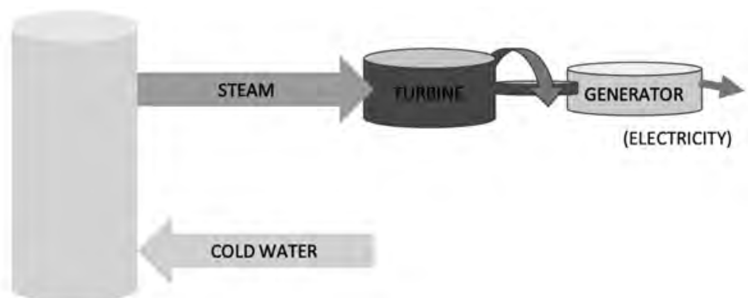


Figure 3.18. A Power System

at the scenario to measure heat loss through the insulation of a cylindrical pipe with a known thermal conductivity k. Consider a typical cross section of a pipe and insulation (see Figure 3.19) with the given thermal conductivity k.

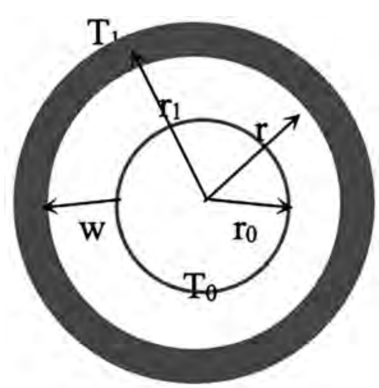


Figure 3.19. A Typical Cross Section of a Pipe and Insulation

We assume the following:

1. all heat flow through the insulation is radial,
2. for the unit length of pipe, all the heat passes into the insulation through its inner surface will eventually pass into the air through its outer surface, and
3. the same amount of constant heat loss per time Q will pass through every cylindrical area between r_0 and r_1 = $r_0 + w$.

If we let T denote the temperature in the insulation at the radius r, it follows that dT/dr is the temperature gradient (temperature change per unit length) in the direction perpendicular to the cylindrical area of radius r. Hence, by

Fourier's law, we can create an equation describing the amount of heat Q flowing through this general area per unit time as the product of thermal conductivity, surface area, and temperature gradient:

$$Q = k \times 2\pi r \times dT/dr$$

1. Rearrange to obtain a differential equation for $T(r)$, then get the general solution for $T(r)$ using separation of variables.
2. Use the initial condition $T(r_0) = T_0$ to calculate the particular solution for this pipe.
3. The figure also shows a boundary condition, $T(r_1) = T_1$. Use this boundary condition to get an expression for the constant Q as a function of r_0, r_1, T_0, T_1.
4. Is it possible to establish the temperature distribution T through the insulation without knowing Q? (yes!)

Solution: We will assume that:

- All heat flow through the insulation (pipe wall) only has a radial dependence.
- Along the length of the pipe, all the heat that passes into the insulation through the inner surface will eventually pass into the air through the outer surface (flowing only in a radial direction).
- The same amount of heat will pass out radially at each point along the length of the line (T_0 and T_1 are the same at each point down the length of the pipe).
- The variable T denotes the temperature in the insulation at any point $r_0 < r < r_1$; the temperature gradient (given condition 1 above) will only have a radial dependency, so is expressed at dT/dr.
- The direction of the temperature gradient is coincident to $\hat{r}$, so it is perpendicular to the cylinder shell (see Figure 3.19).

Fourier's law thus simplifies to:

$$Q = k \times A \times dT/dr$$

Where:

- k is the thermal conductivity of the material
- $A = 2\pi r \times l$ is the surface area of the cylinder at radius r for a length l
- dT/dr is the temperature gradient for $r_0 < r < r_1$

Looking at a cross-section of unit length $l = 1$: $Q = k \times 2\pi r \times dT/dr$.
Since Q is constant, we can rewrite this differential equation as:

$$dT = Q/2\pi k \;\; dr/r$$

which is a separable equation. By integrating, we can calculate T in terms of Q, k, and r:

$$T = Q/2\pi k \ln r + C$$

where C is the integration constant, which must be determined. This will be done by applying the known boundary condition at r_0. (This is a first-order equation, so we only need one boundary condition. We will use the second boundary condition later.)

Applying $T(r_0) = T_0 = Q/2\pi r \ln |r_0 + C|$, and solving for C, we find:

$$C = T_0 - Q/2\pi k \ln |r_0|$$

Substituting into the equation for T, we get:

$$T = T_0 + Q/2\pi k(\ln |r| - \ln |r_0|) = T_0 + Q/2\pi k \ln |r/r_0|.$$

We can use the second boundary condition to calculate Q:

$$T_1 = T_0 + Q/2\pi k \ln |r/r_0|$$

$$Q = 2\pi k(T_1 - T_0)/ \ln |r/r_0|$$

Since k is known, this formula gives the heat loss per unit length and unit time.

Now, we can substitute the equation for Q back into the equation for T to solve for T without knowing Q:

$$T = T_0 + (T_1 - T_0) \ln |r/r_0|/ \ln |r_1/r_0|$$

It is interesting to review a plot of this distribution displayed in Figure 3.20 (note that this is how T changes in r and is shown in one dimension).

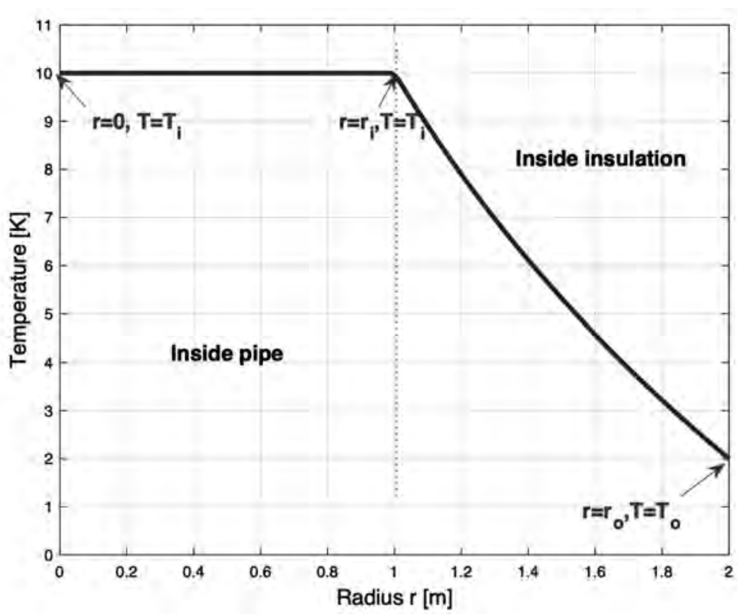


Figure 3.20. The distribution of T along the radius of the pipe insulation. Internal heat is assumed constant.

The Radioactivity Problem

Radioactive activity is equal to the product of the decay constant (λ) and number of nuclei left $N(t)$ at time t; that is, activity = $\lambda N(t)$. It has the unit of the number of disintegrations per second. This unit can also be expressed in curie (Ci). The relationship between Ci and the number of disintegrations per second is 1 Ci $= 3.7 \times 10^{10}$ disintegration/sec. This is equivalent to the activity of roughly 1 g of radium.

Considered a radioactive isotope that is being produced at a constant rate of A_0. The rate of change of $N(t)$ can then be stated as the production A_0 minus depletion due to the rate of reduction of the radioactive nuclei.

1. Create a differential equation that describes the dynamics of the number of nuclei $N(t)$ during the fission process.
2. Using the integrating factor method and the initial condition $N(0) = 0$, solve for $N(t)$.
3. What is the “saturation activity,” that is, the rate of nucleus decay as $t \to \infty$?

Solution:
In the fission process, the rate of reduction of radioactive nuclei is given by:

$$dN/dt = A_0 - \lambda N$$

where: λ is the decay constant (this has been determined statistically), N is the number of radioactive nuclei present at any given time, t.

The integrating factor (IF) in this case is:

$$\text{IF} = e^{-\lambda t}$$

and the general solution is:

$$N(t) = \frac{A_0}{\lambda} e^{-\lambda t}$$

Thus, the number of radioactive isotopes we have at any given time (when isotopes are generated at a rate A_0 and decay at a rate of $A = -dN/dt$ is:

$$A(t) = \lambda N(t) = A_0(1 - e^{-\lambda t})$$

If we take the limit of the above equation as $t \to \infty : A \to A_0$

So, the limiting value of A as $t \to \infty$ is A_0, which is the rate of nucleus production. This limit is called the “saturation activity.”

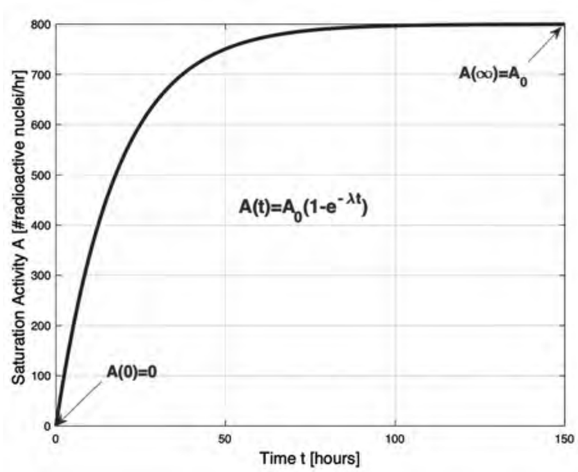


Figure 3.21. Representative curve showing the saturation activity A, which converges exponentially to the rate of nucleus production, A_0.

3.22 STEM Students as Storytellers: Media Journals

Debra Bourdeau and Beverly Wood
Embry-Riddle Aeronautical University
Contact: taylo13f@erau.edu

3.22.1 About the Module

- Course: Mathematics for Liberal Arts (used in How Fiction, Film, and Popular Culture Represent Science and Mathematics at Embry-Riddle Aeronautical University)
- Partner Disciplines: Humanities and Communication
- Required Technology: podcasting software, presentation software and/or video editing software. Examples are provided.

3.22.2 Institutional and Course Contexts

- Type/size of institution: Embry-Riddle Aeronautical University (ERAU) is a private institution with residential campuses in Florida and Arizona as well as a Worldwide campus with over 130 global locations and a nationally recognized online program. Worldwide has over 22,645 students, and about 90% of courses are delivered asynchronously online. The university is known for aviation/aerospace programs.
- Size of Class: up to 30 students
- Characteristics of Students: The students are mostly part-time adult learners in STEM programs.
- Mathematical Content: Varies, depending on student selection
- Purpose/Goal of the Module: The Humanistic STEM (H-STEM) focus of the course is reinforced by having students find blends of humanities and STEM outside of the classroom in their everyday lives. Students are compelled to make connections between these meta-disciplines. Doing so allows them to more clearly understand the transferability of skills and concepts from their general education courses into their program courses then into their careers. Students who make these connections then find value in their complete educational experience.
- After and Before: This module comes after an exploration of the concept of Humanistic STEM as a blending of STEM and humanities content and methods of inquiry. Students have watched introductory videos and participated in discussion board activities meant to reinforce this central focus of the course. As their final project, students will create a "trailer" for a new H-STEM project that they are pitching. Therefore, they will continue to deliver multimedia content in the course.

  In other course contexts, it may be useful for an instructor to show how STEM and the humanities impact each other through a short video or guest appearance with a humanist colleague..
- Other Prerequisites: There are no additional prerequisites for this assignment. Students who register for this upper-level course are required to have at least three semester hours in humanities and three semester hours of mathematics. Most of the students in this course have taken precalculus.
- Inspiration for the Module: The inspiration for this module comes from a desire to reinforce the relevance of course concepts outside of the classroom. Understanding the full intellectual context of a work of art, a piece of architecture, or even a film, provides a richer experience in every aspect of a student's life. Students should see the blend of disciplines in the world around them to underscore the artificiality of academic silos. Specifically referring to the project as a "journal" highlights the storytelling aspect of the assignment, giving students license to tell the tale of H-STEM in their lives.

  The Association of American Colleges and Universities (2021) reports that employers prefer graduates who can think critically, solve problems creatively, reason ethically, and communicate effectively. Students often place these abilities within the domain of their humanities classes. Assignments such as this one are effective in mathematics (and other STEM) courses as they reinforce that disciplinary boundaries are ultimately permeable. Students learn to see how the disciplines impact and influence each other, developing habits of mind that include flexible and interdependent thinking.

3.22.3 Partner Discipline Background

This course is part of ERAU's Humanistic STEM (H-STEM) initiative. We define H-STEM as "blending the study of science, technology, engineering, and mathematics with interest in, and concern for, human affairs, welfare, values, or culture." H-STEM is a way for us to elevate the humanities among our STEM-focused students who often do not value these courses despite the fact that they provide the critical thinking, communication, and creative problem-solving skills that employers value H-STEM compels students to create, to solve problems and to dwell in complexity, resulting in a STEM workforce that has the skills, habits of mind and ways of knowing that foster innovation. H-STEM courses are co-developed, and ideally co-taught, by both humanities and STEM faculty. This specific course (HUMN 333: How Fiction, Film, and Popular Culture Represent Science and Mathematics) is the pilot for the H-STEM concept and also serves as the anchor for a set of courses that uses multiple interdisciplinary lenses to explore important concepts. Other H-STEM courses include digital humanities, data visualization, and the history of communication technologies. Students are able to use HUMN 333 to fulfill the upper-level humanities requirement in the general education program.

STEM-focused students are introduced to the humanities as a meta-discipline that includes languages, literary studies, religious studies, political science, philosophy (specifically ethics), study of visual/performing arts, history, anthropology, archaeology, linguistics, and classics. As such, they learn that the humanities are deeply connected to the human experience. They are also cautioned, however, not to completely conflate "humanities" with "human" and are advised that understanding, investigating, and interpreting are essential elements in the definition of humanities. Through this project, students are asked to determine how the meta-disciplines of STEM and humanities inform and influence each other.

3.22.4 Implementation Plan

Formal Learning Objectives This course requires submissions of four (4) Media Journals by the conclusion of Module 8 (there are a total of nine modules, one per class week). Media journals should be based on examples of the connections between humanities and STEM that students observe in their everyday life. These examples might be from music, television, film, museum visits, newspaper/magazine articles, books, or scenic locations. The goal is to ensure that they see the blend of humanities and STEM that surround us and that they are able to explain those connections. Students also learn to communicate mathematical and scientific concepts in multiple digital formats, a skill that has full utility regardless of discipline, learner level or workforce sector. This assignment is aligned with the following course Student Learning Outcome: "Engage in integrative learning by making connections across disciplines."

Materials and Supplementary Documents Students will need to use podcasting software (e.g. Podbean), presentation software (e.g. PowerPoint) or video editing software (e.g. iMovie). Multiple free options exist. Most students are able to embed their media journals into the Canvas LMS, but others provide links to their YouTube channel or podcasting service so that classmates can "subscribe" to their media journal series.

Time Required Students are required to create four media journals over the nine course weeks. To avoid procrastination, media journals are due in every even-numbered week. There is no specific length requirement for the journals, but they are instructed that length is not a primary concern as long they can be "less than 5 minutes." The time required to complete a media journal varies based on the student's familiarity with the technological tools. Students who record a podcast, for example, may only require approximately two hours to script and record the episode and add theme music. Students who are producing video blogs may need about three hours to gather and organize images, create slides, and record voice overs or add video elements.

Implementation Recommendations This assignment might be different from anything the students have been asked to produce in past courses. Both the technology and content expectations may be unfamiliar to them. As a result, it is important to provide at least one example. In this course (How Fiction, Film and Popular Culture Represent Science and Mathematics), each instructor produced an example media journal. Dr. Bourdeau added a podcast episode explaining Yayoi Kusama's Infinity Mirrors exhibition. Dr. Wood created a voice-over-PowerPoint presentation on the mathematics and architecture of Machu Picchu. These examples help to ease student anxiety about the project. Additionally, instructors should not strive to provide "perfect" examples, as students need to see that some mistakes might be inevitable and should not lead to increased production time.

Because they are an important component of enhancing student engagement, the media journals should be posted on a discussion board or another "public" part of the course. Students should be required to comment on their classmates' work as part of a discussion or participation grade in the class. A requirement to comment on two additional journals, for example, will ensure that students are reviewing the work of their peers.

Encourage students to step outside their comfort zone to use unfamiliar presentation methods. Make sure you always encourage student to explain the *blend* of the disciplines.

Alternative Solutions Ambitious students might wish to produce animated presentations. KeyShot, PowToon, and Animaker all have free trials or even free versions with limited features.

The assignment states that students are required to use at least two different kinds of media. However, exceptions have been made for students who wish to maintain a podcast series or YouTube channel, particularly those who show interest in continuing the project beyond the scope of the class.

Common Errors and Questions Because of the STEM focus of the institution, students typically have no issues with being able to explore and explain the STEM content, but often have to be prompted to include and discuss the humanities blend. Often, students confuse the meta-discipline of humanities with the more basic ideas of "human" or "humanity." To assist students with understanding the humanities as a group of academic disciplines, we created and added an infographic "What are the humanities?" to further define and explain the humanities and to explain what elements to look for in producing their H-STEM projects.

Tips to Handle Student Resistance Students will often initially resist being pushed out of their comfort zone by being compelled to present their work using technologies with which they may not be familiar. It is important to insist that students produce media journals, however, that feature their voices. This element alone increases the sense of a community of scholars in the classroom, particularly when the course is delivered asynchronously.

3.22.5 Additional Information

Media journaling is applicable in *any* mathematics course where drawing on real world examples of mathematical content is important. The search for the blend of humanities and STEM can be targeted more narrowly to mathematics and still give students experience with considering humanities as an important part of mathematics' role in society.

- Architecture is a blend of aesthetics and engineering, both supported by mathematics. Symmetry and proportion are equally important for a successful project.
- Urban planning blends ethics and aesthetics with science and mathematics. The former pair informs the who, what and where; the latter addresses the how and when of the effort.
- Parks, art museums and concert halls all contain elements of the previous two examples that can serve as places to discover mathematics in student environments.

A media assignment looking for visual mathematics is almost certain to uncover it in a humanities context. Being open to discussion of the real life blending of what seems to be diverse disciplines is eye-opening to students and faculty alike.

There is no activity sheet that is needed for reproduction, nor right or wrong answers. Instead, we offer a grading rubric, a breakdown of what students have submitted, and sample screenshots from student submissions. For more details, contact the authors.

3.22.6 Grading Rubric

The grading rubric we use is:

Criterion	**Proficient**	**Satisfactory**	**Unsatisfactory**
Topic is appropriate for assignment	20 - 15	14 - 10	9 - 0
Student selected appropriate audio/visual tool for presentation	20 -15	14 - 10	9 - 0
Student's presentation clearly establishes a link between humanities and STEM	50 - 40	39 - 29	28 - 0
On-time submissions	10 - 8	7 - 6	6 - 0

3.22.7 Media Journal Submissions in 2020 HUMN 333 Offerings

During the year 2020, the breakdown of submissions were:

By Type		
Podcasts	43%	
Screencasts	55%	
Video Logs	3%	
By Subject Subject	Percentage	Example Topics
Technology	24%	Virgin Hyperloop, tech inspired by Star Trek, video games
TV/Movies	15%	*Interstellar*, *Star Wars*, *The Big Bang Theory*
Architecture	12%	Chichen Itza, Sanctuary of Truth, urban planning
Sports	12%	Physical fitness, gymnastics, professional team sports
Arts & Music	15%	Deltron 3000, Space Oddity, Monet, The Rain Room
Other	23%	Art & Eng of Ferrari, Are We Failing our Future?, Storm Chasing

3.22.8 Sample Screenshots

The images below are screenshots of sample student work.

Figure 3.22. Student Screenshot Sample 1

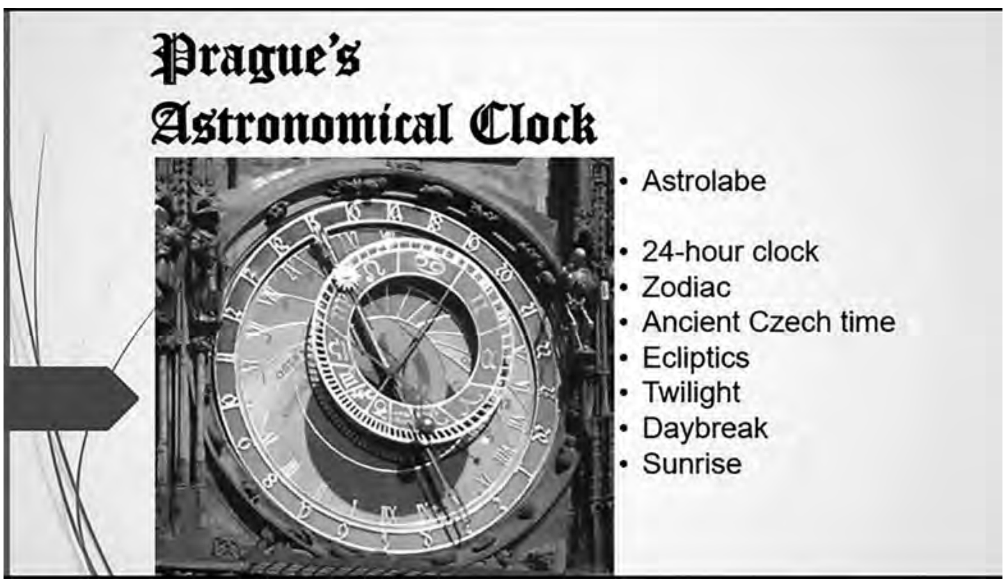


Figure 3.23. Student Screenshot Sample 2

Editors and Acknowledgments

About the Editors

Susan L. Ganter is Provost and Executive Vice President for Academic Affairs at The University of Texas Permian Basin. Her work focuses on innovations in postsecondary STEM curricula, including programs designed to improve success rates for underrepresented students. As an example, Dr. Ganter has been Director (since 1999) for the Curriculum Foundations (CF) Project housed at the Mathematical Association of America, resulting in research on interdisciplinary collaborations designed to support the preparation of students for STEM careers. Currently, she is Lead PI for the NSF-funded SUMMIT-P national consortium of fifteen institutions that are implementing the CF outcomes, and her work has been continuously funded since 1994. Previously, Dr. Ganter was Dean for the College of Arts & Sciences at Embry-Riddle Aeronautical University; a Mathematical Sciences faculty member at Virginia Tech, Clemson University, and Worcester Polytechnic Institute; and a Senior Research Fellow at the National Science Foundation.

Debra Bourdeau is an Associate Dean and Associate Professor in the College of Arts & Sciences at Embry-Riddle Aeronautical University. She is an interdisciplinary humanist holding a PhD in English. She has maintained an interest in intermediality and transmedia storytelling, focused on the intersection of narrative text and visual images and, specifically, on the work of painter/engravers such as William Hogarth and William Blake who sought to construct visual stories through the use of identifiable, recurrent iconography. Her research projects also include Humanistic STEM—a curriculum model that blends humanities and STEM to create a unique interdisciplinary experience that will allow students to integrate ideas, issues, and ways of knowing from diverse academic disciplines in order to expand their capacity for analysis, critical thinking, and creative problem solving.

Victor Piercey is a Professor of Mathematics and the Director of General Education at Ferris State University in Big Rapids, Michigan. His curricular work has involved the interdisciplinary development of a 2-semester sequence of general education courses entitled Quantitative Reasoning for Professionals. Developing this sequence involved collaboration with faculty in business, social work, and health professions, funded by the National Science Foundation. The course integrates social justice, health, and business with mathematics. He also designed and co-facilitated faculty learning community to integrating quantitative reasoning and partner discipline content in math courses and in partner discipline courses. He is currently working on the design and implementation of a faculty learning community to integrate general education and major programs through multi-course student learning communities. He is also currently working on the integration of ethics into undergraduate mathematics and actuarial science courses.

Afroditi V. Filippas is the Commonwealth Center for Advanced Manufacturing (CCAM) Professor at Virginia Commonwealth University (VCU) in Richmond, Virginia. Her role as CCAM Professor is to foster and enable public-private partnerships between CCAM and its industry and academic partners. Her research focuses on data analytics, process modeling and electromagnetic components modeling; her passion is education. Specifically, Dr. Filippas has developed courses geared towards engaging all students in the learning process by blending a variety of teaching modalities and classroom experiences. As part of this coursework development, Dr. Filippas received the "Outstanding Diversity Paper Award" at the 2017 ASEE Annual Conference & Exposition. Dr. Filippas also served as Associate Chair of the Electrical and Computer Engineering department and later as Associate Dean for Undergraduate Studies for the VCU College of Engineering. During this time, Professor Filippas developed and fostered unique undergraduate experiences, such as the da Vinci program and VIP (Vertically Integrated Projects). She also served as mentor to a number of student organizations, and continues to serve as the faculty advisor to the VCU Society of Women Engineers (SWE) and HKN. In 2019, Professor Filippas also took on the role of joint chapter chair of the IEEE Education Society, Richmond and Northern Virginia Chapters.

Acknowledgments

This volume was developed in part through the project Collaborative Research: A National Consortium for Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P, www.summit-p.com) with support from the National Science Foundation, EHR/IUSE Lead Awards 1625771, 1822451. The opinions expressed here are those solely of the authors and do not reflect the opinions of the funding agency.