SHARING AND STORING KNOWLEDGE ABOUT TEACHING UNDERGRADUATE MATHEMATICS:

AN INTRODUCTION TO A WRITTEN GENRE FOR SHARING LESSON-SPECIFIC INSTRUCTIONAL KNOWLEDGE

Douglas Lyman Corey
Steven R. Jones

Editors

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The following individuals were participants in the Conference on Building a Knowledge Base for Teaching Undergraduate Mathematics.

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We would like to dedicate this book to our friend and colleague, the late Karen Keene. Karen was so supportive of this project and was instrumental in getting it funded while working at The National Science Foundation. She also joined the project as a participant and author. She passed away before she could see the finished project.

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We would like to thank all of the conference participants who signed up for one conference, but thanks to COVID-19 restrictions, attended quite another. We appreciate all of the work that each of them put into making the conference a success and this book a reality.

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Contents

1

Introduction to Lesson Analysis and Lesson Analysis Manuscripts

Douglas L. Corey, *Brigham Young University*
Steven R. Jones, *Brigham Young University*

Undergraduate mathematics suffers from the same lament that Dewey [4] pointed out about K–12 schooling in the US around a century ago: teachers take their best ideas with them when they retire, and new teachers are forced to start over when they begin their profession. Although some of this knowledge may live on through verbal exchanges or in small communities or in available print or electronic media, we still struggle to effectively share instructional knowledge in the undergraduate mathematics community. There is a paucity of resources (at least certain kinds of resources) for instructors to help them learn their craft and no coordinated effort to help develop these resources from knowledgeable and experienced practitioners. To be clear, we feel there is a tremendous amount of valuable instructional knowledge within the undergraduate mathematics community, but we lack methods and systems to share the current knowledge and future knowledge among our community.

This book proposes a method for sharing mathematics teaching knowledge and experience at the undergraduate level. This work grew out of a conference of 20 faculty that teach undergraduate mathematics, many of whom belong to the Mathematical Association of America's special interest group on Research in Undergraduate Mathematics Education. The main focus of the conference was to develop and experiment with a new genre of scholarly writing that could capture important knowledge of teaching undergraduate mathematics. We picked this as the focus because we felt that this was a natural next step in improving our ability to generate, store, and share instructional knowledge of undergraduate mathematics.

Our vision is that reflective undergraduate mathematics instructors will engage in the scholarly work of crafting, refining, documenting, and sharing (through publication) insights into specific lessons that address important instructional challenges. Doing so provides the community with useful products that others can use, modify, test, and build on. Other instructors could, in turn, publish their new work that adds on and extends that previous work. These efforts can help our community begin to build what some researchers have referred to as "a knowledge base for teaching" [7].

This chapter is constructed in three sections. In the first section we motivate the need for taking up the issue of sharing instructional knowledge in the undergraduate mathematics community. In the second we build off of the existing work of building a knowledge base for teaching, in order to lay out the kind of knowledge that we take as most important to share. In the final section we offer greater detail about the conference that gave rise to this book.

1.1 Background Leading to our Current Approach

To introduce the problem that this book tries to address we begin with a personal story from Doug (the first author on this chapter).

> As a new professor, I was encouraged to pick up a couple of books from a long reading list of books about teaching. I already loved teaching and was anxious to learn what I could about being a better teacher. One of the books jumped out at me as one that looked promising: *What the Best College Teachers Do* [1]. It was a study of professors from many disciplines, all of which had reputations for being amazing teachers. The book summarized the commonalities across what these teachers did, and the book explains these commonalities.
>
> The book was interesting, and I did learn a lot. The principles seemed sound and I tried to implement some key ideas. But it was hard. It wasn't hard because it took effort. I was putting plenty of time and energy into my teaching. It was hard because the ideas in the book only spoke to a few of the decisions that I needed to make as a teacher. For example, the book explained that these professors "used their knowledge to develop techniques for grasping fundamental principles and organizing concepts that others can use to begin building their own understanding and abilities" [1, p. 16]. But reading the book didn't give me the techniques or organizing principles to do this for the mathematics that I regularly teach. It didn't help me to know how to "simplify complex subjects" or "to cut to the heart of the matter with provocative insights" [1, p. 16]. They pointed me in a general direction, but when it came to making the myriad of instructional decisions I needed to make for each lesson, I did not have a good sense for how to practically use their recommendations for those decisions, or how to judge which decisions would be most beneficial of the many options I might take to try to satisfy the principles in the book. This was a difficult task for a young professor just starting out.

Contrast this experience with one we (both authors) have had (and continue to have) years later. Steven (second author on this chapter) took a position at Brigham Young University (BYU) and was put in an office next to Doug. We had similar interests and taught many of the same courses. Consequently, we had many conversations about our teaching, but in a very different discourse than the book Doug had read years earlier. Our discussion was all about specific decisions or mathematical understanding relating to particular content. We both gained so much knowledge that we could use to improve our practice because the conversations were couched in the specific mathematics and context in which we were teaching. As we have talked to other colleagues around the country, we find ourselves sharing what we have learned.

We believe these personal reflections highlight four characteristics about sharing knowledge and experience of undergraduate mathematics teaching. First, publications tend to discuss teaching with highly general ideas and principles, often illustrated with an example or two. Second, by contrast, much of the knowledge of teaching is grounded in specifics. The content, the students, the institution, the learning goals, and so forth, all impact what could or should happen in a classroom. Third, we usually don't have organized and intentional opportunities to discuss and analyze teaching practice in the context of specific lessons. Fourth, we don't have a method for easily sharing and storing specific and contextualized instructional knowledge and experience that has been accumulated by an individual instructor over time. For example, the knowledge that we (Doug and Steven) gained from each other in our conversations helped us, but others do not have access to what we were learning.

Although we still have a lot to learn, we can at least start by engineering a way for the undergraduate mathematics teaching community to share what we know with each other by creating a system that makes public instructional insights, strategies, and knowledge in an easily accessible way. To that end, this book proposes a means for sharing mathematics teaching knowledge and experience at the lesson level, which we call "Lesson Analysis Manuscripts" (LAM). This term refers to three important aspects: lesson implies that what is shared is specific to a particular lesson, analysis refers to the work of investigating and reflecting upon the specific choices made in planning and implementing a lesson, and manuscript refers to the document that is published to store and share that body of knowledge and experience. Before we get into the nuts and bolts of what a LAM is, and how one might write and publish one (which is the purpose of the next chapter), there are some important background ideas to get on the table first. The rest of this introductory chapter provides this background.

1.2 Building a Knowledge Base for Teaching Undergraduate Mathematics

First, a definition: A knowledge base for teaching is a system that collects and shares practical knowledge about teaching [7]. Practical here does not mean atheoretical. Practical knowledge of teaching can include theoretical tools about learning and teaching that prove to be useful for teachers but requires more than just theoretical ideas or general principles. A knowledge base needs to make available the kinds of knowledge that are used in the profession. Undergraduate mathematics instructors make decisions in a specific instructional context, think about their current students, focus on helping students understand particular mathematical content (among other things), and may draw on a specific textbook or set of materials. One key example of a knowledge base for teaching that has received attention is the junior high and elementary school teaching system in Japan [7]. In this system, as teachers strive to improve their teaching and develop high-quality lessons, they have extensive resources on which to draw. These include:

- Lesson study: A form of professional development where small groups of teachers study and create a lesson to share with others. It is typical to be in one lesson study group a year as a creator, but teachers typically attend 4 to 6 lessons to observe a lesson study lesson.
- Detailed lesson plans: Lesson study groups make their detailed lesson plans available for other teachers, either in a local repository in the school, district, and/or city; or published in conference proceedings or books for teachers.
- Teacher mathematics circles: Monthly evening meetings to discuss teaching and analyze lessons with other teachers.
- Books for teachers: Hundreds of books are published each year for teachers which are readily available in a dedicated section for teachers in any commercial bookstore.

Readers may wonder why we are going all the way to Japan, and down to junior high school or even elementary school to find a model for building a knowledge base for teaching. Aren't there models from the US in K-12 or from already existing materials for undergradute mathematics instructors to build from? The short answer is "No", high-quality resources of the type that are plentiful in Japan (detailed cases of carefully designed lessons, implemented with typical students, focused on teaching specific content, and written for other teachers) are not easy to find in the US at any level. Sharing lesson plans in K-12 is becoming more prevalent with websites like betterlesson.org and teacherspayteachers.com, but some research has shown even the highly rated lessons from lesson-plan-sharing cites (specifically betterlesson.org) tend to be of low quality and don't have the same characteristics of the lesson plans from Japan [3]. Shared instructional materials for teaching undergraduate mathematics also lack the core information and knowledge that makes the Japanese knoweldge base for teaching so useful and powerful [2]. Consider the vision that some researchers (e.g., Ermiling, Hiebert, & Gallimore, [5]) have articulated of a viable knowledge base for teaching and how it compares to instructional materials available to most undergraduate mathematics instructors:

> [T]here's an urgent need to develop a carefully indexed knowledge base of useful cases. Imagine digital libraries stocked with lesson videos accompanied by multiple resources, such as expert teacher analysis, commentary, and interpretation of each lesson; well-specified learning goals coupled to formative assessments for monitoring progress and providing feedback; alternative instructional moves for creating learning opportunities to address specific student needs; and links to resources that support deeper teacher learning of subject-matter content. Not a few lesson videos, and not just those taught by stars, but, instead, many good-although not perfect-lessons all easily accessible through contemporary technologies. (p. 51)

Hiebert and Morris [8], in studying professions that have a knowledge base of teaching, described four features they tend to share:

1. Shared goals across the system
2. Visible, tangible, changeable products
3. Small tests of small changes
4. Multiple sources of innovation from throughout the system

With this book, we hope to accelerate the building of a knowledge base for teaching for undergraduate mathematics. Notice that the second feature (visible, tangible, changeable products) calls for a vehicle for capturing instructional knowledge and experience. The third feature (small tests of small changes) implies a kind of scholarly work that is done to create the products suggested in the second feature. We propose in this book a kind of scholarly work and the resulting published products that could be effective at building a knowledge base for teaching in undergraduate mathematics. We call the work *Lesson Analysis* (LA) and the resulting published documents *Lesson Analysis Manuscripts* (LAMs). This work formalizes what some reflective teachers have already been doing and is a type of design research.

Lesson Analysis is a process where individuals or groups have identified an important challenge in instruction, designed a careful lesson to address that challenge, implemented their lesson in their own classroom(s), and documented the outcomes. Lesson Analysis could also consist of refining a lesson over many iterations of teaching it until one is satisfied with the outcome. Lesson Analysis Manuscripts then share the accumulated knowledge and experience with others through carefully written-up details about the background and context of the lesson, the rationales for decisions in the lesson, the specifics of the lesson itself, how the lesson played out in the classroom, the specific student thinking, and some evidence that the lesson was successful in resolving the instructional challenge. We strongly believe that many in the undergraduate mathematics teaching community are doing much of the work in what we are calling lesson analysis and think more could engage in this work. Providing a space for many professionals across the country at different types of institutions to publish the results of their work could accomplish the fourth feature listed above.

What we are introducing in this book is not that same as Japanese Lesson Study, although the label of Lesson Analysis may imply that there is some connection, which there certainly is. Lesson Study is a process teachers engage in to learn about teaching for themselves (profesional development) and to help build knowledge for the teaching community at-large. The knowledge generated as part of lesson study is often shared or published in a form called the long-form lesson plan. Japanese mathematics teachers make a distinction between a short-form lesson plan and long-form lesson plan, where the latter includes extensive amount of detail about the reason why this topic was choosen, the student thinking in the lesson, the lesson activities, and justificaiton for instructional decisions. The long-form lesson plans include a script of key questions and anticipated student mathematical thinking that details the flow of the lesson. What we are introducing in this book is a genre like the long-form lesson plan used by teachers in Japan, but for instructors of undergraduate mathematics. We are putting forth the LAM genre as a possible way to store and share detailed instructional knowledge. So this book is much more about the product of sharing instructional knowledge (LAM, long-form lesson plans), than it is about the process to create instructional knowlege. The work to create such knowledge, as mentined previously, is already happening formally and informally, at least in part, by many undergraduate mathematics instructors and researchers.

James Hiebert [4] captured the feeling of Lesson Analysis as he described the related idea of "studying teaching." He calls for conversations or resources about teaching to be around specific, lesson-level learning goals where teachers can begin to consider specific actions that are best at achieving those goals in a specific context. He explains that only through focusing on specific details can the cause-effect relationship of teaching and learning come to be understood:

> In addition to unpacking the details of teaching, studying teaching means seeing the cause-effect relationship between teaching and learning that infuse an ordinary lesson. Many teachers do not appreciate that slight changes in lessons – in the ways they interact with students around content – influence directly what students learn. When teachers see the effects of the changes they make on what and how well students learn, they can begin to appreciate the powerful impact of studying the details of teaching [4, p. 53].

In their work, Hiebert and Morris [8] listed some important components to include in sharing instructional knowledge: lesson purpose and learning goals, student thinking, rationale of instructional decisions, curriculum as a connected set of ideas, and the specific instructional activities. A recent study showed that knowledge in these categories were quite rare in existing instructional materials for undergraduate mathematics instructors with only a handful having four categories or more [2]. With so few publicly available good examples, we realized that it would be crucial to establish guidance, for ourselves and for others, on how to do this work and how to write up a publishable LAM. We decided that we needed to convene a group of highly reflective undergraduate teachers in order to flesh out the components of a LAM and to put together a first step toward a "how-to manual" on doing this work. Thus, to finish this introductory chapter, we explain how the components of a LAM came together through a special conference on starting a knowledge base for teaching undergraduate mathematics.

1.3 The Conference on Building a Knowledge Base for Teaching Undergraduate Mathematics

We received a grant to hold a working conference in the summer of 2020 for 20 individuals from the Mathematical Association of America (MAA). The participants were from the MAA and American Mathematical Association of Two-Year Colleges (AMATYC) communities. The participants included the president and vice-president of MAA and a large representation from the MAA-RUME (research on undergraduate mathematics education) special interest group. The focus of the conference was to develop ways to bolster the knowledge base for teaching for the undergraduate mathematics teaching community. The primary task at hand was to develop a genre of professional writing in which we could share a large amount of instructional knowledge in a written manuscript and generate specific high-quality examples. Associated with this primary task was the challenge to describe the scholarly work associated with the process of generating, documenting, and writing up the instructional knowledge and related lesson information in this new genre. To start this, prior to the conference, all participants were asked to attempt to write a document that captured their instructional knowledge and experience in the context of one lesson that addressed an instructional challenge. The items suggested by Hiebert and Morris [8] and a few others were given as a set of preliminary guidelines to help focus the writing. We began the conference by discussing issues and difficulties that arose for us as we developed drafts of these manuscripts.

The conference proceeded with participants reading each others' draft manuscripts in between whole group meetings, discussing them in small groups, and then bringing ideas back to the whole group for consideration. Through this process, we were able to hone in on certain components that manuscripts might contain to make them suited to creating a knowledge base for teaching undergraduate mathematics. We also discussed issues of vetting, publishing, and storing these manuscripts. Afterward, we all worked toward creating a short "how-to manual" that could be shared so that many others in the undergraduate mathematics instructor community could participate in contributing to a knowledge base for teaching. We then each took our own LAM drafts we had written prior to the conference and revised them according to our discussions and our resulting guidelines. The next chapter of this book contains the "how to manual" that we created for the work of writing a Lesson Analysis Manuscript. The bulk of the remaining chapters then consist of all of our revised LAMs from the conference participants. We hope that these LAMs serve as strong examples of how this format can share instructional knowledge and experience with the undergraduate mathematics teacher community.

Finally, of course, we realize that as in any engineering/design task, there may be many solutions to the problem of creating such a knowledge base. As a conference group, we made certain choices in terms of the way we might share instructional knowledge in these LAMs. Our solution is not a catch-all, but, as we describe below, it has many advantages and builds on previous efforts to build a knowledge base for teaching. (Further information on how Lesson Analysis connects to other efforts to improve instruction, namely Scholarship of Teaching and Learning and Design Based Research, is found in Chapter 15 of this book). It is true that other kinds of work and other types of products could also serve as effective vehicles, but manuscripts similar to what we propose in this book have been used before by other groups to share instructional knowledge [8]. These products share knowledge at the grain size of a single lesson, allowing the discussion of a wide breadth of decisions and rationales. These lessons can then be easily adapted and tested by others in practice. The next chapter in this book explains in detail what a LAM is and how someone might do the work in order to write and publish one.

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2

How to Write a Lesson Analysis Manuscript

Participants of the Conference on Building a
Knowledge Base for Teaching College Mathematics

The purpose of this book is to propose a genre of professional writing to college mathematics instructors that can capture the work of reflective teaching [9] at the lesson level. Because we hope that many two-year and four-year college instructors will engage in doing Lesson Analysis (LA) and publishing Lesson Analysis Manuscripts (LAMs), this chapter is meant as a "how to" guide to invite as many college mathematics instructors as possible to participate. We acknowledge up front that there is a lot of information in this chapter. This is because we want this chapter to be comprehensive, both for anyone interested in writing a LAM and for reviewers assessing LAMs. Rather than simply outlining the basic ideas of a LAM, we decided we want this chapter to serve as a detailed resource one can come back to during the LAM writing, or evaluating, process. Because of the amount of information in this chapter, this is how we recommend reading it. First, we recommend reading through the chapter in its entirety once to get a sense for what the essential components of a LAM are. In this first read, though, we would not expect someone to retain all of the explanations, details, and suggestions contained in each section. However, after the first read, we recommend that this chapter then become a reference to come back to, reading a specific section for deeper insight, or for double-checking that the necessary information for that section is contained in a LAM.

As mentioned in the introduction, this book grew out of a conference of 20 faculty who teach college mathematics. This chapter was written jointly by the entire conference group by first dividing up sections and assigning them to two or three participants to write an initial draft, and then afterward reworking the chapter to bring the various sections into flow with each other. We decided to write this "how to" chapter in the second person, in the hopes that it feels inviting to you to engage in writing a LAM.

2.1 Two Guiding Characteristics of a LAM

To start, we first explain two guiding characteristics on what a LAM is. First, a LAM should be ***a complete instance of teaching***. By this we mean that your LAM should be a self-contained, detailed accounting of all of the parts of the lesson, including what you teach, how you teach it, what the context of the lesson is, what you and your students are doing in the lesson, and how the lesson achieves its goals. The focus of a LAM could be on learning a particular mathematical concept or idea (e.g., understanding mathematical induction, motivating why Taylor series are an important topic, or developing a quantitative meaning for definite integrals), or it could be on helping students develop a particular mathematical practice or belief (e.g., creating proofs, being willing to conjecture, or a specific aspect of equitable teaching). In order to contain the detail needed to be a complete instance of teaching, we suggest a LAM cover a lesson that is typically taught in one class session, or perhaps that stretches over two or three class sessions, if necessary. We agree with other scholars that instructional knowledge and experience is made more meaningful and usable if it is couched in a complete instance of teaching that does not stretch over too long of a time period [7]. Trying to take on too much material in a LAM reduces the space to include much of the detail needed to share instructional knowledge and experience in a way that builds a knowledge base for teaching.

Second, a LAM should fulfill a ***dual purpose***. One purpose is to share the details of what you teach and how you teach it. The other key purpose, and arguably the more important purpose, is to share the why behind your instructional decisions [7]. The LAM should effectively reveal the reasoning behind these instructional decisions, not just the instructional decisions themselves. It is this reasoning that other instructors can use to help improve their own teaching, which is a central part of building a knowledge base for teaching undergraduate college mathematics.

2.2 Essential Components of a LAM

In order to achieve the two guiding characteristics given above, we see several components as being essential to include in writing a LAM. Building on previous work in this area, and from the results of our conference, we expect the following components to be part of any published LAM:

1. Articulating the Instructional Challenge (§2.3)
2. Providing Background and Context Information for the Lesson (§2.4)
3. Providing Rationales for Instructional Decisions (at both the whole-lesson and specific-moment levels) (§2.5)
4. Discussing Equitable Teaching Practices (§2.6)
5. Detailing the Tasks, Questions, Activities, and Flow in the Lesson (§2.7)
6. Documenting Student Thinking and Reasoning (§2.8)
7. Discussing Post-Lesson Considerations (§2.9)

The remainder of this chapter goes through each component one by one to explain what we mean by that component, and how you might include it in your LAM. To give examples, we refer to the LAMs in this book by the author's last name. While we certainly expect some freedom in terms of how these components are included in a write up, we hope this chapter serves as a guide for incorporating these major components. By doing so, we hope LAMs can become a powerful genre of professional writing that can widely share instructor experiences and knowledge about teaching specific content or practices that many other college instructors can benefit from.

2.3 Articulating the Instructional Challenge

2.3.1 What are Instructional Challenges?

Put shortly, *an instructional challenge is what prompts you to think carefully about designing or redesigning a lesson in the first place*. For example, suppose that when reflecting about how you introduce matrices in Linear Algebra, you feel like there is little by way of motivation as to why matrices are important to learn about. The instructional challenge your LAM addresses is: "How can I introduce matrices in a way that helps students see why they were developed in the first place and why they are so useful?" Or, as an alternate example, suppose you realize, upon reflection, that in your own instruction you do not feel you emphasize what it means to "communicate mathematically," and you decide you want to take that on in your teaching. The instructional challenge here is: "How can I help my students communicate mathematics more effectively?"

Notice that in these two examples, one instructional challenge is much more local to how a concept is motivated in one specific lesson (or two lessons), while the other revolves around a much longer-term challenge of helping students develop a mathematical practice ([4]; [10]). The longer-term challenge might not be resolvable in a single class period but may be a guiding goal for your entire class. In this case, your LAM might detail the first one or two lessons in which you specifically target this challenge, with some post-lesson considerations on what an instructor might do from there to maintain the practice (see "Post-Lesson Considerations" later in this chapter). For the shorter-term challenge, the LAM would be focused on the single lesson that addresses that challenge, but still must explain why that single lesson is of major importance.

In this way, regardless of whether your instructional challenge is longer-term or shorter-term, the key is that any instructional challenge should focus on important ideas and practices that have real significance to college mathematics teaching. The two examples given here — developing intellectual motivation for the study of matrices and learning to communicate mathematically — are both highly significant endeavors. Even though motivating matrices might only cover one or two lessons, it has implications for how students think about matrices in general, which is a foundational topic in undergraduate mathematics. In this vein, instructional challenges are not things like a different way to prove a theorem, a practical logistical issue, a fun idea, memory tricks for remembering formulas, or simply making a personal lesson plan. Rather, *instructional challenges should be focused on fundamental ideas, key meanings, important student practices, or impactful instructional practices that other teachers will be able to draw from in improving their own practice.*

Instructional challenges can be about major concepts, teaching practices, mathematical behaviors, student identities (Hill, 2008), teaching norms, or other important areas. These instructional challenges might aim to address students' ways of thinking or doing mathematics, meanings or motivations associated with a concept, students' dispositions toward mathematics or learning, or students' identities as mathematical learners, among other possibilities. The following are a few examples of instructional challenges pulled from the LAMs included in this book, and we encourage you to look through all of the LAMs in the book for additional examples. First, Melhuish (chapter 11) claims that while it is important for students to engage in producing proofs, it is also critical they engage in analyzing and altering proofs. Their LAM is centered on a particular theorem that provides a meaningful and accessible setting to do this. As a second example, Wangberg (chapter 12) claims it is crucial for students to conjecture during class, even though they might not naturally feel comfortable doing so. His LAM focuses on instructional episodes where he helps students start the process of feeling comfortable in conjecturing. Next, Rogers (chapter 5) explains that some mathematics courses that pre-service teachers take might not strongly connect the college mathematics they are learning with high school mathematics they may teach in the future. Her LAM consequently explores how to make some of those connections explicit. As another example, Jones, in chapter 4, identified that typical calculus classes promote meanings for definite integrals that do not align well with their subsequent uses in STEM courses. His LAM focuses on foregrounding quantitative meanings for the definite integral concept that are more broadly applicable to subsequent STEM coursework. As a last example, Katz (chapter 8) explains that viewing mathematics as independent from people is problematic in that it creates unequal access to participating in mathematics. BK's LAM focuses on a lesson that helps students start a process of critiquing mathematical knowledge consciously and to learn about its co-constructed nature.

2.3.2 Avoid Deficit Framing

We strongly *discourage* deficit views of students in describing the instructional challenge or the learning objectives ([1]; [8]). For example, stating "Students don't understand proof by induction very well" is a pejorative way to describe students. Instead, we hope that reflective teachers would frame a challenge more in terms of, "Instruction (or textbooks) tends to make proof by induction feel circular" (from Rabin's LAM). This places the challenge on the instructor to improve, rather than on creating negative views of students. The focus of instructional challenges should be kept on aspects of teaching and learning over which you, as the instructor, have some sense of agency and opportunities for making positive change.

2.3.3 Lesson-specific Learning Objectives

Although instructional challenges are not the same thing as "learning objectives," the two are closely connected, so we take a moment here to discuss being explicit about the learning objectives for the lesson as well. Whereas the instructional challenge is the thing that prompts you to think carefully about redesigning or updating a lesson in the first place, learning objectives typically address the immediate, content-specific, learner-focused, instructional goals. Lesson-specific learning objectives provide a focus for student learning and a target for assessments, and help convey the instructional intent to others reading the LAM. To understand some of the features of these objectives, it may be helpful to consider non-examples. First, these objectives should not be focused on the instructor's performance. Stating a goal such as, "I will teach students about derivatives," would not inform the reader regarding what *students* should be able to do with their learning of derivatives. By focusing on something you want students to be able to do, instead, you can modify such an objective to be learner-focused, such as, "Students will use the limit definition of derivatives to determine derivative properties." Second, phrasing objectives in a way that focuses solely on the learning process can be problematic, as it necessitates the ability to open students' heads and peer inside. It is best to avoid objectives stated like, "Students will understand the limit definition of the derivative," because you cannot know if they understand it. Rather, focus on actionable, measurable objectives that clearly identify desired results. For instance, Rogers' LAM in chapter 5, whose instructional challenge is about connecting college mathematics to high school mathematics, contains three explicit learning objectives: (1) Students will be able to define and explain rational numbers and irrational numbers; (2) Students will be able to explain in more than one way why $0.999\ldots = 1$; and (3) Students will be able to apply the idea of uniqueness of decimal representation to problem situations. These three learning objectives help define the content and also describe how to begin making some headway toward addressing her overarching instructional challenge.

Overall, we feel it critical for a LAM to explicitly articulate the instructional challenge that led to the LAM in the first place, and to articulate the learning objectives for the lesson. The instructional challenge is necessary so that the origins of the LAM and its objectives, activities, and rationales make more sense to the reader. The learning objectives can then help the reader know what targets and goals they should be aiming for if they were to implement your lesson.

2.4 Providing Background and Context Information for the Lesson

When writing a LAM centered around a particular instructional challenge, including details of your particular setting and context can help readers assess what aspects of your LAM may be relevant for them. In addition to providing contextual information about the course and instructor, it could be helpful for you to briefly provide related information about the curriculum, teaching practices, and perspectives on the mathematics relevant for the LAM. The following are aspects of background and context that should be included in a LAM:

- Institutional context and student audience
- Curricular and instructional context
- Teaching practices and norms (including equitable practices) central to the lesson
- Your perspectives on key mathematical ideas in the lesson
- Instructor positionality
- Other useful contextual items or factors that would help a reader understand your lesson better.

In writing your LAM, each of these bullets does *not* require its own subsection, as some bullets can be discussed together. Also, while all of the bullets should be brought up in some way, some bullets might need more description for some LAMs than others, and addressing some bullets might only need a couple of sentences. The two main things to consider are: Are each of these bullets addressed in some way in your LAM, even if briefly? Have you given the necessary background for an outside reader to understand how specific contextual factors influence your lesson?

To provide some guidance on what might be included for these bullets, the following paragraphs give some additional explanation. However, we again emphasize that each bullet does not necessarily need its own subsection, and the following is rather to help you have a sense for the kinds of things that might go within the background and context section of your LAM.

For institutional context and student audience, it is often beneficial to include a very brief description of the course, the institution, and student audience where you have implemented your LAM. This brief description could include the course name (e.g., linear algebra or introduction to proofs) and how it is situated in the curriculum (e.g., an elective linear algebra with emphasis on computation or a required linear algebra taught after the intro-to-proofs course). Also, the reader may benefit from knowing whether you are teaching at a community college, liberal arts college, or research university. Lastly, information about the student audience might consist of the number of students in the course, the students' majors (e.g., general education students, pre-nursing or pre-engineering students, science, engineering, or business majors), and relevant demographic information (e.g., gender, race/ethnicity).

For curricular and instructional context, it is important to identify how your LAM fits into your course's overall curriculum and any essential features of the class structure. How it fits into the overall curriculum might include the key ideas students need to know prior to the lesson, or what some of the big ideas or practices are that come later in the course that this lesson is meant to connect to. For course structure, please indicate an amount of time needed for the lesson. If relevant, you may also explain if certain technologies are available that you use, the physical makeup of the classroom (large lecture seating vs. small classroom with tables), how long the class meets for, or the textbook you use. (For more information see the discussion of *Knowledge of the Curriculum* in [5].

For teaching practices and norms, it is essential to briefly explain how different instructional practices are used in the lesson, or what different classroom norms are that would be important for a reader to know about your class. Instructional practices might include whether individual work, group work, partner work, whole class discussion, or lecture is used. It might be important to also justify briefly why those teaching practices are used. Classroom norms might consist of explaining, for example, that your students are accustomed to "think-pair-share", or that your students are already comfortable with the expectation to share tentative ideas that could be critiqued. Of course, some of these details might naturally come up during the description of the lesson itself, so this context does not necessarily need to explain every detail. Rather, it is helpful to simply provide a basic feel for how your class runs and the underlying rationale for how these practices are used. Some questions you can think of when deciding what information to put here could be: (a) Why do you use individual work, group work, whole class discussion, or lecture? (b) What norms are your students used to that are important for how this lesson runs?

For your perspectives on key mathematical ideas, it is important to realize that each instructor might have their own ideas about what to emphasize for a mathematical concept. For example, an instructor might focus on the "instantaneous rate" interpretation of derivatives in a lesson over the "slope of a tangent line" interpretation. Or, one might want their students to think of vectors as "packets of information" rather than "literal, visual arrows." It may be important for you to reflect on how you think of the concepts, and for your LAM to convey that knowledge and focus up front to the reader. That way, they can follow what you are doing in the lesson and potentially help their own students develop that same nuanced understanding of the underlying mathematics. This is particularly important for subtle or deeper aspects of the mathematics that may not be covered well in textbooks. It also may be helpful to highlight situations where it is okay for students to be uncomfortable with the underlying mathematical approach, such as if an activity were to ask students to construct mathematical models for an idealized situation.

For instructor positionality, your beliefs, training, and approach to student learning likely influence the design of your LAM. Conveying information about yourself and your teaching orientation could help readers understand the instructional choices you make in the lesson. For example, you might choose to include a brief statement of your teaching experience or philosophy, or career/research experiences that inform the lesson you are presenting. Some authors may choose to include how they self-identify in terms of gender, race/ethnicity, or other demographic information if they feel doing so would help the reader understand the instructional choices made in the lesson.

Finally, since LAMs will have quite different instructional challenges, have different objectives from each other, or be focused on different types of outcomes (e.g., content versus engaging in a practice versus developing a belief), it is not possible to list every item a LAM *might* need to include in the background and context in order to help the reader understand the details of instructional choices and moves. Thus, we end with a few other possibilities for consideration. These may include research findings related to the LAM, helpful theoretical ideas for making sense of your approach, or previously documented related student thinking. Of course, not all of these need to be in every LAM, and we encourage you to think carefully about what information will be most useful to your reader in understanding what is happening in your specific lesson.

2.5 Providing Rationales for Instructional Decisions

Essentially every choice you make in designing and implementing your lesson is based on some explicit or implicit rationale. In this section we consider one of the most important actions a reflective teacher can do in building a knowledge base for teaching: making your rationales for your instructional decisions explicit for others to see [5]. In fact, *we consider carefully unpacking the rationales for your teaching decisions and making them visible to others to be one of the defining characteristics of a LAM.* Your lesson is designed and implemented in the context of your own teaching perspective, and so writing an effective LAM requires your perspectives to be made explicit for at least two reasons. First, since a written document does not directly facilitate a dialog between you and the reader, your LAM needs to make your reasons clearly known. Any elements of your rationales left implicit will be filled in by readers' own perspectives, which may not align sufficiently with your own for the readers to implement the LAM successfully. Second, no reader will have exactly the same context, goals, resources, and skills as you, so they will be reading your LAM in light of their own viewpoint and understanding. Externalizing your decision-making process is key for helping readers build new ways of thinking about teaching and will help readers better adapt their lesson to their own context. We see two main types of instructional decisions whose rationales need to be made explicit in the LAM: decisions made at the whole-lesson level and decisions made at specific moments during the lesson.

2.5.1 Rationales for Instructional Decisions at the Whole-lesson Level

Some of your decisions relate to the lesson as a whole. For example, you might have an overarching idea you are trying to develop within the lesson, or you might have a major guiding activity that scaffolds most of the lesson, or you might use a particular pedagogical approach throughout your lesson. The rationales for these decisions should be explained toward the beginning of your LAM to provide rationales for these bigger-picture decisions.

To give some examples, consider Dawkins' LAM, in chapter 3, where he explains his choice to bypass propositional logic during earlier stages and to pursue predicate logic from the start. His rationale is based on previous findings that students' engagement with propositional logic tends to interact with their expectations about language in a way that obscures or interferes with the learning goals for this lesson. By making that rationale clear, other instructors can better understand why that choice was made, as well as whether they might want to make a similar choice. As another example, Rasmussen and Keene explain in chapter 6 how their philosophical stance on using guided reinvention influences the types of tasks and activities their students engage in. Knowing this stance helps the reader understand why they use the tasks they do in their LAM, what students are expected to do during those tasks, and what the instructor may be doing during the task.

Of course, every LAM will have a different set of "whole-lesson level decisions" that need rationales provided for them. The point is that every lesson does have such whole-lesson decisions and we expect a LAM to make the rationales for these decisions explicit.

2.5.2 Rationales for Instructional Decisions at the Specific-moment Level

In addition to decisions made about the entire lesson as a whole, you also make many decisions at specific moments during the lesson. Some of these decisions may have been planned out ahead of time, and some of these decisions may come as the result of perceiving an in-the-moment need or opportunity. Making explicit a few of these decisions can help reveal the whys behind the choices you have made in your lesson, as well as help provide ideas toward managing student-centered classrooms. Of course, it might not be practical to fully explore every decision made, but key decisions made at important moments should certainly be discussed.

Some possibilities for specific-moment level decisions that you might want to provide some brief rationale for could be (a) how you know students understand the idea well enough before pivoting to the next idea, (b) when you decide to reconvene students from individual or group work and who you invite to share their work, (c) whose voices are privileged when you choose to select students to share their work, (d) which student ideas to discuss as a class and in what order, (e) when to vary the lesson in a certain way (possibly depending on current student understanding or engagement), and so on. As an example, at one point in Dawkins' LAM, he explains how he took advantage of the in-the-moment opportunity to ask the class to apply one student's truth conditions for one statement to other statements, in order to continue to develop the ideas of the "and" and "or" operators. The student had compared "and" to "or" and the instructor took the opportunity to have the other students discuss that comparison. Katz' LAM, in chapter 8, explains how they decide which students to select in beginning critical-oriented discussions. BK tries to select students who they think will respond well to their ideas being challenged, based on their knowledge of how students have responded previously. Rogers has several examples in her LAM where she explains possible modifications she could use depending on the circumstances, such as continuing a whole-class discussion if the students are ready versus returning the students to their small groups if they need more time to think on the idea.

As with whole-class decisions, every LAM will have a different set of "specific-moment" decisions that need rationales provided. We encourage you to consider different points in your lesson and whether it would be useful or beneficial for other instructors to be able to understand your rationale for why you made the decisions you did.

2.6 Discussing Equitable Teaching Practices

We expect that issues of equitable teaching practices (Schools, 2019) should be considered by all LAM writers in some way. As we plan and analyze our lessons, we need to consider our role as instructors in providing a space where all students are positioned as valuable contributors to mathematics. Mathematics classrooms are microcosms of our larger society – our lessons do not take place in a neutral space. As such, the impetus is on us to develop the classrooms norms where student voices from diverse backgrounds aren't just heard but valued. Speaking as mathematics instructors ourselves, we recognize it can be challenging to attend to our content goals in tandem with issues of equity and inclusion. We also acknowledge that we are all "works in progress" on this front. The goal of a LAM is not to provide a perfect instance of teaching, but a meaningful, reflective instance of teaching.

In a LAM, equitable practices might be included within the background and context or may be best described during the flow of the lesson itself, or may happen in post-lesson reflection about better incorporating equitable practices in future iterations of the lesson (they do not need to be placed under their own header, though they can be if appropriate for a specific LAM). Thus, we decided to discuss equitable teaching practices in this chapter as its own section, since each LAM may incorporate them in different ways. Also, because some instructors might not have much of a background in equitable teaching practices, this section is structured differently than the others. Here, we provide ideas about how one might incorporate equitable practices into their own teaching, in addition to giving direction on what to actually include in a LAM write-up. Thus, to be clear, not all of the following elements need to be included in every LAM, but each LAM should contain reflection on equitable practices in some way, which can be brought up within background and context, or the lesson itself, or in post-lesson considerations.

First, in addition to not positioning students as deficient (see Rationale for Instructional Decisions), we encourage you to go further and think about the *assets students bring* into a lesson [3]. This involves designing tasks and content that provide opportunity for students to leverage their knowledge. Some questions to consider to incorporate this into your teaching are: (a) Do your tasks and content tap into knowledge funds of your students? (b) Did *all* of your students seem to make sense of the tasks and content they engaged with? (c) Did your students make any important connections to their life outside of the classroom or bring in any unexpected knowledge that shaped the lesson? If you consider these questions, you might wish to include some brief descriptions about them in your LAM. As one example, Burn's LAM describes in her background section how she and her colleagues try to identify ways to contextualize problems that can increase a sense of relevancy and meaningfulness for their students, especially their minority students. Because of that, her lesson is better suited to *all* of the students being able to engage with the mathematics and to draw on their knowledge funds.

Next, a powerful way to provide increased *equity of opportunity* is incorporating classroom structures and in-the-moment teaching moves that allow students sufficient time to think and make sense of lesson components. Mechanisms include allowing sufficient wait time, providing private think time before group work, or using quick turn-and-talks with neighbors. Without sufficient time, certain students may be more likely to dominate group work or respond to teacher questions. If you use these tools, you may wish to explain that in your LAM, possibly even as part of your rationale for an in-the-moment instructional decision.

The norms in our classrooms also can be powerful in situating students as contributors to mathematics. If you develop norms in your classroom that situate students in this way, we suggest sharing them in your LAM. This may be included in the background section when you explain general norms, or perhaps within your lesson at a specific moment. Further, you might explain how you promoted respectful interactions (i.e., *relational equity*, [2] amongst your students, or how you have developed a space where students are comfortable sharing ideas that may be different, early conjectures, or rough drafts. If relevant to your LAM, you may wish to explain pre-planned activities to emphasize norms of respect, or evidence that your students did engage with each other in ways that were status-free and respectful, or moves you made to mediate student interactions or to value non-normative strategies and ideas.

Another important aspect of promoting an equitable classroom is how students are *positioned* by us and by each other. Students come into the classroom with a wide range of experiences, which often include conscious or unconscious notions of who is capable of doing mathematics. Many stereotypes exist related to gender, race, and other elements of students' background. We can work to position students as contributors to mathematics. Some questions to consider to incorporate this idea into your teaching are: (a) Have you addressed and amplified the work of mathematicians from different backgrounds, races, and genders? (b) Did you amplify contributions from students that may otherwise be perceived as having low-status in your class? (c) Do you make explicit moves to emphasize that everyone is capable of mathematics and you expect success from all students? If this is a part of your teaching, we suggest including some description of it in your LAM.

A final dimension of equity that we discuss in this section is *participatory equity* [13]. As you plan and enact lessons, it is vital to consider what students have contributed, whose voices were taken up in the class, and for what purposes.

Ideally, all students contribute meaningfully, with no particular group of students (e.g., a particular race or gender, "A" students, or other group) making all the substantial mathematical contributions. If you are trying to include this idea in your teaching, you might report in your LAM how you decided who would share their thinking, or whose voices were heard, or what you did to encourage further participation from some students. Even if, upon reflecting afterward, you realize a certain group of students had not participated, you could state in your post-lesson considerations what could be done the next time to address the issue. As an example, Hall's LAM contains a "walking activity" and he explains in his post-lesson considerations that he is still thinking through how to implement this lesson for students who may not be able to walk. This brings up an important issue of equity that others can now consider as they build on his lesson.

These are just a few of many ways we can attend to equity in our teaching [11]. We hope this section helps you consider both how you can incorporate equitable teaching in your classrooms, and how you can include parts of those equitable practices in your LAM. If you hadn't explicitly considered some of these components while enacting your lesson, you can use them now as a tool for reflecting back on your lesson in your post-lesson considerations. Thus, in including equitable practices in your LAM, some of this discussion may happen in background and context, some may happen during the description of the lesson flow, and some may happen in post-lesson considerations. Regardless of how it fits into your LAM, this type of information can help us all move towards mathematically productive classrooms for all of our students.

2.7 Detailing the Tasks, Questions, Activities, and Flow in the Lesson

In the previous sections, you have thought about an instructional challenge and reflected on information relevant to writing a LAM that addresses that challenge. Now it is time to write a major part of your LAM: describing the components of your lesson in a manner that conveys the complex and subtle aspects of your teaching. This is no trivial matter! Teaching is something that many of us have been doing for a while, some as early as childhood. We have instincts and intuitions that, during lesson analysis, should be made available for reflection. In a LAM, therefore, authors should clearly describe the various components and flow of their lesson in a way that illustrates this reflective practice. To do this, we expect that LAMs (a) include the exact problems/tasks/activities as presented or given to the students, (b) describe the follow-up questions, connections, and examples that you plan on using to develop understanding as the lesson progresses, and (c) provide details on what the in-class discussions look like in order to help others know how to have similar discussions.

2.7.1 Describe the Tasks, Problems, Activities, Questions, Connections, and Examples Utilized in the Lesson Enactment

One of the most important aspects of your LAM is the description of the tasks, contexts, questions, examples, and activities utilized in your lesson enactment [5]. What specific activities do students work on? What questions do you ask students and when? What are the key concepts to highlight? There are many ways to present this information – just remember to present enough detail regarding your implementation that readers are not left guessing at important elements.

To provide an example of carefully articulating a task or activity, consider Ström's LAM in chapter 13. As she recounts her lesson, she provides the clear prompt that she gives her students in which she shows them a video of Usain Bolt running and instructs them: "Sketch a graph that could represent Bolt's distance from the start as a function of time. Be sure to label the axes!" However, she doesn't just provide the prompt in her LAM, but she also explains what the students are expected to do, what she (the instructor) does, and how the activity flows. She clarifies that she asks students to pair up shoulder to shoulder with a partner and find a location at the whiteboard. She is clear that she plays the video two or three times. She explains how the instructions are given to the students, and she indicates in her LAM what decisions she allows the students to make on their own. She gives information on how long she lets the students work (5–7 minutes) and how she transitions to the discussion after student work. She explains that she often starts with a "linear" graph, and then moves on to an "accelerating" graph, and so on. Such details help other instructors be able to conceptualize how such a lesson might run and how they might enact such a lesson in their own classes. If Ström had simply provided the task without the supporting details of the lesson flow, then other instructors might not have a clear mental image of what a discussion might look like or how to navigate the discussion.

To provide an example of being explicit about what questions the students are asked, consider Melhuish's LAM in chapter 11. Prior to their lesson, their students attempt to prove a theorem at home regarding two isomorphic groups. They explain that for one in-class activity, the students are asked to exchange the proof they wrote at home with another student. They clearly state the questions given to the students at this point: "What is one thing about this proof approach that makes sense to you?" and "What is one thing that you have a question about?" Notice that Melhuish included the discussion questions verbatim as they would convey them to students. By doing so, other instructors may better be able to implement such an activity in their own classroom. By contrast, if Melhuish had simply stated in their LAM, "I ask the students to exchange their proofs and discuss them," then the reader would have a less clear idea of how exactly that discussion might unfold.

2.7.2 Provide Details on Key Classroom Discussions

One defining aspect of a LAM is the high level of detail that is shared regarding a specific lesson, written from the point of view of the instructor. Thus, LAMs should include not only *what* happened in an enactment but *how* it happened, or at least *how it tends to happen* for LAMs that summarize over multiple instances of enacting the lesson. For example, a LAM should not just state the concept that was discussed in class by the students and/or instructor, but it should also explain how it was discussed in class. Although there are many approaches to accomplish this, a guiding set of questions that may help you include: (a) What are the key concepts you highlight during the whole class discussion, and how do you accomplish that? (b) When facilitating small-group or whole-class discussion, what specific questions do you ask students? When? Why? (c) When students share their ideas during class (either in writing or verbally), how do you respond to those student contributions? What guided your decisions?

For example, Corey's LAM in chapter 7 provides a nice summary table of key dialogue that occurs in the classroom. He provides detailed sample excerpts from students that you might hear if you were to teach this lesson. He explains what ideas students come up with fairly naturally, as well as what ideas take a little longer for the students to explore. He also has helpful notes to the instructor about how to manage the discussion. Such details on how the classroom discussion occurs can help other instructors visualize how such a lesson might play out. Since the point of LAMs is to help build a knowledge base for teaching, in which reflective instructors can share and learn with each other, the more insight you can give your fellow instructors into what your classroom discussion actually looks like and how it unfolds, the more it can help them feel more confident in using your ideas to improve their own instruction. In fact, because capturing student thinking is such a central part of showing the flow of the lesson, we now shift to an entire section devoted to this single idea.

2.8 Documenting Student Thinking and Reasoning

In this section we discuss capturing student thinking and reasoning in your LAM. Your LAM should describe common patterns of student thinking and reasoning related to your lesson [5]. In addition, it should address how to build on those ways of thinking to guide students forward in pursuit of your goals. One key way to embed these patterns of student reasoning and their ways of responding into your LAM is through included dialogue (actual dialogue or hypothetical dialogue summarized from experience). We also explain how you can use this dialogue to help the reader anticipate some questions and conversations that will arise during the lesson, allowing them to plan ahead about how to respond productively.

While assessment questions, like exams, can provide insight into student thinking, we believe there is no real replacement for discussing directly with our students about how they are thinking about the mathematical topics at hand. We can be most responsive to student thinking and reasoning, and use it to advance our goals, when we talk with them, or listen to them as they talk with each other, *during* their learning process. This is why we expect that the *most effective LAMs will include some interactions or dialogue between students and instructor, or among students*. Interactions or dialogue can consist of actual conversation that may have been captured during a class ("actual" dialogue) or could consist of a hypothetical conversation that is based on experience having taught the lesson multiple times ("hypothetical" dialogue). In either case the dialogue should be based on what students really say, not on how an instructor imagines a "perfect" conversation as happening, even if the hypothetical dialogue is condensed for space constraint reasons. These dialogues are useful ways to express the patterns of student thinking and reasoning you see in your students, as well as your choices in responding to student ideas in the moment. This is not because every effective instructor should or would respond in the exact same way, but because it allows us to describe instances of interacting with student thinking that can help other instructors reflect on and learn from. You do not need to have the absolute best response to the student question but sharing their common way of responding will help you convey detailed information about how you handle student contributions. For example, in chapter 12, Wangberg provides several instances of student thinking in his LAM. At one point, he clearly states different conjectures made by the students, and then explained how he used those different conjectures to further his lesson. At another point, he describes how students' prior conceptions of integrals (as area under a curve) led to certain (incorrect) assertions about the line integrals being examined in the lesson. To provide another example, Ström's LAM (chapter 13) shows student work that could help other instructors anticipate typical student responses and know how to productively use those responses. In one place, the students are expected to create a graph and she provides several examples of different graphs the students produce. She then discusses them in terms of how students were likely thinking when they created them.

We recognize that not every reflective instructor organizes their classroom in such a way that they have extended interactions with students during class. This may make it harder for them to envision how to describe student thinking and reasoning in this way. We assume that most instructors can at least draw upon short question and answer dialogues they have in their class or the more extensive question and answer exchanges they may have during office hours. Regardless of exactly what sources of insight they have to draw upon, a LAM will benefit from the author reflecting on and distilling: (a) what they know about student thinking and reasoning and (b) how they respond to pursue their goals for the lesson. Thinking about these specifics and capturing them in some form is part of the value that writing a LAM should have for others.

Having explained this, a natural question might be: What are the benefits of sharing student thinking? We often privately spend time reflecting on our interactions with students to make sense of their reasoning, and to make our teaching more effective, but we do not always have opportunities to share those insights with other instructors. Many of our rationales for instructional decisions will be intimately tied to the ways we want students to think about the key ideas in our lesson and the patterns of student thinking and reasoning that we anticipate will arise. One goal of writing a LAM is to address the minute decisions we make in how to interact productively with students and respond to their questions and comments. A LAM allows a reader to hear student responses and feedback even before teaching the lesson. Sharing specific insights about student thinking and reasoning in a LAM helps accomplish this for the reader. Naturally, our students are diverse and draw upon many interesting and unique ways of thinking as they learn, so we do not mean that we can anticipate them all or that we should not listen carefully to the students in each separate class. Nevertheless, experienced teachers know that for a particular topic there are usually some common approaches students tend to take. You likely have certain topics you teach that you can predict many of the ways students will think or act. Others would benefit from knowing that, too. With this knowledge, those instructors will enter their classrooms better prepared to address some of those ideas, questions, and debates that are likely to arise during instruction. In writing a LAM, please consider how to provide that kind of experience to the reader as they plan their lessons. Even if their students reason about the given topic in slightly different ways or give different responses than yours, having a chance to reflect on what you wrote about your students' thinking will help them better guide their own teaching.

2.9 Post-Lesson Considerations

After teaching a lesson, especially one designed to address a challenge like in a LAM, it is natural for the instructor to reflect on the lesson in multiple ways. In this reflection, you might think about (a) what directions upcoming lessons could take that would build on the lesson you just taught, (b) if certain aspects of the lesson could be altered based on the student thinking or student engagement you observed, or (c) whether the lesson satisfactorily met the learning objectives and (at least partially) addressed the instructional challenge. We expect LAMs to make explicit some of these post-lesson considerations. LAMs do not need to address all post-lesson considerations, but it is important to highlight those post-lesson considerations that you deem useful for the reader to make the best possible use of your LAM. We now discuss some of these possibilities and what including them might look like.

One of the most important post-lesson considerations is your lesson's impacts on subsequent lessons. You want your LAM to provide guidance for an instructor about what they might do after this lesson, if they were to try it in their own class. You may outline how to proceed in the next lesson to maintain the conceptual focus or student practice you began to develop in this lesson. You may have some information on how students' engagement with material in your LAM will impact subsequent classes that build on it. As one example, in Jones' LAM (chapter 4), quantitative meanings for definite integrals are developed. He explains in his post-lesson considerations that, if after this lesson, one reverts back to standard area and antiderivative conceptions of definite integrals, without maintaining a quantitative focus, students will likely abandon the quantitative meanings carefully constructed in that lesson. He provides ideas on how to continue to develop the definite integral concept while still maintaining those crucial quantitative meanings.

Next, your post-lesson considerations may suggest variations to address different but related goals, or adaptations to the lesson for a different context to make your LAM useful to a broader audience. These variations or adaptations are useful for both an instructor teaching a lesson for the first time and for an experienced instructor improving or customizing their approach. You can also explain what you might do differently yourself the next time you teach this lesson. In one example in chapter 7, Corey explains how he realized that not all of his students had the same level of background understanding of "linear approximation," which he builds his lesson on. He discusses how in the future he may need to either provide more details of this idea himself or provide outside resources to make sure the students are all on par with each other. Thus, despite the success or the failure of a lesson, a LAM can help identify where to improve.

Lastly, it is important for a LAM to suggest that it had some impact toward the instructional challenge. For many LAMs, this might be put directly in the Lesson Components and Flow to show how students are thinking at specific moments and how that thinking is evidence of addressing the instructional challenge. However, at times it may make sense for you to provide some brief summative information that strengthens the claim that the goals of your LAM were met. For example, you may have had a homework question, or an in-class iClicker quiz, or an exam question, or other type of assessment, that helps you show that your students were thinking or acting in the way you hoped for based on the lesson.

2.10 LAM Format Structure

For this book, we have tried to follow a particular format for our LAMs, which we believe will make it easier for instructors to identify LAMs that fit their needs and to know what to expect as they read the LAMs. But we also have tried not to impose so much structure that they can still be flexible and adaptable for different lessons and purposes. Overall, we have decided on a structure with four major ordered sections:

I. Summary Table

II. Background Information

III. Lesson Implementation

IV. Post-Lesson Considerations

The Summary Table at the beginning gives a quick overview with key information, serving a similar purpose to the abstract of a research article. We find this summary table quite important so that instructors can easily get a sense of what a LAM contains to know whether they want to continue reading the rest of the LAM. The ordering of the other sections emerged from our experience writing LAMs, where we found it was natural to give important background information about the lesson first, then to lay out how the lesson itself plays out, and then to conclude with post-lesson considerations.

I. Summary Table

The Summary Table has five rows that allow the instructor to briefly share key information to the reader, so they can quickly assess if this is a LAM they want to read in depth. The table shown here states what should be in each row, and you can look at the LAMs in this book for examples on what to put in each row. Each description in the summary table should be brief but informative. The "brief overview of instructional approach" should provide a very concise summary of the nature of the lesson so readers can judge whether they would like to further explore your LAM.

Topic of lesson	1–2 sentences that state the topic of the lesson examined in this LAM.
Course context	1–2 sentences that state what kind of course and institution the lesson was taught in, or other key contextual items.
Instructional Challenge	1–2 sentences that clearly state the instructional challenge this LAM is addressing in this lesson.
Brief overview of instructional approach	A short paragraph that quickly recaps the highlights of the lesson, to give potential readers a sense of what they might find in this lesson.
Keywords	3-5 keywords that might help potential readers search for and find your LAM.

II. Background Information

After the summary table, we find it useful for LAMs to start with the background information, which includes:

- The instructional challenge and learning goals
- Background and context (including the elements discussed in that section)
- Rationale for instructional decisions at the whole-lesson level

As needed, some LAMs may find it helpful to include information about:

- Research findings related to the LAM
- Alternative approaches or techniques that could also be used
- Equitable teaching practices used throughout the lesson

III. Lesson Implementation

After the background section, the lesson implementation section contains the details of the lesson so instructors can clearly understand what to do (if they are trying to follow the lesson closely) but also the purposes and justifications for the actions. The topics discussed earlier that would be included in this section (in whole or in part) would be:

- The tasks, questions, activities, and flow of the lesson, with rationales
- Specific-moment instructional decisions and rationales for them
- The students' mathematical thinking and reasoning

Authors of the LAMs included in this book used a variety of formats and strategies to portray the information in this section. Some used a narrative approach, others used a table showing the lesson flow with important commentary/justification, and still others used the natural pieces of the lesson and discussed each piece in detail before moving to the next piece.

IV. Post-Lesson Considerations

As described in detail above, there are important ideas to discuss as instructors reflect on the lesson and think about how to use the ideas in the lesson moving forward, or for what could be done differently. Although this section does not need to be long, a LAM should end with some discussion about post-lesson considerations, as described earlier.

2.11 Conclusion

So far in this book we have explained the need for a knowledge base for teaching college mathematics and have discussed the background on how our proposed LAM format came to be. In this chapter we have explained two guiding characteristics for what a LAM is (i.e., including a complete instance of teaching and fulfilling a dual purpose), and given detailed information on the major essential components that should be included in a LAM. The following chapters of the book now give several examples of LAMs, written and revised by the conference participants, that can aid interested people in successfully writing and publishing a LAM. The book then concludes with a chapter that provides guidance on where to publish LAMs and commentary that reflects on our overall process.

The LAMs in this book are just a sample of the kind of topics a LAM could address. For example, LAMs could focus on general education or developmental courses, although none of the examples happen to in this book. Many of us teach mathematics courses for future teachers that differ from core mathematics classes. These mathematics-for-educators courses are often a different flavor with additional goals and layers of content. Teaching these courses well are vital and LAMs provide one way to share key instructional knowledge related to these courses. We note that the majority of LAMs contained in this book are from classrooms that take an active-learning approach, though a few come from interactive lecture classrooms, suggesting that different modes of classroom teaching could be appropriate for a LAM. Some of the LAM authors in this book even tried to comment at the end of their chapter how an instructor might be able to adapt the lesson to other classroom formats. We envision LAMs to be a flexible genre that could help to share and store instructional knowledge independent of the level, type, and format of instruction, as long as the focus remains on the core knowledge we have explained in this chapter — e.g., describing student thinking and providing strong rationales for decisions — noting that even an interactive lecture can be senstive to and carefully make use of student mathematical thinking.

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3

Reinventing the Logic of Mathematical Disjunctions

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Topic of lesson	Beginning logic for learning to prove; specifically, • the truth-conditions for disjunctions (*or* statements), • meaning of logical negation, and • proof by cases and contradiction
Course content	University transition to proof course or initial proof-based course such as linear algebra or abstract algebra
Instructional challenge	How to help students understand what questions logic answers and how they can develop logical ideas from their own use of mathematical language.
Brief overview of instructional approach	This lesson provides students with a series of mathematical disjunctions that vary in their mathematical content (divisibility, order relations, categories of polygons), usually universally quantified. Students must determine whether they are true or false and explain why. By comparing their thinking across the various statements, students should begin to think about their own interpretation and see content-general patterns (i.e., the same in number theory, geometry, etc.), which are the foundations of logic. By encouraging students to think about the truth-sets and falsehood sets of compound predicates of the form "$P(x)$ or $Q(x)$," students can develop a meaning for the negation of a predicate and state the standard negation for a disjunction. Students also produce arguments for the truth of a universally quantified disjunction that form the foundation for proof by cases and proof by contradiction. This lesson builds logic on the foundation of mathematics, since students are more familiar with the latter, even though formally the relationship runs the other way. Also, this lesson simultaneously addresses truth conditions, set relations, and proof techniques, rather than treating these as three separate topics as is commonly done.
Keywords	Logic; disjunctions; truth-conditions; negation; proof by cases; proof by contradiction

3.1 Background information

3.1.1 The Instructional challenge

This lesson is meant to provide students' first introduction to logic for the purposes of learning how to read and write mathematical proofs. This lesson focuses on one statement type (*or* statements, i.e. disjunctions) that serves as a good starting point for students who have had no exposure to logic. It is part of a lesson sequence that mirrors the activities with other statement types (conditional, multiply quantified) and should appear in an introduction to proof class or at the beginning of an early proof course such as linear algebra. I call it one lesson, though it may take up to 2-3 class days to complete. There are a few ways this teaching approach deviates from the way that basic mathematical logic is traditionally treated in undergraduate mathematics classes. It is necessary that the instructor understand the viewpoint taken on the nature of logic and how it can develop for students. First, note that we intend to develop in an integrated fashion some ideas that are traditionally separated into different chapters:

- definitions for the *logical connectives* such as *or* and *if then* and the operator *not*,
- some *set concepts* such as union, subset, and complement, and
- beginning *proof techniques* such as proof by cases, contradiction, and contraposition.

The way we mesh these different goals together is to let them all arise as tools for accomplishing a common purpose. Throughout these lessons, students' primary goal will be to determine *whether mathematical statements are true or false and to explain why*. When logic is approached in more conventional ways, students may learn logical tools without knowing what questions they really answer (in this case, what are truth conditions for all universally quantified mathematical disjunctions). This lesson sequence intends to help students understand the question on the path to developing some of the standard answers.

Logic entails generalizations about mathematical statements of the same type. For instance, when we replace parts of a statement with linguistic variables such as "$\forall x \in S$, $P(x)$ or $Q(x)$" or "$\forall x \in S$, if $P(x)$, then $Q(x)$," we are assuming that there is something shared by all such statements. I encourage the reader to notice how odd this is compared to the ways we use grammar in everyday speech. Most of the information we get from other people's statements are in the $P(x)$ and $Q(x)$ parts of the sentences. When someone says, "I am going to the store," I do not think about all sentences of the form "[Subject] is [verb in present progressive tense] [article] [object noun]," but this is similar to what formal logic asks us to do. If you have never studied logic, it is hard to understand what will be gained removing so much essential information from the sentence. In these lessons, we will delay talking about any mathematical statement of a given form (using linguistic variables such as p or $P(x)$) and instead talk about some particular examples of statements of the same form. For instance, what is shared between the way I interpret the following?

- "Given any integer x, x is a multiple of 2 or x is not a multiple of 4." and
- "Given any quadrilateral $ABCD$, $ABCD$ is not a rhombus or ABCD is a parallelogram."

Students will need some time and guidance to attend to what is common between the two in terms of the language/grammar, the conditions for being true or false, and the techniques by which we prove them true or false. This is at least in part because students' will initially focus on the aspects that are not shared – numbers and multiples versus polygons and rhombi – before they focus on what is shared – *or*, *not*, and *given any*.

A key motivation for reframing logic as is done in this lesson sequence is to move away from the sense that mathematical logic is somehow right, natural, and implicit in the ways people naturally reason. Many mathematics books that teach logic express those assumptions and it is problematic for multiple reasons. First, psychological research has shown that mathematical logic is a bad model for everyday reasoning, and this is not because people reason irrationally but rather because their reasoning is much more probabilistic rather than Boolean [3]. This is also because people reason contextually and mathematical logic abstracts across contexts. I think mathematical logic is a useful tool, but it is not the only account of how language can or should work. Second, I worry about what messages we send if we treat these matters as obvious. If students do not already reason according to logic, are they illogical? I think the best way to provide equitable access to understanding logic is to engage students in the process of reasoning about how language works. It is not the same for all of us and it is not the same across all language, but doing mathematics means we need to agree on some useful conventions. By letting everyone talk about the same sentences and reflect on how we understand them, we can come to some shared ideas about how we will use mathematical language in proof-oriented mathematics. I find that students' different ways of interpreting mathematical statements make a lot of sense once I understand them, even if they differ from the interpretations agreed upon in the mathematical community. By bringing our interpretations into the conversation so that we can reflect and compare, we provide students with better access to the conventions by which mathematicians read and use mathematical language. I think this kind of reflective awareness of language can help students not only in their mathematical studies, but also in their broader endeavors.

3.1.2 Learning goals: Two ways of reasoning that should emerge

There are two key *ways of thinking* we want students to develop from reading multiple mathematical statements of the same form throughout this lesson:

1. students need to begin to attend to the ways they read the statements to compare whether they read them the same way (i.e., use consistent truth-conditions) and
2. students need to develop a sense of how these statements refer to sets of objects.

The first way of thinking is our operative definition of what it means for a student to *reason about logic*. Research shows that students do not always interpret statements of the same form in the same way. For instance, when reading a disjunction students may declare the statement false because one of the two parts is false, declare the statement true because one of the two parts is true, or declare it true because everything that makes the first part false makes the second part true[1]. More importantly, they often do not notice that their truth-conditions are changing or conflict with one another. Researchers Dubinsky and Yiparaki (2000) [2] said the following about their observations of undergraduate students reading statements of the same form:

> "Our students were not so much conscious of the written statement [they read]. It was as though the statement was a window from which they were looking out. The students described what they saw looking out the window, but they did not see the window itself. When coming up with an interpretation of a statement, they were not aware of the process they followed in order to reach their interpretation" (p. 23).

As suggested above, reasoning about logic entails drawing generalizations across statements with diverse semantic contents. *If you give students a number of statements of the same logical form, they will notice the pattern and (with prompting) begin to make comparisons or analogies between them. This kind of conscious attention to how they interpret statements is precisely what Dubinsky and Yiparaki observed as lacking among their undergraduate research subjects and what this lesson is built to foster*.

The second way of thinking emerged during research using these lessons. The way that most proof-based undergraduate textbooks that teach logic approach matters is to begin with propositional logic, which serves as the foundation for predicate logic. Propositional logic deals with claims that are either true or false ("15 is odd") rather than claims that depend on a variable ("x is odd"). Propositional logic thus does not include quantifiers ("for any integer x"). Predicate logic includes quantifiers and truth-functional statements whose truth-value depend on a variable (called predicates). Research revealed there are some deep issues with this approach for novice undergraduate students learning proving for the first time. The reason is that quantified statements were actually easier for them to make sense of than non-quantified statements. Consider the following:

A. "Given any integer x, x is odd or x is even."

B. "15 is odd or 15 is even."

C. "Given any integer x, if x is odd, then $x + 1$ is even."

D. "If 15 is odd, then 16 is even."

Undergraduate research participants (who had not yet taken any logic or proof courses) often affirmed Statement A as true and rejected Statement B as false. For some of them, their reasons for this are defensible and insightful. Most everyday uses of the word or convey some sense of alternatives or uncertainty. For instance, when I say "I will order the fish or the chicken tonight" the hearer is reasonable to conclude that I have not yet decided. This would be a strange sentence if I knew my order already. Students similarly find Statement B very unnatural because we know that 15 is odd. Notice by comparison that the use of *or* in Statement A matches the everyday use of or because both even and odd are possible alternatives depending upon the value of x. Statements C and D exhibit a similar pattern. Statement C uses *if then* in a rather natural way. Though mathematicians affirm Statement D as an instance of Statement C, Statement D by itself is strange to students and "If 16 is odd, then 17 is even" bothers them even more. Recognizing that students do not connect quantified and non-quantified statements the way conventional logic does, I largely bypass propositional logic and pursue predicate logic from the beginning. The statements in the lessons include just a few non-quantified *or* statements, always paired with their quantified counterpart. This lets students discuss the relationship between them. In the spirit of helping students reflect on their own interpretation, it may be helpful to let students consider why these statements read so differently to them. Another rationale for bypassing propositional logic is that quantified statements are the ones of most interest for doing proof-based mathematics.

3.1.3 Learning goals continued: Set-based truth conditions and proof techniques

This lesson, in pursuing predicate logic, prioritizes students associating the predicates in mathematical statements with the set of things that make them true (the *truth set of a predicate*) and the set of things that make them false (the *the falsehood set of a predicate*). Using this idea, we can develop the following definitions:

- ***Or* statement truth-conditions** — A statement of the form "$\forall\, x \in S$, $P(x)$ or $Q(x)$" is true whenever the union of the truth set of $P(x)$ and the truth set of $Q(x)$ is all of S.
- **Negation of a predicate** — The negation of a predicate $P(x)$ is a predicate that is defined on the same universal set and has the opposite truth value as $P(x)$ for each value of x.

In general, reasoning about truth sets and falsehood sets requires two major shifts for novice students. First, they often do not reason about whole sets of objects, so this needs to be encouraged by the instructor. Second, students often think of phrases referring only to the objects that make them true (this is another reason students do not think Statement B above is similar to or an instance of Statement A). Reasoning about the falsehood set is non-obvious, but is important to develop since it will serve as our foundation for proof by contradiction.

How do proof techniques emerge from reasoning about truth sets and falsehood sets?

- **Proof by cases** — Students often justify certain *or* statements by showing that everything in the falsehood set of $P(x)$ is in the truth set of $Q(x)$. This can be organized into a proof by cases with the cases depending on the truth-value of $P(x)$.
- **Proof by contradiction** — Once students know how to characterize the falsehood set for a compound predicate such as "$P(x)$ or $Q(x)$," then they can justify why this set may be empty. Proving that the falsehood set is empty shows that all examples are in the truth set, which develops the classic approach to proof by contradiction for disjunctions.

I hope that these two examples demonstrate how thinking about truth sets and falsehood sets can support students developing proof techniques out of their own efforts to justify why particular statements are true or false.

3.2 Lesson Implementation

3.2.1 Initial statement set — Assigning truth-values to *or* statements (1–2 class days)

In the list below you can see the first set of statements that I provide to students in this sequence of lessons. I do not present the definition of or before this because I want it to arise from the ways students read the statements on their own. Their goals at this point are to "determine whether each statement is true or false and explain why." Since the task is to determine truth-values, I may alternate between calling these statements and tasks. You may also want to specify that their only choices are *true* and *false*, though they may want to pick something else for some statements. I usually have students work on this in small groups of 2–4 while I go around the room and listen to their explanations. I can easily take most of a class period for them to talk through all of these. I recommend that you alternate between 10–15 minutes in small groups for them to make progress and 7–10 minutes segments in large group to let them share their ideas and develop some shared insights and strategies. In the following sections, I will explain my reasoning behind this set of statements, provide some guidance about how the lesson can/should proceed, and share some common student ways of reasoning.

1. Given any integer number x, x is even or x is odd.
2. The integer 15 is even or 15 is odd.
3. The integer 16 is even or the integer 15 is odd.
4. For every pair of real numbers x and y, $x < y$ or $y < x$.
5. Given any two real numbers x and y, $x \leq y$ or $y \leq x$.
6. Given any real number y, $y < 3$ or $y > 5$.
7. Given any real number, $y > 3$ or $y > 5$.
8. Given any real number y, $y = 0$ or y has a reciprocal $\frac{1}{y}$ such that $y * \frac{1}{y} = 1$.
9. $\pi = 0$ or π has a reciprocal $\frac{1}{\pi}$ such that $\pi * \frac{1}{\pi} = 1$.
10. $0 = 0$ or 0 has a reciprocal $\frac{1}{0}$ such that $0 * \frac{1}{0} = 1$.
11. Given any even number z, z is a multiple of 2 or z is a multiple of 3.
12. For each even number z, z is not a multiple of 4 or z is not a multiple of 3.
13. Given any even number z, z is a multiple of 4 or $z + 2$ is a multiple of 4.
14. For each integer number z, z is a multiple of 2 or z is not a multiple of 4.
15. For every triangle, it is equilateral or it is not acute.
16. For every triangle, it is acute or it is not equilateral.
17. Given any quadrilateral, it is a square or it is not a rectangle.
18. Given any quadrilateral, it is not a square or it is a triangle.

3.2.2 The Choice of statements

To get students to generalize the ways they read these statements, the mathematical context systematically changes (divisibility relations, order relations, geometric categories). I make sure that the predicates in the statements are somewhat familiar, but that does not mean that students will find these tasks easy or straightforward. In particular, many students find the geometric statements (15–18) challenging, in part because they often struggle to know how to deal with negative predicates productively. However, the familiar content is important for students to be able to imagine constructing the truth sets as we eventually want them to do.

Mathematics versus everyday language

Instructors will notice that I only give mathematical statements. I think that the mathematical logic that we want students to learn makes the most sense in mathematics, not in everyday language. Well-chosen statements from everyday speech can be useful, but I generally avoid them. Formal logic arose in the modern era among mathematicians because the precision of mathematics requires formal logic. Each instructor ultimately needs to do what makes sense to them.

The instructor should feel free to drop or add statements as they see fit. An easy guide to building the statements is to make sure you intentionally vary the relationships between the truth sets in the quantified statements. If $P = \{x \in S \mid P(x) \text{ is true}\}$ and $Q = \{x \in S \mid Q(x) \text{ is true}\}$, then some examples should have the following:

- $P \cup Q = S$ – true statements such as Example 5, 7, 14, 16, and 18,
- $P \cup Q \neq S$ – false statements such as Example 4, 11, 12, 15, and 17,
- $P \cup Q = S$ with $P \cap Q = \emptyset$ – true statements like Example 1, 8, and 13, and
- $P = S$ – true statements like Example 11.

One easy way to produce true statements is to begin with a subset relation such as $\{\Delta ABC : \Delta ABC \text{ is equilateral}\} \subseteq \{\Delta ABC : \Delta ABC \text{ is acute}\}$ and then form a disjunction between one of these predicates and the negation of the other (e.g., statements 15 and 16).

3.2.3 Working through the lesson

Letting students interpret all of these examples can take most of a class period for the groups that move the slowest. For 50-minute sessions, you may spend as much as two class periods letting students develop and share their ways of reading these statements. I usually dedicate one class session if I have 80-minute sessions. This may feel like a lot of time, but recall that we are going to develop logic, set ideas, and proof techniques simultaneously from this sequence of activities. The geometric items produce some of the most interesting arguments, so make sure that students have a chance to think through those even if they skip some others.

The goal early on is to let this task remain very open so students can do what makes sense to them. You will notice that students often read and reword the various statements to find some way to make sense of them. They do not yet have stable ways of interpreting every *or* statement and their interpretations may continue to shift. When students share a clear statement of their thinking about one statement, it is good to write that down so that there is a public record. The key learning opportunities arise when they are able to compare their interpretation to their classmate's and notice how they are different or when a student is able to intentionally compare their own interpretations for different statements. This is usually a key step in developing first target way of thinking described in section 3.1.2: consciously reflecting on their own reading. A good way you can advance this process is to ask students to try to read one statement *in the same way* they read another. This invites them to adapt their reading strategy to a different statement. This is also how I get students to revisit statements when their initial interpretation is not mathematically accurate. This is when the public records of thinking are important.

As students explain why each statement is true or false, encourage them to articulate their interpretations in general terms. For instance, they may develop rules such as "something is a counterexample when it makes both parts of the statement false" or "the statement is true when one of the two parts is true, or both." This is general enough to allow them to adapt their reasoning from one statement to another and see what is shared between them. However, do not be too quick to assume they are already thinking generally. For instance, you may hear students saying something like "Statement 17 is false because a rectangle is not a square and is a rectangle." To the expert, this sounds a lot like stating the negation, but I will show in the next sections why this may not be what the student means.

Troubleshooting

Inevitably if you are teaching this to students who are in their second or third year, some students in the class will have been taught logic in another class. This is usually easy to spot because they will write p's and q's or try to recall truth-tables. I encourage you not to let them dominate. Other students need a chance to interpret these statements on their own and often students have learned truth tables in very superficial ways. Generally, students do not understand the truth table as a way to organize the objects referred to in the statement. For instance, on Example 17, students often do not recognize that the line "T – F – T" in the truth table means that the statement is true for all quadrilaterals that are a square and a rectangle. If a student is writing p's and q's or T's and F's, ask them to explain what those symbols mean in terms of the predicates in a particular statement. If they can, encourage them to listen to their neighbor's reasoning and see how others are thinking. If they cannot, then encourage them to come up with an explanation for each statement based on the properties therein ("what does the truth table tell me about squares and rectangles?").

3.2.4 Idealized dialogue from the early part of the lesson

Student A: I think statement 2 is false because 15 is not even. The statement clearly says something false.

Student B: Yeah, it doesn't make sense because these two are not really possible. I guess we could change what we mean by "odd" and "even" or something, but we know that 15 is odd

Instructor: So, do you see any relationship between Statement 1 and Statement 2?

Student A: Not really. I mean they have some of the same words, but we know that x could be odd or even, so that makes sense. Each x is going to fit one of those two, but not both of them.

Student B: Yeah both even and odd are possible for all integers, but not for 15.

Student C: I guess I can see how Statements 1 and 2 are similar because 15 is one of the values of x. Isn't that sort of what *or* means is that only one of them has to be true? If both parts need to be true, that is what *and* means, but this says *or*. That's why I said that Statement 2 could be true.

Student A: Yeah, I guess I wasn't thinking about *or* like that, I just saw that Statement 2 said something false.

Instructor: Do you see how those ways of thinking about Statement 2 could be used on Statements 9 and 10?

Student A: Statement 9 was even worse because we know π is not 0. I don't even know where to start with that.

Student B: And 0 does not have a reciprocal, so Statement 10 can't be right.

Instructor: Can you apply Student C's thinking about *or* to those statements?

Student A: So, each statement has two parts and one of them is ok. I guess we could technically use *or* to mean that as long as one of the parts is true, then the statement is ok.

Student B: I still don't know why you would say that instead of just saying, π has a reciprocal or that $0 = 0$.

Instructor: Try to distinguish the question "would I say this" or "does this sound weird" from "is it true or false." Our goal is to think about when we call statements true or false, and why.

Student C: I think Statement 8 is true, so doesn't that mean that Statements 9 and 10 should be true? It is like you put in numbers π and 0 for y.

Student B: Yeah but all of the numbers except 0 go into the part about reciprocals and only 0 goes into the part saying $y = 0$. It does not make sense to put the same y in for both parts of Statement 8.

Students' actual thinking will not always be this direct, but I have tried to share lots of the common patterns of student thinking in a short dialogue here. Two main things to notice in the dialogue, and in your own students' reasoning, is when they use comparisons to think about the meaning of the statements. Student C compared *or* to *and*, which is a helpful strategy. The instructor invited the students to apply Student C's truth condition for Statement 2 to Statements 9 and 10. This kind of comparison is the heart of what we want students to work on. This helps them think about the conditions for declaring statements true and false and to formulate them generally so they can be applied to different disjunctions. When students begin to see the strategy as applying generally, then they are beginning to think about logic, by which I mean content-general conditions for truth.

3.2.5 Common student patterns of thinking

One way to divide the common student approaches to making sense of these statements is between thinking focused on examples, properties, or sets. None of these is unproductive, but we will prioritize sets in the lesson.

1. *Example reasoning* occurs when students think about the conditions in the statement using one or a few examples. With integers, they often iterate through the small numbers. For shapes, they often pick examples from the familiar categories.

2. *Property reasoning* occurs when students make arguments based on the properties that define the terms in the statement. This occurs most obviously on the order relation tasks since students know that any two numbers satisfy one of three properties: $x < y$, $x > y$, or $x = y$.

3. *Set reasoning* occurs when students think about the (entire) group of objects that satisfies part of the statement (the "truth set"). Students do this more readily in some cases than others depending on the mathematical context. Students tend to think "even numbers" is a set while "non-multiples of 6" may not be as associated easily with the whole collection of numbers. Students think of "rectangles" as a set much more readily than they do "non-rectangles." Students may need more time before they spontaneously think of something non-obvious like "even numbers that are not multiples of 4" as forming a set.

It may help you in listening to various students to ask which of these broad categories their reasoning falls into. Research has found that set-based reasoning is the most challenging for some students (depending upon context), but also the most productive for reasoning about logical patterns across the statements.

As alluded to above, you may observe that for false statements students will say something that sounds very similar to the negation. For instance, for Statement 15 they may say "this triangle is not equilateral, but it is acute." Research has found that many of those students are not thinking about the negation, if nothing else because they do not yet know what that means. Rather, they are describing a counterexample. This is a subtle, but important distinction. A counterexample is a relationship between a statement and a mathematical object (number or polygon). Negation is a relationship between two statements. Part of learning logic is to treat the statement itself as the focus. Students rather focus on the mathematical objects referred to (numbers, triangles, etc.). When the student says "is not equilateral, but is acute," they likely are describing why the shape they have in mind is a counterexample, they are not stating the *conditions for any shape to be a counterexample* (which is the definition I use for the negation of a predicate). In the next part of the lesson, we will develop falsehood sets as a way to introduce negation. For now, write down the students' reasoning for why this was a counterexample, but put off making the connection to negation (some students may see this connection, but almost certainly some in your class will not be ready for that generalization yet).

Another pattern of student reasoning that will likely emerge that is worth focusing on is "if not, then" reasoning. This is the foundation I use for building proof by cases since it partitions the set of examples using one of the given predicates. This is a really nice strategy for interpreting the geometry statements (16 and 18), which students may find difficult to fully justify. These tasks appear later because this is a productive challenge. Some of the "if not, then" arguments for statement 16 are as follows:

A. If the triangle is acute, then it makes the statement true. If the triangle is not acute, then it will have an angle that is at least 90°. Equilateral triangles have three 60° angles. Thus, the triangle will not be equilateral and the statement is true.

B. If the triangle is not equilateral, then the statement is true. If the triangle is equilateral, then it will have three 60° angles. That means it is acute. Thus, the statement is true.

When "false" does not mean untrue

One thing you should be prepared for is that students may sometimes say that a statement is "false" not to mean it is untrue, but rather to say it is unnatural and they would not say it. This is one way of making sense of why students reject "15 is even or 15 is odd." Some students are less rejecting this as untrue (though some think that), but rather they are using "false" to mean, "I would rather say '15 is odd' instead." If you think someone is using "false" in this way, you can simply ask them which they mean and invite them to consider the difference between a "weird" statement and an "untrue" statement.

What about "exclusive or"?

I usually find that some of the most precocious students will raise the question of whether an *or* statement is true when both parts are true. I generally just tell them the mathematical convention that *or* statements are true in those cases. I have not found a clever way within this task sequence to help them decide that this is the "better" choice on their own, so I simply tell them that the inclusive meaning is standard.

Naturally, students may not state all of the parts of those arguments, especially the first sentence. It is up to you how much you want to help them make all the parts of this explicit. The way students explain it, they turn the statement "$\forall x \in S$, $P(x)$ or $Q(x)$" into the equivalent "$\forall x \in S$, if not $P(x)$, then $Q(x)$." I think it is fascinating that students develop this kind of argument intuitively. For instruction, I turn this into an argument by cases. They are dividing all the triangles into two groups that cover all the cases, either acute/not acute or equilateral/not equilateral. By bringing out the usually "hidden case" in the first sentence of each argument, you help students explain why the argument really covers all the possibilities.

Another task I sometimes give for homework after this set of activities is "Given any integer x, x is a multiple of 3 or $x^2 - 1$ is a multiple of 3." The proof by cases approach works well here, but students will struggle to know how to deal with numbers that are "not a multiple of 3." I introduce the representations $x = 3n + 1$ and $x = 3n + 2$, which also provides a nice opportunity to discuss how we deal with negative predicates. In some cases, we have a good way to substitute a negative category with a positive one (e.g., "not acute" means "right or obtuse") and in others we do not (e.g., "not a rectangle" means. . .).

3.2.6 Idealized dialogue from the middle part of the lesson

Student D: For Statement 17, it says the quadrilateral could be a square or not a rectangle, which means it is like a parallelogram. So, it only says it could be a square or a parallelogram, and I think there are other shapes.

Student E: Yeah, I was thinking of like a trapezoid, but I think that is not a rectangle.

Student F: I think it could still be a rectangle, which is what would make it false. A rectangle is not a square and it *is* a rectangle, so that makes both parts false.

Student D: But aren't all squares rectangles?

Student F: I am thinking of a rectangle that is longer than it is tall.

Student E: So if all squares are rectangles, does Statement 17 say "it is a rectangle or it is not a rectangle," which would be true?

Student F: I am not sure I follow that, but look at this rectangle I drew right here. This one is not a square, so the first part is false, and it is not a rectangle, so the second part is false. If both parts are false, this quadrilateral makes Statement 17 false.

Student D: Like I said at first, this statement says there are only two kinds of quadrilaterals, squares and parallelograms, and really there are others like the one you drew.

Student E: Ok. I see that works, so Statement 17 is false.

Student D: So, it seems like Statement 18 should be false too because it only lists two kinds of quadrilaterals, not squares and rectangles, so it should be leaving something out.

Student E: Does the parallelogram make it false because it is not a square or a rectangle?

Student D: But the parallelogram makes "not a square" true, so it makes the statement true.

Student E: Oh right.

Student F: It seems like when it says "not a square" that covers a whole lot. I mean, most quadrilaterals I can think of are not squares. The first part of the statement covers almost everything. The only other examples are the squares, and those are going to fit the second part of the statement, since all squares are rectangles.

Student E: I am having trouble thinking about what it means to not be a square.

Instructor: It sounds like you are splitting everything into two groups: squares and non-squares. Explain again why you think this statement is true for both of those groups.

Student E: So, I guess the non-squares make the first part true since that is just what it says. Like we said above, *or* statements are true when one of the two parts are true, so we don't have to worry about the rest of the statement for those. Then if the quadrilateral is a square, then the first part is false, but the second part is true. Cause squares are always rectangles.

Student D: If it is a rectangle that is not a square, both parts are true. Is that a problem?

Instructor: Did you have any other statements where both parts were true at the same time?

Student F: Statement 5 when x and y are equal. I really feel like Statement 5 is true though. I guess it would be better if we said "$x < y$ or $x \geq y$."

Instructor: I agree that Statement 5 as it is written is true. Any pair of numbers will make one or both of those conditions true. Generally, in mathematics we say that *or* statements are true even if both parts are true in some cases. This is not the same for everyday statements. When I tell my daughter "you may have pie or cake," I do not mean both. In mathematics we allow for both parts to be true. I like your argument for why Statement 18 is true using squares and non-squares. Try to write that argument down as clearly as you can and I will have you share it with the class when we come back to large group.

What I hope to portray with this bit of dialogue is how students will struggle to find productive ways to think about the geometric statements. This is related to their common tendency to pick examples to represent the conditions in the statement, sometimes replacing negative conditions (not a rectangle) with positive ones (is a parallelogram). The final argument confirming Statement 18 shows the power of thinking about whole groups of shapes (all non-squares). The instructor can use their own judgment when and how to push for this kind of thinking. That is ultimately the focus of the third and final part of the lesson.

3.2.7 Second sequence of tasks-focusing on truth sets and falsehood sets (1 class day)

The predicates in this lesson are not really new. They are the statements from the previous part of the lesson with the quantification removed. By removing the universal quantifiers, the statements become compound predicates. The task assigned to students at this point is "describe the set of objects that make this condition true and the set that makes it false." The big idea here is that the truth set of "$P(x)$ or $Q(x)$" is the union of the truth sets of the two component predicates. Given that these students have not taken proof-based courses yet, they may or may not be familiar with set notation or with set operations such as union. I am not particularly interested in them writing this out in formal notation (though the instructor may choose to introduce set-builder notation to express students' ideas). The primary goal is for students to see the relationship. Maybe even more importantly, describing the falsehood set will help students define the negation of a predicate and develop proof by contradiction.

1. z is an even number; "z is a multiple of 2 or z is a multiple of 3."
2. z is an even number; "z is not a multiple of 4 or z is not a multiple of 3."
3. z is an integer; "z is a multiple of 2 or z is not a multiple of 4."
4. y is a real number; "$y < 3$ or $y > 5$."
5. y is a real number; "$y > 3$ or $y < 5$."

6. ΔABC is a triangle; "ΔABC is equilateral or ΔABC is not acute."
7. ΔABC is a triangle; "ΔABC is not equilateral or ΔABC is acute."
8. $\blacksquare RSTU$ is a quadrilateral; "$\blacksquare RSTU$ is a square or $\blacksquare RSTU$ is not a rectangle."
9. $\blacksquare RSTU$ is a quadrilateral; "$\blacksquare RSTU$ is a not square or $\blacksquare RSTU$ is a rectangle."

I originally introduced this extension of the previous tasks because I realized that students did not know what a negation of a statement was and I did not know an easy way to describe it to them. I looked through a bunch of textbooks and found that they do not really give a general meaning for negation either! Some say that the negation of a statement T is the statement "It is not the case that T," but this does not get us very far. This is why this lesson focuses on negating predicates, where we can give the meaning described above: one predicate is the negation of another whenever they are defined over the same universal set and the negation gives the opposite truth value as the original for every example. Stated another way, the truth-set of the negation is the falsehood set of the original and vice versa.

As stated above, students' task regarding these conditions is to "describe the set of objects that make each statement true and the set of objects that make each statement false." Students answers to the second question emerge as the negation of the predicate. This is the point where the lesson explicitly pushes students to reason about sets and to consider how to use the complement of a set to give meaning to negative predicates ("not a rectangle" refers to the complement of the set of rectangles and "not equilateral" refers to the complement of the set of equilateral triangles).

The hard part of the first task for some students is how to describe the truth set without simply restating the statement itself. They will often begin using *and* to link the two sets, such as "the multiples of 2 and the multiples of 3" (for Predicate 1). This raises a natural difficulty because mathematicians use *or* between to predicates to represent the union of the truth sets, but students often use *and* between the two sets to express that idea. You may want to introduce set-builder notation at this point to be able to talk about the set of objects and introduce the union notation to express putting the two sets together. This allows them a new and technical way to express what they want to say. It is good that this notation come up in class as a solution to a problem they experience in trying to communicate clearly (rather than simply another thing we expect them to memorize). I think it is better not to let a student who has already learned it share this because you want to avoid the sense that you are giving them an advantage that is unfair to the other learners.

For the predicates that are always true (Examples 1, 3, 5, 7, and 9) students will often describe the falsehood set by simply saying "nothing makes it false" or "it is empty." This is another opportunity to invite students to adapt their reasoning from one statement to another. Once students have clearly stated the negation of predicates that have a non-empty falsehood set, invite them to use the same strategy on something that has no examples that make it false. This will feel non-obvious to some of them, but they can do it. Once you have written down the conditions for something to be a counterexample, ask them to prove why there are no counterexamples. This will elicit proofs by contradiction. For instance, students can generally show why there is no triangle that "is equilateral and is not acute" or no quadrilateral that "is a square and is not a rectangle." You can work then to make the argument explicit: since there can be no counterexamples, all examples make the predicate true (the truth set is the universal set) and this proves why the universally quantified statement from the previous session was true. This same procedure works to motivate proof by contradiction for conditionals as well.

This lesson should take about 1 class session if you explored the tasks on the first day adequately. You may also want to go back to the examples from the previous lessons and explain how the negation of a universally quantified statement can be stated with an existential quantifier. This lays the foundation for negating multiply quantified statements, which will not be covered in this lesson sequence.

3.3 Post-lesson considerations

The heart of teaching logic using mathematical statements is to help students reflect on their own language use so they can see what problems logic helps us address and how mathematical language may differ from other kinds of language. By focusing on predicate logic instead of propositional logic, we reduce the distance between student language and formal language so it is easier for them to adapt to the way mathematicians use language. Obviously, this claim relies on an extrapolation from the language use of students I have taught both in experiments and the classroom. This way of teaching has the added benefit of giving the instructor access to student interpretations. By listening to students, instructors can adapt to the linguistic needs and opportunities among various students. Indeed, I anticipate that teaching this lesson in a different language would work though the particular challenges might vary quite widely.

Furthermore, we try to keep a tight connection between logic (conditions for when statements are true or false), proof techniques (mathematical arguments that justify whether a statement is true or false), and sets (the collections of objects referred to by mathematical statements). These principles of studying logic by reflecting on example statements and example arguments extends to other topics such as conditional statements, multiply quantified statements, etc. You can also justify indirect proof techniques such as proof by contradiction and proof by contraposition (justifying "If $P(x)$, then $Q(x)$" by proving "if not $Q(x)$, then not $P(x)$") using set-based truth-conditions for conditional statements. The goal as much as possible is to always begin with students' own processes of reading, interpreting, explaining, and arguing about these kinds of statements. Try to encourage the generality in their understanding of logic to arise from their generalizing across many statements of the same form. This runs alternative to the common tradition of working with symbols that refer to nothing in particular (p's and q's). The main goal of learning logic in the undergraduate curriculum is learning how to use mathematical language effectively for reading and writing proofs. Students will read and write proofs in particular contexts (number theory, analysis, or algebra) so it seems strange to me to only teach de-contextualized logic. If the "statements" they see in the logic section are all symbolic, non-mathematical, or nonsense then I fear that what they learn in that section will be inert in their later learning because it will have no connection with their ways of reading meaningful mathematics.

A key part of implementing this lesson well is getting away from the idea that the way mathematicians interpret language is right and correct, which implies that students are irrational in some way for interpreting in other ways and needing to learn these ways of using language. Not only does this fail to acknowledge our students' rationality, it is refuted by psychological research on how people reason in the everyday. I think it is important that we think of the way mathematicians use mathematical language as a very specialized set of conventions that solve problems faced in doing proof-based mathematics. It is only fair to help students see how the conventions of mathematical logic solve problems of ambiguity and make our language precise enough to prove claims conclusively. We should only expect students to see the value in mathematical logic if we can show them the power it has for allowing us to communicate clearly and precisely and to construct finite proofs that have infinite scope. Students are capable of engaging in these things, but they need the right opportunities to learn the game being played and the coherence of the "rules" for that game.

3.4 Bibliography

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4

Introducing Definite Integrals Quantitatively through Adding Up Pieces

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Topic	This lesson examines the introduction to definite integrals.
Course Context	This lesson is taught in a first-semester calculus course at a large, private university.
Instructional Challenge	Typical approaches to definite integrals focus on area and antiderivatives, though subsequent uses of integrals in science, engineering, and even mathematics are best supported with quantities-centered meanings based on the Riemann sum.
Brief Overview of Approach	Rather than focus on the standard "area" problem, this lesson is couched in the familiar real-world context of fuel flowing through a pipe that allows students to use quantitative reasoning to develop the Riemann sum structure. Students are asked to find the total amount of fuel that flowed into a tank given a fuel flow rate. A constant rate is first used to develop the ideas of product and accumulation, and attention is given to the amount that flowed into the tank versus the actual amount in the tank. A variable rate is then used to develop the idea of partitioning the domain (time), finding a product within each domain segment (rate$\times$time), and summing the results of the products together to get an overall total, corresponding to a Riemann sum. Questions of accuracy are then used to develop the limit idea that goes from a Riemann sum to a Riemann integral, where the definite integral is defined. Throughout the lesson, quantities remain at the forefront, with symbolic expressions and area used as ways to represent the quantitative structure. This LAM ends with attention to what can be done in subsequent lessons to ensure that the chop-quantity-sum structure meaning remains at the forefront, as those lessons progress through the remainder of the integration chapter.
Keywords	Calculus, Definite Integrals, Quantitative Reasoning, Connections between Math and Science

4.1 Background information

4.1.1 The Instructional challenge

In typical curriculum, definite integrals are commonly introduced through finding the area of a shape bounded by the graph of a function (e.g., [16]; [17]). Riemann sums are usually used as a calculational method for approximating this area, but research has shown that students do not tend take up the Riemann sum as the central meaning for integrals

([11]; [5]). Rather, students may believe area to be the true underlying meaning of integrals, with Riemann sums being nothing more than crude approximations of this area that can be discarded after the more efficient antiderivative technique is learned. Thus, areas and antiderivatives tend to form the central meanings for students, with Riemann sums being a cumbersome sidenote ([11]; [5]). While area and antiderivatives are certainly important components of the overall definite integral concept, research has clearly shown that quantitative meanings for definite integrals based squarely on the Riemann sum structure are much more powerful for understanding, using, and setting up integrals in a wide range of contexts. This is true both within mathematics, for topics such as arc length [2], volumes of revolution [15], and line integrals [10], as well as outside of mathematics, for disciplines such as physics and engineering [3]; [19] or chemistry [9]. One such Riemann-sum based quantitative structure is called *adding up pieces* [4], which can give students' stronger abilities to model contexts with definite integrals ([1]; [2]; [6]; [13]; [15]). Adding up pieces will be described in detail in the Key Mathematical Ideas subsection. This lesson focuses on developing definite integrals quantitatively through the Riemann sum structure, with area as a secondary representation.

4.1.2 Learning Goal of this Lesson

Students will interpret definite integrals as a quantitative Riemann sum structure and identify the individual parts of that structure. The students can use this structure to define a definite integral for a given context. Note that "areas" are meant only to be representations of this structure, without becoming the central meaning itself.

4.1.3 Context

This LAM reports on a lesson taught in first-semester calculus classes at a large, private university in the United States. In these classes, the students are mostly White, with only few racial and ethnic minorities. The classes tend to be about one-third Female. Thus, if open-ended questions are answered only by volunteers, it is likely that White Males will make up a disproportionate amount of the responses. Because of this, I feel it important to call on Minority and Female students to have a voice in the class and to provide their thinking. Of course, rather than cold-calling on these students, I do allow students time to discuss with their partners and I look for minority and female students with ideas I want to be brought up to the entire class. Thus, in those places in the lesson where I ask students to respond to questions, calling on these students is an important consideration.

Additionally, the majority of the students in these classes have already taken calculus. Yet, this lesson is still intended to be able to be used in any calculus class with any group of students, even in classes where many students have had calculus before. This is because it is likely that the students have not been previously taught the strong quantitative meanings for the definite integral this lesson develops. This is important from an equity standpoint, because students who have not had calculus before could be seen to be at a disadvantage. Yet, given the standard approach, this lesson works well to make students' access to the content more equitable.

Because this lesson is meant to be the introductory lesson on integrals, it is not specific to any particular curriculum. It can be used to begin integrals regardless of textbook used, though it likely differs from how the textbook introduces integrals. Thus, the instructor may need to adjust their usage of the initial parts of the integral chapter to use this lesson and connect back into the remaining book sections. It may be helpful if, in the class at other times, the instructor has talked to the students about what a "quantity" is, such as when discussing derivatives or limits or other ideas. If not, that is still okay, because that can happen at the beginning of this lesson, too.

4.1.4 Instructor Background

It may be useful to relate that I have worked on education research that aims to bridge the gap between mathematics education and science/engineering education. I care deeply about helping students productively use their mathematical knowledge beyond the math class in which it is learned, such as in their future science, engineering, or mathematics coursework. My research has focused on gaps that exist between calculus education and fields within science and engineering, and how curriculum and instruction could be revised to help strengthen connections across this divide (e.g., [6], [7], [8], [9]). In fact, the lesson described in this chapter is an elaboration of and solidification of earlier ideas I have discussed about how to introduce integrals.

As far as my personal teaching goes, I often operate with a mix of students discussing briefly with a partner, followed by whole-class discussion. In this lesson, most questions are posed to the students with the expectation that they discuss with a neighbor for a short period of time. Afterward, I call the class back together and either open-endedly ask students to volunteer ideas, or I call on specific students who discussed ideas I want to be reported to the class or to ensure Minority and Female students are having their voices heard.

4.1.5 Key Mathematical Ideas for this Lesson

To be as clear as possible, the following are ways I conceptualize important ideas related to the definite integral concept.

1. *Riemann sums* are mostly used in the usual way: $\sum_{k=1}^{n} f(x_k)\Delta x_k$. However, I also allow the term "Riemann sum" to be used for structures that are more than just the product $f(x)\Delta x$. For example, I allow $\sum_{k=1}^{n} \pi f^2(x_k)\Delta x_k$ to be a "Riemann sum" for volumes of revolution without needing to redefine "$\pi f^2(x)$" as a separate new function.

2. *Definition of definite integrals*: I use the common textbook definition for the definition, that the integral is the limit of Riemann sums as $n \to \infty$ (or $\Delta x \to 0$).

3. *Quantity*: Lastly, a quantity is an attribute of an object or system that can be assigned a value, through some chosen unit of measure [18]. For the purposes of this lesson, the actual measurement process itself is much less important than the idea that every quantity has some measurement unit associated with it (e.g, liters, liters/min, feet, tons, meters/second).

4. *Adding up pieces* [4, 6] is a particular way to conceptualize definite integrals based on the definition of the definite integral. It comprises three key aspects: (a) chop, (b) figure out how much of the quantity is in each piece, and (c) sum. These are described as follows:

   (a) In *chop*, the domain is partitioned into small segments. For example, consider the integral $\int_{t_1}^{t_2} R(t)dt$, where $R(t)$ is the rate at which fuel flows into a tank with respect to time. The "chop" step would be to partition the time interval $[t_1, t_2]$ into very short time segments, each being Δt. Or, to use volumes of revolution as another example, it would be to partition the solid into thin disk slices, each with width Δx. However, in this conception, it is allowable to imagine the Δt or Δx pieces as already approaching infinitesimal in size, in accordance with the limit $n \to \infty$. When imagined as such, the notation dx is used instead of Δx (or dt instead of Δt), which is very much in keeping with Leibniz' original intention with this notation [12]. That is, Δx is a symbol used for "macroscopic" x segments, whereas dx is a symbol used for "microscopic" x segments. This allowance departs from the definition, which completes a summation for every natural number n and takes the limit of this constructed sequence. The complexities involved in forming a completed sum for every n and taking the limit of the resulting sequence is why this conception allows the limit to be viewed more intuitively, with the pieces being conceived of as approaching infinitesimal in size *before* the sum is taken.

   (b) In *quantity in each piece*, the quantity of interest is determined within each piece from part (a). In the fuel example, the quantity of interest is "amount of fuel" and this step would be to determine that the amount of fuel that flowed into the tank over one very small dt time interval is approximately $R(t)dt$, derived from rate $\times$ time. In the volumes of revolution example, the quantity of interest is "volume" and this step would be to determine that the volume of a single disk is approximately $\pi f^2(x)dx$, derived from $\pi r^2 h$. Note the explicit use of "d" instead of "Δ" in accordance with imagining the pieces from part (a) as approaching infinitesimal in size.

   (c) The third step, *sum*, is to conceptualize that the quantity of interest from (b) can now be summed across all of the pieces to capture the total amount of that quantity across the domain of integration. The integral symbol itself, $\int$, in accordance with Leibniz' original intention, is thought of as a "sum" symbol that is specific to adding up these infinitesimally-sized pieces. In the fuel example, it would be to realize that the small amounts of fuel that flowed into the tank within each dt time interval can be added together notated

$\int_{t_1}^{t_2} R(t)dt$, to get the total amount of fuel that flowed into the tank over $[t_1, t_2]$. In volumes of revolution, it would be to realize that the volumes from each thin disk can now be added, notated $\int_a^b \pi f^2(x)dx$, to constitute the total volume of the solid from a to b. This same *chop-quantity-sum* structure can be used as a template for taking essentially *any* context to which definite integrals apply and constructing a definite integral expression for it.

4.1.6 Rationales for Whole-lesson Decisions

As discussed in the "instructional challenge," my reason for prioritizing quantitative meanings for definite integrals through adding up pieces is because this meaning has been shown to be quite useful for making sense of and setting up integrals across a variety of contexts. Thus, I base my introduction to definite integrals on a real-world quantitative context in which students can use quantitative reasoning. I use a fuel-flow problem because it is intuitive to the students but contains all the needed parts to develop the ideas of chop, product, sum, limit, accumulation, and total vs. additional amount.

I purposefully delay the area conception, because prior experience has taught me that if it is used too early, students latch onto it and begin to neglect the quantitative structures (see also [11]). I very intentionally want symbolic expressions and graphical area images to be *representations* of the central quantitative meanings I am trying to develop.

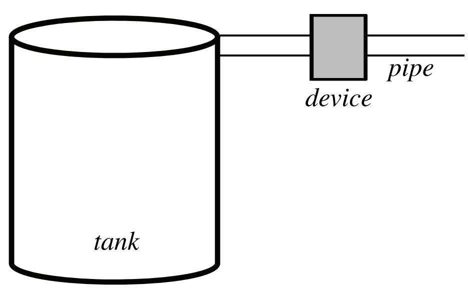
Because there are many ideas that need to be developed in this lesson (chop, product, sum, limit, accumulate, initial amount) across different representation types (numeric, symbolic, graphical), it is important to structure the lesson to attend to these systematically. I start with numeric values/computations because it is the best way to keep the focus on the quantities and the relationships between the quantities. Once the quantitative structure is developed through the numeric, I then proceed to help the students create symbolic and graphical-area representations.

4.2 Lesson Implementation

4.2.1 The Initial Fuel-Flow Context

(About 3 minutes)

To begin this lesson, I project the following on the board for students to see. Note that in my actual class, I use images of a real tank, pipe, and device, but to avoid any copyright issues, I made simple reproductions of them here. "A pipe bringing fuel into a tank has a device on it that records the fuel's flow rate through the pipe. Over a 4-minute interval, the flow rate is 10 liters per minute. What quantities are part of this context? What are their units?"



I do not spend too much time on this question, but allow students about thirty seconds to briefly discuss with their neighbor. I then ask students to call out answers. As students call out quantities, we label them with symbols, such as t for "time," R for "flow rate," and A for "amount of fuel." These are the three key quantities to bring up, and students usually identify them at this time. If students do not recognize "amount" as its own quantity, the instructor could prompt this with: "What about the tank itself?"

I then ask, "Can we determine the amount of fuel in the tank after the 4 minutes? Talk briefly to your neighbor about this." After a brief moment to discuss, I ask students to report. Two major responses are given at this point:

- Student 1: Yes, it's 40 liters.
- Student 2: We aren't sure if there was already fuel in the tank.

I build on both student contributions by first asking Student 1 to recap why 40 L is a sensible response. This ensures that we have the quantitative structure rate $\times$ time on the table, which is the precursor to the structure $R\Delta t$ and Rdt within the Riemann sum. I then ask Student 2 to recap their concern, in order to get at the idea that "40 L" is the *additional* amount of fuel in the tank. This allows us to distinguish between the quantities "amount," "initial amount," and "additional amount." This is crucial for definite integrals, which fundamentally represent additional or net amounts. Lower bounds can correspond to different initial amounts.

4.2.2 Developing the Riemann Sum Structure Quantitatively

(About 9-10 minutes)

I then project the following add-on for the students to see: "The fuel might not be passing through the pipe at a constant rate. If so, then R would be different at different times and we can think of R as a function of time: $R(t)$. Suppose at $t = 0$ the rate is $R(0) = 18$ L/min." After reading this, I first ask a quick question: "Can I multiply 18 by 4 minutes to get the amount of fuel?" Students readily call back, "No," and I select a student to explain why. I then follow this up with a question I want the students to discuss with each other briefly: "What might I need to do to better approximate how much fuel passed through the pipe over these 4 minutes?" After a short one minute, I ask the students to share ideas with the whole class and I engage in a short whole-class discussion, such as:

- Student 1: We need to know the rate at different times.
- Teacher: How does that help us? [Ask different student to answer.]
- Student 2: Because then we'll know better what the rate is during the four minutes.
- Teacher: OK, let's say I did that. Here's a table of rate values at different times [project table]. Work with your neighbor to estimate the amount of fuel that flowed through the pipe over the four minutes.

t (min)	0	1.25	2	2.5	3	3.75	4
$R(t)$(L/min)	18	12	7	6	4	3	2.5

There are two important items to explain about this part of the lesson. First, so far the students tend to focus on rates at different times, with partitioning the time interval as being more implicit. It is important in the following discussion that I call explicit attention to the partitioning of the time interval. Second, the time spacing in the table provided is not uniform. This is on purpose to later show the need for all intervals to shrink toward zero in the limit.

- Teacher: OK, [calls on student], explain to me how you and your partner estimated the amount of fuel that passed through the pipe.
- Student 1: We took the rate, 18, and multiplied it by the 1.25 to get 18.75. Then we took the next rate, 12, and multiplied it by the 0.75 to get 9. [And so on until 3×0.25.]
- Teacher: What are your units? Why can you multiply 18 by 1.25 and 12 by 0.75?
- Student 1: Rate is liters per minute, and each time is minutes. So that gives liters.
- Teacher: I want everyone to notice that in doing that, you stopped using the entire four-minute interval, but instead broke that interval down into smaller time chunks.
- Student 2: We did it that way, but we weren't sure why you don't use the 2.5 at the end.
- Teacher: What does everything think? Why not use it? [Call on student.]
- Student 3: When you do it that way, it's the rate at the start of each time interval you use. 2.5 is at the very end, so it doesn't start another time interval.
- Teacher: OK, good. [Call on different student.] I noticed you did it a different way. Can you explain your way?
- Student 4: Well, we thought that since 18 is only at the beginning of the first time interval, and it ends at 12, it might be better to average them. So we averaged 18 and 12 and got 15 and used that as the rate for the first time interval. [And so on.] We did use 2.5 at the end, because it was averaged with 3 to get 2.75.
- Teacher: Great. Notice that while they used a different method for the rates, they also broke the 4 minutes down into smaller intervals. In fact, let's call that process the idea of *chopping* or *partitioning* the interval into smaller pieces.

It is useful to find students who used different methods, such as the two methods here, and to label these: the "left-hand" method and, perhaps, the "averaging" method (which is the same as the "trapezoid" method). Interestingly, students rarely invoke the "right-hand" method on their own, and it is important as a teacher to interject it as another possibility, especially since it can be notationally convenient at times. Now, to build on these ideas, and to try to make the Riemann sum structure more salient, I highlight for the students that all of the methods have the following in common: (1) they chop or partition the time interval, (2) they multiply a rate to each time interval, and (3) they sum up the resulting amounts of fuel to get a total amount.

4.2.3 Symbols as Representations of the Quantitative Structure

(About 6–7 minutes)

Once the students have worked with the numeric data, and we have called attention to the chop, quantity, and sum structure, it is time to add symbolic and graphical representations. It is important at this point to continue to keep the focus on the quantitative structure, and not let the symbols or areas become the primary meaning. I tell the students I first want to write what we're doing using symbols instead of numbers. I tell them to go back to the initial constant flow rate problem and, using the symbols r, R, and A, to re-write their calculation using symbols instead. Students tend to quickly write: $A = R \times t$. I then tell them to go to the table of different rate values and work with a partner to try to write their entire calculation using symbols instead of numbers. After giving them a minute, I have a couple of students provide their symbolic inscriptions. Some examples include:

- $A = Rt + Rt + Rt + \cdots$
- $A = R_1 \cdot t_1 + R_2 \cdot t_2 + R_3 \cdot t_3 + \cdots$
- $A = R_1\Delta t + R_2\Delta t + R_3\Delta t + \cdots$

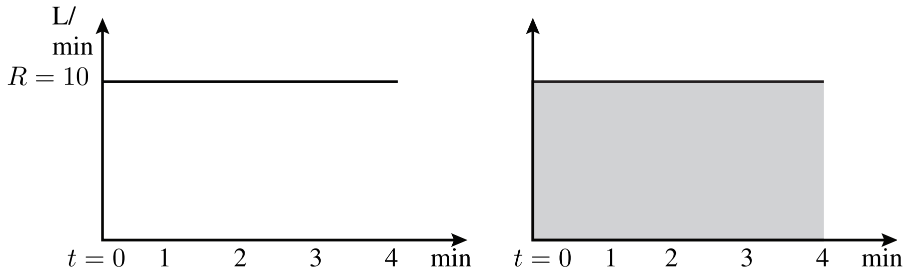
I express that each of these are valid ways for someone to personally inscribe their thinking. However, I explain that for effective symbolic communication, we could tweak these to be more precise. I ask the students what they like or would change about these expressions in order for us to have an agreed upon way of writing it. Students' thoughts include: (1) the first one has the issue of appearing like all R's and t's are the same, (2) the Δ's in the third show the chop/partition aspect, but that (3) the third one also makes it seem like all Δt's were the same. I also add that it might be nice to suggest that R is a function, as in $R(t)$, and that each different rate we used happened at a specific time value, $t_1, t_2, t_3, \ldots$. Building on these ideas, I suggest the following expression: $A = R(t_1)\Delta t_1 + R(t_2)\Delta t_2 + R(t_3)\Delta t_3 + \cdots$.

I then state that it is a little cumbersome to have to write the "$R\Delta t$" so many times and I ask them if there is a symbol that can suggest that a summation is happening across many terms. The students *reluctantly* state "Σ." I say reluctantly because most students have had bad experiences with summation notation before. To help manage the cognitive load in adapting the intuitive quantitative structure to the less-understood summation notation, I first start with a simpler version of the notation, without indices. I also find it helpful to take the lead here. I rewrite our expression as $A = \sum R(t)\Delta t$, but then remind them of the issue that it kind of looks like all of the R's and Δt's are the same, so we need a way to tell them apart. I explain that we can do this using indices. For example, if we call $[0, 1.25]$ the 1st interval, $[1.25, 2]$ the 2nd interval, $[2, 2.5]$ the 3rd interval, and so on, we can use $k = 1, 2, 3, 4, 5$, and 6 to represent the six intervals. The Δt for each interval can then be labelled $\Delta t_1, \Delta t_2, \Delta t_3$ and so on. The $R(t)$ value chosen for each interval can be labelled as $R(t_1)$, $R(t_2)$, $R(t_3)$, and so on. The basic $A = \sum R(t)\Delta t$ expression can now be embellished to the full expression: $A = \sum_{k=1}^{6} R(t_k)\Delta t_k$. It is incredibly useful as this point to again highlight the *chop* (Δt)*, quantity in each piece* ($R(t)\Delta t$), and *sum* ($\sum R(t)\Delta t$) structure inside the symbolic expression as well. This helps maintain a focus on the quantitative structure, despite abstract symbols. Once this is done, I explicitly name this summation expression as a "Riemann sum" from the mathematician Bernhard Riemann.

4.2.4 Area as Representation of the Quantitative Structure

(About 9–10 minutes)

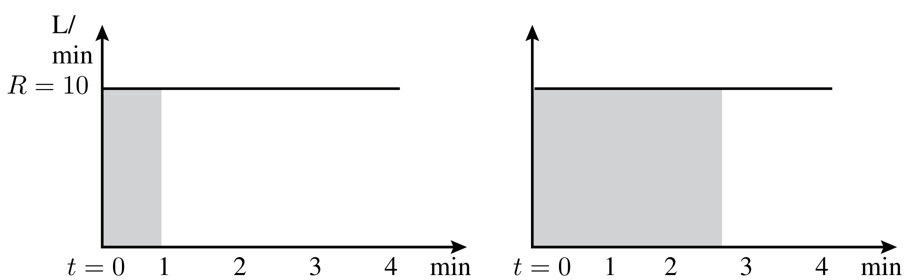
I recap, "So far, we *numerically* calculated approximations to the amount of fuel that flowed into the tank, and we have *symbolically* represented those calculations. It's good to have different ways to represent what's going on, because each way to represent it can show us different things. Since we have a symbolic way to represent what's going on with the quantities, I also want to develop a visual way we can represent it. Graphs are a good way to visually represent function information. Let's start back with the constant rate example, $R = 10$ and graph R as a function of time. What would the graph of R for this constant rate look like? Visually, how could you represent the calculation you did to get the 40 L?" I give the students a minute to work together, with the goal that they produce the following images:



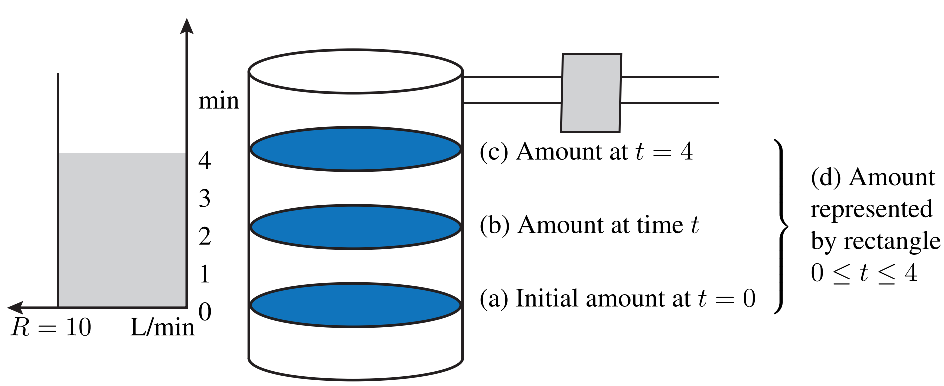
Some students will want to graph R as the "amount of fuel" instead, creating an increasing linear function. This incorrect graph should be brought up so that students can make sure they are thinking of the *rate* itself as the function, which is constant.

The unit associated with the area in this case is L/min $\times$ min = L. Some students will be unsure as to why the area can represent "liters" as opposed to "normal" area units. To make sure this comes up, if a student has not asked about it already, I ask: "Multiplying 10, this height here, and 4, this length here, corresponds to the area of this rectangle. But, isn't area supposed to have units like ft^2 or m^2 or in^2? What's the unit of *this* rectangle's area? Do those units make sense?" I call on a student to explain how the vertical side of the rectangle has units L/min, the horizontal side has units min, and multiplying these results in the unit L.

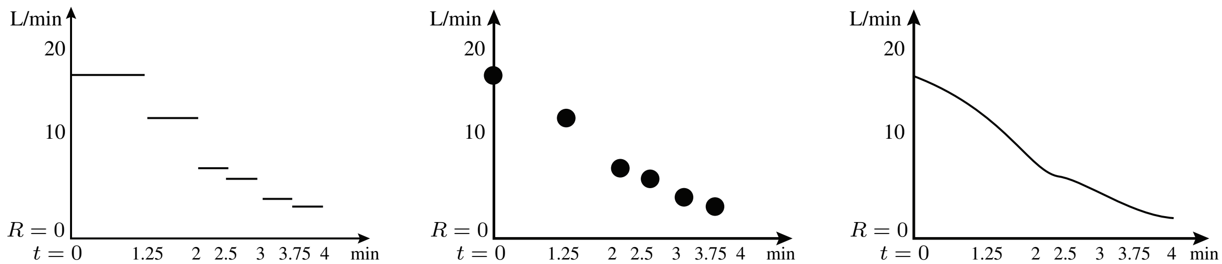
To leverage the area representation to think more about continuous accumulation, I ask a series of questions: "What would be different with our rectangle if we just looked at the first 1 minute? The first 2 minutes? The first 2.75 minutes? What's happening with the rectangle if we let time *continuously* increase from $t = 0$ to $t = 4$? What's the relationship between the rectangle and what's happening in the tank?" As students respond, it is useful to have a dynamic image in which the rectangle can be "stretched out" to the right, as implied by this figure:



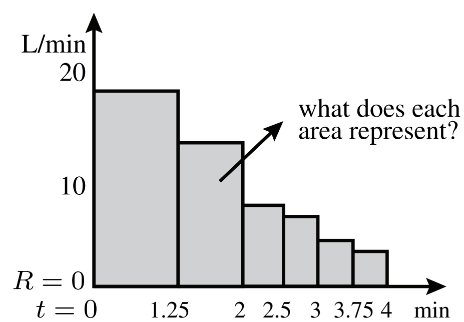
Before continuing, I find it helpful to emphasize a few ideas. I show the following image, with the graph tilted on its side, where the rectangle can dynamically be stretched upward. I recap: (a) the initial amount in the tank at $t = 0$, (b) that the size of the rectangle corresponds to the amount of *additional* fuel in the tank at any time t, (c) the completed amount at $t = 4$, and (d) the completed rectangle over $0 \leq t \leq 4$ represents the *total* additional amount of fuel in the tank.



Next, I tell the students to go back to the table of data with changing R values. I give them the instructions: "Let's draw this new $R(t)$ graphically. Draw the graph of $R(t)$ if we use the left endpoint as the function value for each entire short time interval." I choose the left-hand because it is a method the students themselves created, though one could opt to use the righthand instead. While the goal is to create a piece-wise function (left image below), the middle and right images below are graphs that might also be created. These can be discussed by asking whether the middle graph has a defined rate for all time values and whether the graph on the right accurately depicts the calculations that we used.



I then ask the students, "You also did some calculations to determine the amount of fuel over each short time interval. Like we did for the constant rate case, how can you visually represent those calculation in your graph?" The goal is to depict each part of the calculation as its own rectangle over that particular time segment. To ensure that students see these rectangles as representing the quantitative structure, I ask, "What does the area of one of these rectangles represent again? Is the unit of one of these rectangles ft^2 or m^2 or something like that? What is it in our case?" Finally, I ask, "In our computations, we also added the amount of fuel across the different pieces. What does that correspond to in this image?" This helps the students see that the total area of all rectangles is equivalent to the total amount of fuel that passed through the pipe.



4.2.5 Extending the Context with Refined Partitions

(About 9–10 minutes)

At this point in the lesson, we have examined a quantitative structure at a simple constant rate and at a changing rate, have numerically worked with the structure, and have used symbols and areas to represent that structure. To start advancing toward the notion of the limit, I ask, "To move on, what's not quite exact about the amount of fuel we calculated from that table? What would be needed to make it more exact?" Student fairly readily identify that more data points would help us be more accurate. To build on that, I show the following two tables and ask, "Here are two possible tables with more data points. For improved accuracy, which is better? Why?"

t	0	1.25	2	2.25	2.5	3	3.25	3.5	3.6	3.75	3.9	4
R	18	12	7	6	6	4	4	4	3	3	3	2.5

t	0	0.5	1	1.25	1.75	2	2.5	2.75	3	3.25	3.75	4
R	18	21	16	12	9	7	6	5	4	4	3	2.5

Here are a couple of typical student responses:

- They both have the same number of data points, so it's the same.
- The second table is better, because it's more evenly spaced across the four minutes.

With students providing the second option, the class pretty quickly comes to consensus that the second table is better. If the class does not come to a consensus, I ask a student to articulate why they feel the second table is better. The student can explain that the first table has one very wide time interval, making that portion of the data less accurate. They can also explain that the smaller time intervals in the second table are more evenly spaced and do not contain any wide intervals, making the data more accurate across the entire four minutes. This is important in developing the idea that *all* time intervals need to be getting smaller to hone in on the exact value, not just some of the time intervals. In fact, the first table gives a nearly identical estimate to the previous table, where the second gives a better one.

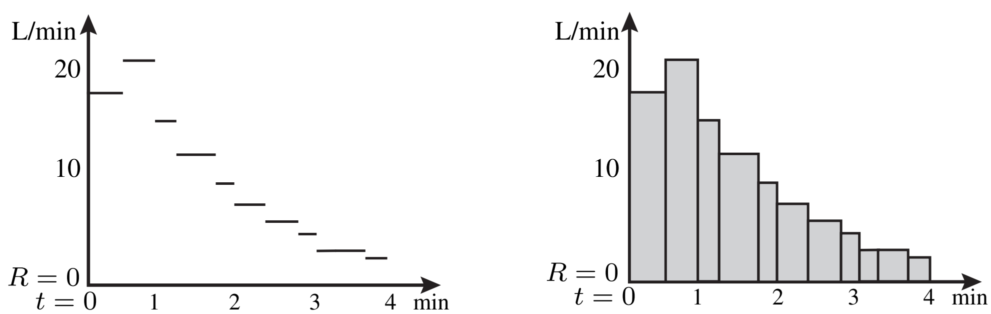
I discard the first table and have the students work out the new estimate of fuel amount based on the second table. I permit them to use whatever method they want, and some use the left-hand, some the "average" (trapezoid), with even a couple sometimes using the right-hand. Once they have finished their calculations, I ask, "Did we find the total amount in the tank?" The purpose of this is to remind students that the result is the *additional* amount of in fuel in the tank, not necessarily *the* amount.

I tell the students, "I think by now we have a pretty good sense of how we'd numerically calculate an estimate, given more and more data points. But, how long does it take to write all that out? That's the advantage of symbols, is that it gives us a shorthand way to represent all those numeric calculations. For our previous data table, we built the symbolic expression $A = \sum_{k=1}^{6} R(t_k)\Delta t_k$. Talk to your neighbor about whether this same symbolic expression applies to the calculations we did with this updated data set with more data points or if anything needs to change." At this point, some student responses often include:

- Yes, because we're still doing rate times time, and adding those up.
- No, because that has "6", but now we have "11" time intervals.

Based on the second response, I ask, "OK, what could we change about this expression that *would* make it match the calculations for our new data set?" The students begin to recognize that a simple change from "6" to "11" accomplishes that: $A = \sum_{k=1}^{11} R(t_k)\Delta t_k$. Again, I explicitly refer to this summation structure as a "Riemann sum."

I then follow this up with, "Let's see if we can also create a visual way to represent this. Like before, work with your neighbor to sketch a rough graph for the updated $R(t)$ function, using the left-hand method. How can you represent the numeric calculations you did in this graph? Also, talk to your partner about how the symbolic expression $\sum_{k=1}^{11} R(t_k)\Delta t_k$ relates to what you drew in your new graph." Because this is similar to the previous data set, students will fairly reliably create the graph in the image here, on the left, and will represent the multiplication as rectangles, as on the right.



It is important for the students to see the following connections between the symbolic and the visual: (1) Δt is related to partitioning the t-axis into small segments, (2) $R\Delta t$ is seen as each rectangle over that segment, and (3) "Σ" represents the total area of the rectangles combined. If any of these are not noticed by the students already, ask them to consider that specific symbol in connection with what is on the graph. Questions such as the following can be used: "What calculation coincided with the area of one rectangle? Where is that in the symbolic expression? What does the summation symbol correspond to in terms of the areas of the rectangles?"

4.2.6 Defining the Definite Integral

(About 6–7 minutes)

Here I reiterate, “Was our calculation from this new data set perfectly exact? What should we do to continue to get better accuracy for the amount of fuel?” Students again quickly acknowledge that more and more data points on smaller and smaller time intervals are needed. I follow this up with, “How would our symbolic expression change if we used 50 short time intervals? 100 short time intervals? Is there a way to write it for some arbitrary ‘n’ number of time intervals?” I let students work this out together, with the goal of becoming comfortable switching the upper summation index to be $\sum_{k=1}^{n} R(t_k)\Delta t_k$. Because of the challenges with summation notation, students will need a moment to think together to realize that simply switching the upper index represents a switch to more time intervals.

I then pose to the students, “Let’s suppose we could get infinitely accurate in our time, rate, and amount measurements, ignoring real-world physical limitations for now. What should we do to essentially get a perfect degree of accuracy for the amount of fuel that passed through the pipe? What concept is this like that we’ve seen before in the class?” Some students will notice the connection to the limit concept. To let the students think about it, I ask, “What could we add on to our symbolic expression to show that we are getting infinitely many time intervals in there?” There are two ways to write this limit and I look for students to possibly write both, so that we can put both on board: $\lim\limits_{\Delta t \to 0} \sum_{k=1}^{n} R(t_k)\Delta t_k$ and $\lim\limits_{n \to \infty} \sum_{k=1}^{n} R(t_k)\Delta t_k$. With both up, I can ask students to briefly discuss if the two versions mean the same thing. I listen for a student who can explain that as the number of partition intervals goes to infinity, the size of each Δt interval has to shrink in size. If the student does not mention that *all* intervals have to be decreasing, I recall the first table from earlier to add the assumption that we can’t let one interval remain large. This helps some students connect with the fact that *all* time intervals have to be shrinking for this to be true. I finish by telling the students that this is the definition of what is called the “*definite integral*.” Any time they chop or partition a domain into small segments, evaluate a quantity within each of those pieces, and sum the results together, they have essentially defined a definite integral.

At this point, it is useful for me to lecture for a couple minutes on some important background ideas based on history and specific notational uses that the students would not necessarily have access to. Also, there is a tendency that I address for students to want the limit to make the Δt or dt collapse into nothing [14], which entirely removes the time quantity from the context. For this explanation, I state that this idea of tiny intervals getting smaller and smaller was a crucial idea in the historical development of calculus and was discussed by many major mathematicians in the 1600s, including Fermat, Wallis, Newton, and Leibniz. In particular, Gottfried Leibniz called these “infinitesimals.” Informally, the idea of an infinitesimal is that a quantity’s value gets increasingly close to 0, but *without being* 0. That is, the quantity does *not* disappear. As we shrink the time intervals smaller and smaller, they are heading *toward* 0, but the time interval never is 0. Leibniz used a “d” to represent these infinitesimal quantities, much like we today use “Δ” to mean the change in a quantity. So, to help us distinguish between larger intervals versus infinitesimal intervals, we can write dt or dx or dy to mean Δt or Δx or Δy at the infinitesimal scale. In other words, whenever we see a dt or dx or dy, we can think, this is a Δt or Δx or Δy as it is going *toward* zero under a limit.

4.2.7 Definite Integral Notation

(About 5 minutes)

To wrap up, I provide the students the definite integral notation and I make explicit connections between the previous summation expressions the students have created and this new integral notation. I start by explaining that if we let the time intervals become infinitesimally small, the basic summation structure goes from $\sum R(t)\Delta t$ to $\sum R(t)dt$, where “Δ” changed to “d.” I explain that when Leibniz was developing these ideas, the summation symbol as we know it, Σ, was not yet in use. Thus, his choice for a “summation” symbol was to use an “s” for “summation.” However, in the 1600s, an “s” at the beginning of a word looked like “$\int$” [12], so instead of writing it $\sum R(t)dt$, Leibniz wrote it as $\int R(t)dt$. And similar to how we use indices $k = 1$ to $k = n$ in our summation to denote the first and last time intervals, Leibniz denoted the first and last time intervals by the first and last t values. In our example, that would be from $t = 0$ to $t = 4$. Leibniz attached these values to the “sum” symbol, written as $\int_{t=0}^{t=4} R(t)dt$. Because this summation takes all the little amounts of fuel over each time interval and adds them together, we use a word that means “whole.” This word is “integral.” I then project the following on the board and go over the connections between modern “sum” notation and Leibniz’ integral notation.

$$\underbrace{\lim_{\Delta t \to 0} \sum_{k=1}^{n}} \underbrace{R(t)\Delta t}$$

$$\int_a^b R(t)dt$$

1. dt is a “tiny” Δt.
2. $R(t)dt$ represents multiplication between $R(t)$ and dt.
3. $\int$ represents a summation between a and b.

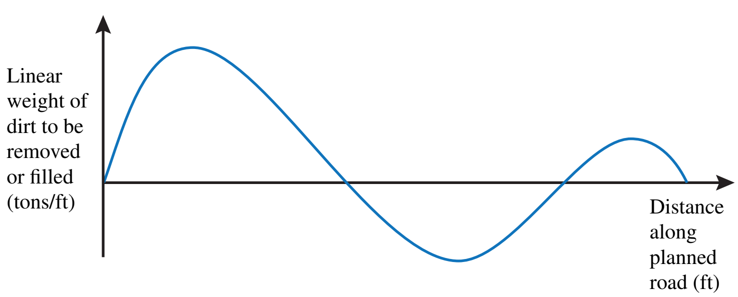
To end, I ask one last question for the students to discuss for the remaining few moments of class: “Discuss with a partner how Leibniz’ integral notation contains the exact same *chop*, *quantity in each piece*, *sum* structure that’s in the Riemann sum expression.” This question gives them a chance to see for themselves how the quantitative *adding up pieces* structure is directly visible in the definite integral expression.

4.3 Post-Lesson Considerations

This lesson is focused on developing quantitative meanings for the definite integral. It is useful to end this LAM with a few ideas toward continuing to highlight the quantitative structure for integrals in subsequent lessons, as the lessons progress to net area, integral properties, accumulation functions, and the FTC. If, after this single lesson, one leaves the quantitative structure to focus on area or antiderivatives, the students will very likely adopt the area or antiderivatives meanings and discard the quantitative structure almost as if this first lesson didn’t happen (see [11]). This outcome would make this entire lesson moot.

First, the fuel-flow context itself is quite amenable to extending to these subsequent lessons. If one introduces an outflow as well, then negative rates, and negative integral values can be discussed. If one provides the students with a continuous graph of the entire flow rate function, $R(t)$, then one can create accumulation graphs (i.e., antiderivative graphs) of the amount of fuel that has flowed through the pipe at any given moment in time. By extending the end-time from four minutes onward, one can develop the idea of a variable upper bound, which can even lead to the FTC as one creates an “amount” function that is derived from a given rate function

Of course, one would not want to use only this same context for the entire integral chapter. Thus, it is useful to supply here some other contexts that can be used as well. One very nice context for retaining quantitative meanings is an adaptation of the “mass-haul diagram” that engineers might use when paving new roads. I originally found this context in a YouTube video discussing road work, where I was surprised to see an antiderivative graph being sketched before my eyes: https://www.youtube.com/watch?v=PIK6I6Q58Ec. In this context, a construction crew is building a straight road segment through uneven terrain. They have to dig away any earth that is above where the road will be and have to fill in any low areas below where the road will be. The following diagram shows the density of tons of dirt per foot that need to be removed (positive) or filled in (negative) at each point along the length of the new road. The idea here is to identify what the total “leftover” amount of dirt is that needs to be hauled away after creating this portion of the road. One way this context complements the fuel-flow context is that the fuel-flow context is with respect to time, t, whereas this context is with respect to linear distance, x. It is beneficial for students to experience different types of quantities, especially non-time quantities, in order to build a more robust quantitative meaning.



The same chop, quantity, and add structure from before can be applied here. One can partition the road length into short Δx segments, and then find out the amount of dirt to be removed or filled in over each Δx segment by multiplying

$W(x)\Delta x$ (where W is the linear weight). The total amount can be determined by adding up all of the $W(x)\Delta x$ results: $A = \sum W(x)\Delta x$. As one allows the road segments to become infinitesimally small, this structure defines a definite integral: $\int_a^b W(x)dx$. The graph shows that part of the summation will deal with negative W values, meaning that they contribute negatively to the overall amount of dirt-haul at the end. One can also imagine sweeping continuously from left to right to figure out the dirt-haul that exists up to any point so far along the road, developing antiderivative functions. One could also extend the diagram further to the right to imagine different upper bounds for the integral, $\int_a^X W(x)dx$, leading to explorations that build to the FTC. Thus, the fuel-flow and mass-haul contexts can work in parallel to build many of the ideas of integrals and allow students two different types of contexts with different quantities from which to abstract the definite integral structure.

One way I know that this lesson (and subsequent lessons) is reasonably successful in helping students develop this quantitative meaning for integrals is by asking them in homework or on tests to explain how the chop, quantity, and sum structure applies to various contexts. Here, I borrow from some contexts that will later be seen in second-semester calculus, or even multivariable calculus. The point is *not* for the students to actually work out integral calculations for these contexts, but simply to explain how the chop, quantity, and sum structure applies. For example, after this lesson I give my students a homework question that asks them to explain how the integral structure applies to finding the mass of an object, M, with variable density, ρ, $M = \int \rho dV$. Again, the point is simply to explain that the object is partitioned into small dV pieces, the density of each piece is multiplied to that dV volume to find a little bit of mass, ρdV, and the masses across all the pieces are added up to find the total mass, $\int \rho dV$. I similarly ask students in homework or on a test to apply the chop, quantity, sum structure to the volume of a solid object (chop=partition the object into thin slices, evaluate=find volume in each slice, sum=add volumes to get total value), or to arc length (chop=partition the curve into tiny segments, evaluate=find the length of straight-line approximations to each tiny segment, sum=add lengths to get total length), or to other rates (like growth rate of a child or the rate at which trees are cut down in a forest). The homework and test results from my own students show they do tend to develop productive “adding up pieces” understandings for integrals from these lessons and that they retain this quantitative structure, even as we encounter other ideas in the integral chapter.

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5

Geometric Series, Rational Numbers, and Mathematical Justifications with Prospective Secondary Mathematics Teachers

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Topic of lesson	This lesson aims to refine students' understanding of equivalent representations of rational numbers while they also compare and contrast features of different approaches for justifying mathematical claims about rational numbers.
Course Context	This lesson takes place in a Capstone mathematics course for prospective secondary mathematics teachers. The course was taught at a rural, public, liberal arts institution in the Midwest.
Instructional Challenge	How can mathematics content courses help prospective secondary mathematics teachers develop a profound understanding of mathematics to make explicit connections between collegiate mathematics content and secondary content they will teach in the future?
Brief overview of instructional approach	By first posing a hypothetical student question about whether or not $0.999\ldots = 1$, this lesson gives students an opportunity to respond to that notion. Students explore explicit reasons why they might think $0.999\ldots \neq 1$ and how this way of thinking leads to the necessity of thinking about limits of sequences of partial sums. Students then have opportunities to generate their own justifications for why $0.999\ldots = 1$. The instructor also provides guided proofs to facilitate opportunities for students to explore explicit connections between geometric series and rational numbers. After exploring multiple explanations for justifying this relationship, students can work through a few example problems where they apply the idea of uniqueness of decimal representation. Specifically, given a decimal representation, they must determine if the given decimal is unique or can be rewritten as an equivalent infinite or terminal decimal. Addressing these learning objectives typically takes approximately 50 minutes, depending how much time students spend generating their own justifications. Descriptions of the incorporation of some specific formative assessment strategies during the lesson are also included.
Keywords	Capstone Mathematics Course, Exit Ticket, Formative Assessment, Geometric Series, Proof and Proving, Prospective Secondary Mathematics Teachers, Quick Poll, Rational Numbers, Reasoning, Series, and Sequences

5.1 Background Information

5.1.1 Instructor and Classroom Background

I first describe my positionality as an instructor and then background information about the course around which the LAM is situated to help the reader understand the instructional challenge and content focus. When I teach prospective secondary mathematics teachers who are taking mathematics content courses, I value providing opportunities for students to uncover, examine, and refine their own preconceived notions about mathematics content that includes common misconceptions or partially formed ideas. Any time I address a partially formed idea or potential misconception, I have two components to my strategic teaching approach: (a) Introduce the topic in a way that potentially generates a point of disequilibrium for students and (b) provide opportunities for students to produce mathematical justifications related to the content. More specifically, I begin the class discussion by posing a hypothetical student question, justification, or solution and ask the class to respond. Hypothetical scenarios give everyone an opportunity to critique an idea while helping individual students realize if they might have some misconceptions we need to address. The class time typically revolves around addressing both sides of the issue: namely, unpacking why someone might agree as well as why someone might disagree with the hypothetical student. This unpacking discussion leads into the second component of my strategy because I ask students to generate, critique, and refine mathematical justifications to support their reasoning. When I facilitate discussions about why a hypothetical student is mathematically correct or not, the goal is for the undergraduates to develop more than one way to *explain why* so they (informally or formally) prove why something is mathematically valid. This *something* they are proving is a topic or idea that directly addresses the misconception or area that is only partially formed. I believe that by experiencing the process of proving to themselves, and one another, undergraduates develop a profound understanding of that mathematical idea.

In the context of this chapter, undergraduates are expected to explain, in more than one way, *why* $0.999\ldots = 1$, which can refine their understanding of infinite processes, series, limits, and rational numbers. The university, course, and student context for this LAM is as follows: Bowling Green State University (BGSU) is a large, rural, public, liberal arts institution in the Midwest. The lesson I am focusing on is from a Capstone course for prospective secondary mathematics teachers, which serves as the last required collegiate mathematics course in their undergraduate career. As a capstone course, prerequisites are that students must have earned a C or better in Elementary Linear Algebra and Fundamental Concepts of Modern Algebra. At BGSU, I have found that the prospective secondary math teachers are highly motivated learners who typically want to understand mathematics well enough to teach it to others. They often enter into the Capstone course expressing some complaints about having to take so many mathematics courses, but then quickly realize that they need to dive deeper into some of the content they have previously learned to understand why procedures they use make mathematical sense (to ultimately teach others in multiple ways). In recent semesters, this course has transitioned to a distance-learning environment to make it easier for practicing teachers to enroll in the course, but I describe instances from when I was teaching it in person. We met three times a week for 50 minutes for a 15-week semester. There is typically one section of the course each year, with a maximum of 29 students enrolled, but the average is around 24 students. The course is also cross-listed as a lower-level graduate course, which means that practicing teachers or mathematics education graduate students can enroll in the course and be given differentiated learning opportunities or assignments to further enhance their learning experiences in the course.

5.1.2 Instructional Challenge

When teaching prospective secondary mathematics teachers mathematics content, my two overarching goals are for students to (a) develop their ability to communicate mathematical ideas and justifications coherently while they (b) make explicit connections between high school content they will teach and collegiate content they have previously (or recently) studied. I try to find more ways to encourage students to explore rich, meaningful mathematical tasks, while providing opportunities for them to articulate connections across mathematical content. I believe that in so doing prospective secondary mathematics teachers can develop a profound understanding of collegiate-level mathematics and become better equipped to explain and teach secondary-level mathematics content. In this chapter, I focus on a content area from the secondary level (i.e., rational numbers) that intersects with some potential misconceptions or partially formed ideas undergraduates may have related to that content (i.e., limits of series). That is, I present this content area as an example of how I address the overarching instructional challenge of *how to help prospective* *secondary mathematics teachers make connections between collegiate-level mathematics content and secondary-grade mathematics content*. In the next section on the learning goals of this LAM, I explain in greater detail how this LAM addresses a common misconception students (in college or secondary settings) can have about whether it is true or false that $0.999\ldots = 1$. By focusing on this mathematical context, prospective secondary mathematics teachers can address many misconceptions they may have about infinity or rational numbers and generate multiple mathematical justifications for why that equality is true.

5.1.3 Learning Goals and Important Mathematical Ideas

To address this instructional challenge in this context, the three learning objectives I focus on are:

1. students will be able to define and explain the meaning of rational numbers,
2. students will be able to explain in more than one way why $0.999\ldots = 1$, and
3. students will be able to apply the idea of the uniqueness of decimal representation to problem situations.

There are two mathematical ideas that are prerequisite knowledge, and we will have already reviewed these in lessons prior this LAM. Those topics include, *Rational Numbers and Sequences and Series*. I first provide specific information about these two prerequisite topics to situate the content that is the focus of this lesson. And then provide examples of explanations and justifications students generate and encounter in the context of this lesson.

Prior Knowledge about Rational Numbers I base our definition of rational numbers on the textbook we use for the course by Sultan and Artzt [6]. Specifically, we write the following definitions together:

q is a *rational number* if and only if q can be written as the quotient of two integers $\frac{a}{b}$ such that $\gcd(a, b) = 1$ and $b \neq 0$.

z is an *irrational number* if and only if z cannot be written as the quotient of two integers $\frac{a}{b}$ such that $\gcd(a, b) = 1$ and $b \neq 0$.

When unpacking these definitions, students sometimes discuss how the definition for a rational number requires that we can find at least one quotient of two integers that satisfies the stated requirements, but the definition for irrational numbers is saying there does not exist any such quotient of any two integers equivalent to z. So, it may seem easier to students to rewrite a given number as the quotient of two integers than to prove that no such quotient exists for a given number. Focusing on that idea, students explore ways to convert between decimal and fractional representations of numbers. To do that, we often make some explicit connections back to the Division Algorithm.

To convert between decimal and fractional representations of numbers, undergrads explore both directions. First, students can explore what happens when they are given a decimal representation of a number and expected to convert it to a fractional representation. If the given number terminates, they can use place value to write the decimal as a fraction. For example, $2.345 = 2 + \frac{3}{10} + \frac{4}{100} + \frac{5}{1000} = \frac{2345}{1000}$. Otherwise, if the given number is rational and does not terminate, then it has a *repeatend* (a string of digits that repeats, or the "repeating part" as Sultan & Artzt calls it [6]), the students can algebraically manipulate the decimal to subtract off the repeating part and determine a fractional representation. Consider the following example with two similar solution methods included:

Repeating Decimal Example with Solution 1:

Let $N = 2.34555\ldots$, then $100N = 234.555\ldots$ and $1000N = 2345.555\ldots$.

Subtracting $1000N - 100N = 900N = 2111$.

Therefore, $N = \frac{2111}{900}$.

Repeating Decimal Example with Solution 2:

Let $N = 2.34555\ldots$, then $N = 2+\frac{3}{10}+\frac{4}{100}+\frac{5}{1000}+\frac{5}{10000}+\cdots$. Since we know how to add together the first three terms, we just need to figure out how to represent the repeatend as a fraction. So, let $M = 0.00555\ldots$, then $1000M - 100M = 900M = 5$. We have $N = \frac{213}{100} + \frac{5}{900} + \frac{2106}{900} + \frac{5}{900} = \frac{2111}{900}$.

```
           .53813
1875 |1009.00000   (f)
      -9375
        7150       (a)
      -5625
        15250      (b)
      -15000
          2500     (c)
         -1875
           6250    (d)
          -5625
            625    (e)
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(a) $10090 = 1875(5) + 715$
(b) $7150 = 1875(3) + 1525$
(c) $15250 = 1875(8) + 250$
(d) $2500 = 1875(1) + 625$
(e) $6250 = 1875(3) + 625$
(f) $1009 = 1875\left(\frac{5}{10} + \frac{3}{100} + \frac{8}{1000} + \frac{1}{10000} + \frac{3}{100000}\right) + \left(\frac{625}{100000}\right)$

Figure 5.1: Division process example with Division Algorithm unpacking aspects of the process.

Second, students can explore what happens when they are given a rational number written as the quotient of two integers and asked to convert it to a decimal representation. This task may seem trivial, as we are simply asking students to divide the numerator by the denominator. However, if they are asked to approach this problem using long division, it can provide an opportunity for them to examine the relationship between the divisor, dividend, quotient, and remainder. Namely, we consider the Division Algorithm: *If an integer N is divided by a positive integer b, then there is always some integer q and some remainder r where $0 \leq r < b$ such that $N = bq + r$. Furthermore, q and r are unique*. Students can apply this notion to understand what is going on mathematically at each step of a long division process. For instance, suppose we have $\frac{1009}{1875}$ and want to write this number as a decimal using long division, what would the work look like to complete this process? I use colors to show how the Division Algorithm could help students understand the mathematics that takes place at each step of the division process in Figure 5.1. By exploring this idea, students can get a better understanding of how they know when to stop dividing when converting a given fraction into its decimal representation. That is, in Figure 5.1, you will notice that the quotient has a 3 in the hundredths place and in the hundred-thousandths place (and then onto infinity). Students should be able to explain how they know the repeatend is only 3 and not the entire string of 3813. If they focus on the remainder and the pattern of remainders obtained in the division process, it could be clarified. Students or readers who are curious about exploring this question of "how do I know when to stop dividing?" might consider exploring seemingly tame rational numbers like 21/34, and converting them to their decimal representation using long division. See if you have clarity on what each step of the long division process is telling you and what digits from the quotient are included in the repeatend. Then, consider checking your final answer using some technological tool like wolframalpha.com

Prior Knowledge about Sequences and Series[1] Since a rational number can be written as a series it is important to note that students (and teachers) can sometimes experience confusion when working with series language and notation. For instance, the notation $\sum a_n$ means the limit of a sequence of partial sums and at least four important pieces are connected to this meaning.

1. The process of adding up the terms of the sequence $\{a_n\}$.
2. The creation of a new sequence, $\{s_n\}$, the sequence of partial sums.
3. The limit, S, of the sequence, $\{s_n\}$, if it exists.
4. The value of S, if S exists, as the value of the series.

[1] The examples and ideas in this Sequences and Series section are informed by annotated lecture notes from emeritus faculty member from Michigan State University (MSU), Dr. Richard O. Hill. He shared them with me in 2009 when I was teaching the Capstone mathematics course for prospective secondary mathematics teachers at MSU.

To add to the confusion, sometimes mathematicians use the series notation, or word *series*, as a short-hand to represent the sequence of partial sums, not just the limit of the sequence of partial sums. Being able to think of the sequence of partial sums is fundamental in understanding the meaning of a series, but students can focus on the sequence of partial sums without considering the limit of the sequence, which directly relates to the potential misconception underscoring why a student could think $0.999... \neq 1$.

Additionally, the relationship between a rational number and its corresponding series is seldom spelled out and often misconstrued. For example, if we ask what the first several terms of the infinite series that corresponds to $2\,\frac{34}{99} = 2.343434\ldots$ are, we might be met with a number of incorrect responses including (a) $2, 3, 4, 3, 4, 3, 4, \ldots$ or (b) $2, 3/10, 4/100, 3/1000, 4/10000, \ldots$. The next sub-section reviews background information for the reader clarifying prerequisite knowledge about sequences and series that the undergraduate students should be aware of prior to addressing the uniqueness of decimal representation question this LAM focuses on.

Review of Sequences. Intuitively, a *sequence* is a bunch of things where there is a first thing, a second thing, a third thing, and so on. More rigorously, a sequence is an ordered set of mathematical objects or a function whose domain is the set of natural numbers (or a set of the form $\{k \mid k \in \mathbb{Z},\, k \geq \text{some positive integer}\}$). For example, $\{2n\}_{n=1}^{\infty}$ is the sequence of positive even integers. To explore another sequence, let $a_n = \frac{1}{n}$. Then, we can consider what are the first several terms of $\{a_n\}_{n=1}^{\infty}$, does this sequence have a limit, and, if so, what is the value of that limit? The first several terms of this sequence are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \ldots$; the limit exists; it is zero. This example begs the question, how do we define the limit (if it exists) of a sequence of real numbers, $\lim\limits_{n\to\infty} a_n = L$? Before directly answering this question, it is helpful to note two important potential traps students may fall into when informally answering this type of question. First, it is not sufficient to say, "the sequence converges to 0 because it is getting closer and closer to 0." Consider $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ again and notice that this sequence is getting closer and closer to the number -2 (and to every negative number) as well as to the number 0. However, neither -2 (nor any other negative number) is the limit of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. Second, a sequence that has a limit need not be getting steadily closer to that limit, as in $\left\{1 + \frac{2+(-1)^n}{n}\right\}_{n=1}^{\infty}$, which has a limit of 1. Therefore, if we let $\{a_n\}_{n=1}^{\infty}$ be a sequence of rational numbers, we say that $\{a_n\}$ has the rational number L as a *limit*, or that $\{a_n\}$ *converges* to L, if, given any positive number ϵ, there is an integer N such that $|a_n - L| < \epsilon$ for all $n \geq N$. We can also think of the neighborhood of L as being an open interval about L, and that a sequence converges to L if it is eventually in every neighborhood of L. Or there are only a finite number of elements of the sequence outside any neighborhood of L.

Review of Series of Real Numbers. Building off of our understanding of sequences, I now review series of real numbers. Briefly, a *series* is a limit of a sequence of partial sums. More elaborately, if $\{a_k\}$ is a sequence of real numbers, its associated series is the limit of the sequence $\{s_n\}$ where $s_n = a_1 + a_2 + \cdots + a_n$ is the nth partial sum of $\{a_k\}$. Thus, a series $\{s_n\}$ is a limit of a special sequence. If a series $\{s_n\}$ has a *limit* S, we say that the infinite series $\sum\limits_{k=1}^{\infty} a_k$ *converges* and has sum S. For example, $\sum\limits_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots = e$. As prerequisite knowledge for this lesson, my students and I have also previously defined *Geometric Series* with the following information ([6, p. 278]):

- A **Geometric Series** is a series (infinite sum) of the form $\sum_{i=0}^{\infty} a(r^i) = a + ar + ar^2 + ar^3 + \cdots$, where a and r are non-zero real numbers.
- The sum of the first n terms of a geometric series is given by $s_n = \frac{a - ar^n}{1-r}$.
- A geometric series $\sum_{i=0}^{\infty} a(r^i)$ converges when $|r| < 1$ and the sum of this infinite series is $\lim\limits_{n\to\infty} s_n = \frac{a}{1-r}$.
- A geometric series diverges if $|r| \geq 1$.

To apply these ideas, we can consider one of the example sequences from the Review of Sequences subsection: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$. As one explores the series $\sum_{k=1}^{\infty} \frac{1}{k}$, one can see that the partial sums would include $s_1 = 1$, $s_2 = 1 + \frac{1}{2}$, $s_3 = 1 + \frac{1}{2} + \frac{1}{3}$, $s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$, $s_n = 1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n}$. And using the integral test from Calculus 2 or comparison test from earlier mathematical content, one can prove that this series (known as the harmonic series) diverges. This result can be surprising for students since the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ converges to zero, but the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Using place value, one can write rational numbers as series: 5678.912 could be written as a finite series, $5000 + 600 + 70 + 8 + \frac{9}{10} + \frac{1}{100} + \frac{1}{1000}$. And if we had an infinite decimal, we could write it as an infinite series, consider its sequence of partial sums, and the limit of that sequence of partial sums. For instance, $2\,\frac{34}{99} = 2 + \frac{3}{10} + \frac{4}{100} + \frac{3}{1000} + \frac{4}{1000} + \cdots$. Its sequence of partial sums is $2, 2.3, 2.34, 2.343, 2.3434, \ldots$. And the limit of that sequence is $2\,\frac{34}{99}$. In understanding an infinite decimal, students should be able to consider both the sequence of partial sums and the limit of that sequence, and understanding both is an important piece of navigating the question that is the focus of this LAM; specifically, "Is it true that $0.999\ldots = 1$?"

The Mathematics Content and Student Thinking Related to Justifying that $0.999\ldots = 1$ Exploring the question "is it really true that $0.999\ldots = 1$?" can provide an opportunity to investigate how students understand infinite processes, limits, series, and rational numbers. This is the specific content focus of the lesson that this LAM centers around.

Why Might Students Think $0.999\ldots \neq 1$? Specifically, the reason why students might think it is not true (i.e., why a student would say $0.999\ldots$ does not equal 1) is if they are focusing on the sequence of partial sums $0.9, 0.99, 0.999, \ldots$ and comparing that sequence of partial sums, or values in that sequence, to 1. When comparing the individual values obtained in the sequence of partial sums to 1, then the students reason that the answer is, "No," because each term in the sequnce is less than 1. In essence, if we plot this sequence of partial sums on a number line, we can see that there is always a little bit of *space* on the number line between any value in the sequence of partial sums with the trailing nines and the integer 1. However, to close this gap we can encourage students to go from this finite process (investigating specific partial sums) to an infinite process (that determines the limit of this sequence of partial sums), which can be challenging for students, but would directly answer the intended question.

Some Approaches for Justifying that $0.999\ldots = 1$ Focusing on the intended question, "is the limit of $\sum_{n=0}^{\infty} \frac{9}{10}(\frac{1}{10})^n$ equal to 1" we can consider a number of ways to prove why it is true that $0.999\ldots = 1$. Tables 1, 2, 3, & 4 display four example justifications that are modified from a homework problem from a supplemental textbook ([7, p. 33]) where these justifications are provided to students and they are asked to provide rationales to explain each step. More specifically, when using these justifications in class, I include the algebraic representations in a left column, and ask students to fill in the middle and right columns. Procedural explanations for each step in the justification are requested in a middle column. And the right column is where students should include a mathematical explanation for what content or ideas are behind each algebraic step in the argument.

Table 1: Argument 1. Using Geometric Series Convergence

Equation We Currently Have	**What Did You Do To Get This Equation from the Previous Equation?**	**Why Could You Do This?**
$0.99999\ldots = \frac{9}{10} + \frac{9}{10}\left(\frac{1}{10}\right) + \cdots + \frac{9}{10}\left(\frac{1}{10}\right)^n + \cdots$	Decimal expansion of $0.999\ldots$	Any decimal can be written as an infinite series
$0.99999\ldots = \frac{9}{10}\Big(1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 + \cdots + \left(\frac{1}{10}\right)^{n-1} + \cdots\Big)$	Factored $\frac{9}{10}$ out from each term on the right-hand side	Distributive Property
$0.99999\ldots = \frac{9}{10}\left(\frac{1}{1-\frac{1}{10}}\right)$	Applying the theorem for convergence of a geometric series with $a = 1$ and $r = \frac{1}{10}$	Since $\left\lvert\frac{1}{10}\right\rvert < 1$, we can apply the theorem for convergence of geometric series. So the series converges to $\frac{a}{1-r}$
$0.99999\ldots = 1$, as desired.	Simplify the right-hand side	Combining like terms

In my course we do not prove everything using this 3-column approach, but I use it here to provide an opportunity for undergraduates to unpack why each step in the justifications make sense procedurally and conceptually. Having that focus on *why* seemed to help convince them that this statement was true. These four examples (Tables 1, 2, 3, & 4) are not exhaustive, as students could provide other approaches, but they are typically similar in style to these four.

In Table 1, I explicitly use the notions of limits of series, so I make sure students spend some class time discussing the rationales for this argument if no one comes up this approach on their own or in small groups. Table 2 is included because it builds on the approaches we previously explored for ways to convert from a decimal representation to a fractional representation of a rational number. And Tables 3 and 4 utilize fractions and decimals the students may be fairly comfortable with (like $\frac{1}{3}$ and such) while efficiently justifying the desired mathematical statement. Students' first

Table 2: Argument 2. Converting a Decimal Representation to a Fractional Representation

Equation We Currently Have	**What Did You Do To Get This Equation from the Previous Equation?**	**Why Could You Do This?**
$n = 0.999\ldots$	Assumed	Because we want to use algebra to show $0.999\ldots = n = 1$. So we're starting with LHS of that equation to reach the RHS.
$10n = 9.999\ldots$	Multiplied both sides of the equation by 10	Preserving equation equality: multiplying by a nonzero quantity on both sides of the equation
$9n = 9.000\ldots$, as desired.	Subtract $10n - n$	Combining like terms on the LHS and RHS
$9n = 9$	Rewrote $9.000\ldots$ as 9	A number with trailing zeros in the decimal is equivalent to the number without the trailing zeros, because the zeros go on forever and we're tacking on $\sum_{k=1}^{\infty} \frac{0}{10^k}$
$n = 1$, as desired.	Divide both sides of the equation by 9	Preserving Equation Equality: Dividing both sides of the equation by the same non-zero quantity

Table 3: Argument 3. Using Different Representations of $\frac{1}{3}$ and Algebraic Properties

Equation We Currently Have	**What Did You Do To Get This Equation from the Previous Equation?**	**Why Could You Do This?**
$\frac{1}{3} = 0.3333\ldots$,	Given/we know this information	We can use long division to show this fraction and decimal are equivalent, or note that $\frac{1}{3} = \frac{3}{9} = 0.333\ldots$ because in base 10 a fraction with only nines in the denominator is simple periodic.
$1 = 3 \times \frac{1}{3}$	Rewrite 1 as $3 \times \frac{1}{3}$	Multiplicative Identity
$1 = 3 \times (0.3333\ldots)$	Substitute in $\frac{1}{3} = 0.333\ldots$	Property of Substitution
$1 = 0.9999\ldots$, as desired.	Multiply $3 \times 0.333\ldots$ on RHS	Combining like terms

inclination tends to be to produce arguments similar to Tables 3 and 4, and sometimes 2, but the Geometric-Series-based justification is included first in this chapter to make sure to build on these related ideas we have discussed of series and limits.

Mathematical Content Summary By providing an opportunity to explore first why some people may argue that $0.999\ldots \neq 1$, anyone who held that misconception can feel heard and they can begin to understand what mathematical ideas contributed to their misconception. I provided four ways one could prove why $0.999\ldots = 1$ because different proofs may be more explanatory or convincing to different students. And students are encouraged to come up with other justifications that I do not include here. By providing opportunities for students to develop their own proofs for why $0.999\ldots = 1$ they can also explore different ways to represent numbers that are equivalent.

5.2 Lesson Implementation

This lesson implementation occurs during one 50-minute lesson. The structure of the lesson follows the approach outlined in the Background section and now detailed further in the Lesson Flow section below. Specifically, first, I propose *Thea's Question:* "Is it really true that $0.999\ldots = 1$?" I ask students to think about the question for no more than one minute then I conduct a quick poll to count how many would tell Thea it is true and how many would argue

Table 4: Argument 4. Using Different Representations of $\frac{4}{9}$ and $\frac{5}{9}$ and Algebraic Properties

Equation We Currently Have	**What Did You Do To Get This Equation from the Previous Equation?**	**Why Could You Do This?**
$\frac{4}{9} = 0.444\ldots$ and $\frac{5}{9} = 0.555\ldots$	Given/known information	We can use long division to show this fraction and decimal are equivalent
$1 = \frac{4}{9} + \frac{5}{9}$	$1 = \frac{9}{9}$ and we wrote $\frac{9}{9}$ as $\frac{4}{5} + \frac{5}{9}$	$4 + 5 = 9$ and adding fractions with like denominations we know $\frac{4}{9} + \frac{5}{9} + \frac{9}{9}$
$1 = 0.444\ldots + 0.555\ldots$	Substitution: $\frac{4}{9} = 0.444\ldots$ and $\frac{5}{9} = 0.555\ldots$	Property of Substitution
$1 = 0.9999\ldots$, as desired.	Add the RHS together	Combining like terms

for false. Specifically, I ask everyone in the class to respond at the same time by either putting a thumb up (for Yes), thumb down (for No), and (if they really need this option) thumb sideways for (I don't know). By having everyone vote at the same time, I typically have at least 1/3 of the class saying "No" and I think it is a fairly honest response since they do not see what their classmates are voting before they vote (I rarely have a plethora of "I don't know" responses, but I use quick polls a lot in my teaching, so students are often willing to choose a specific answer after the first couple quick polls). Having a good number of students select "No," leads to some engaging discussions of mathematical ideas related to Thea's question.

As explained in Part III, Table 6 below, after the quick poll I give anyone an opportunity to explain why someone might argue that the answer to Thea's question is "No." In this way, anyone (even someone who thinks the answer is "Yes") can provide a rationale for why someone might say "No." After unpacking the mathematical reasoning related to the "No" answer together, I then give everyone an opportunity to explain why someone might say the answer is "Yes." The structure of the exploration of this "Yes" answer can provide individual work time, small group work-time without guided proofs, and/or class time focused on the guided proofs. Depending how much individual and small-group work time the instructor wants to allow, Parts IV and V can vary in terms of their structure. As detailed below, more or less time can be spent on students generating their own proofs or exploring connections between justifications that students produced and those included on the guided notes. The goal is for students to experience explaining why the answer to Thea's question is true mathematically in more than one way. As an optional Part VI for this material, applications of this idea to the notion of uniqueness of decimal representation can be explored (for instance, we can explore questions such as: *Can we write 14.26 as a non-terminal decimal using this idea?* Answer: Yes, $14.2599999\ldots$). More specifically, students can explore examples of numbers that do and do not have unique decimal representations. Ultimately, they should realize that any terminal decimal can be re-written as an equivalent decimal with trailing nines (and vice versa), and if an infinite decimal repeats, but it is not with just trailing nines, then that decimal has a unique representation.

Lesson Flow, Including Student Reasoning and Thinking In this section, I include specific questions, responses, and explanations in relation to the progression and flow of the lesson as I have experienced it. Feel free to make modifications as you and your students may need. I provide example dialogue and implementation commentary by breaking the lesson up into seven major parts (Tables 5–10): (1) Introduction of Learning Goals and Prior Knowledge Review, (2) Setting up the Focus Problem, (3) Discussion of Common Misconceptions, (4) Proving why $0.999\ldots = 1$, (5) Additional Guided Notes for Proving Why $0.999\ldots = 1$, (6) Optional Application Problems, and (7) Closure. This example dialogue and implementation or modification commentary intends to provide a means for the reader to envision how this lesson can play out while also making explicit my rationale for methods used during the lesson so others may choose to adapt portions of this LAM, if not the entire approach.

Table 5: Introducing and Setting up the Problem

Key Teacher and Student Dialogue and Actions	**Points to Consider During Implementation**
Part I. Learning Goals and Review, 5 minutes *Modified Learning Objectives Written on the Board and Read Aloud by Teacher* (*modified* to not reveal the answer to Thea's Question too early): Students will be able to 1. Define and explain rational numbers and irrational numbers 2. Explain in more than one way why Thea's question is true or false 3. Apply the idea of uniqueness of decimal representation to problem situations *Teacher*: In relation to our first learning goal today, talk with a nearby classmate or two about how you would define an irrational number and a rational number. *Students*: Students talk to one or two people near them, and teacher has a few people write definitions on the board. *Class Discussion*: Focused on the definitions on the board students and teacher refine them for clarity and precision of mathematical language. *Teacher*: Now that we are all working from clear definitions of rational and irrational numbers, recall that we spent time last class and, in the homework, exploring converting between fractional and decimal representations of rational numbers. Specifically, about decimal representations of numbers, let's assume that a high school student of yours has a question.	Students have typically already been exploring the definitions of rational and irrational numbers and applying them prior to this review. I include it as the first objective because it comes directly from the Common Core State Standards [5] and helps the prospective teachers articulate how the ideas from class relate to high school content. So, this section could be shortened and the definitions could be already on the board at the start of class so a really short discussion can take place to ground the lesson. Ideally, this is a brief discussion where definitions similar to these are written/refined: q is a *rational number* if and only if q can be written as the quotient of two integers $\frac{a}{b}$ such that $\gcd(a, b) = 1$ and $b \neq 0$. z is an *irrational number* if and only if z cannot be written as the quotient of two integers $\frac{a}{b}$ such that $\gcd(a, b) = 1$ and $b \neq 0$. It can be helpful to clarify with students that the fact that b is not equal to zero is what makes it mathematically possible to divide a by b when converting from a fractional representation of a number to its decimal representation. Since this introduction section is meant to be a brief review, when implementing think-pair-share an instructor could also choose to save a little time by inviting students to share their definitions verbally rather than writing them on the board, if timing is a concern.
Part II. Posing Thea's Question and Quick Poll, 3 minutes *Teacher*: (reads aloud hypothetical question projected on board or written on board) Suppose a hypothetical student, Thea, asks you the following question, "Is it really true that $0.999\ldots = 1.000\ldots$?" Think to yourself for 45 seconds if you would say "yes" or "no" to Thea. *Teacher*: Times up. We are all going to vote at the same time. When you vote if you would say "yes" to Thea's question you will hold your thumb up, if you would say "no" to Thea's question you will hold your thumb down, and if you are not sure you may hold your thumb sideways. So, everyone first put your fist in the air in front of you, and on the count of three you will vote. Here we go. . . *Teacher*: (counts up how many thumbs of each type and writes the totals on the board) All thumbs can go down. We notice there is some disagreement in our class based on these results. For those of you who voted "I don't know" we'll discuss the "no" and "yes" arguments and then check back in at the end of class to see if you have been convinced one way or the other. First let's discuss together, no matter how you voted, let's come up with an argument for why someone might think the answer to Thea's question is "no".	When using this voting method, typically at least 1/3 of my class will vote "no" and some combination will say "I don't know" and "yes". Using a *quick poll* means getting everyone's vote at the same time. This poll can be done with or without technology. The basic way I propose is with a thumb up, down, and sideways. But having a note card with different colored or numbered edges could be a voting tool. If the class is meeting virtually, using the polling feature of the video conference software or having everyone enter their vote at the same time in the chat can also be effective ways of gathering students votes simultaneously. It is vital to ask the question in a way that everyone can respond with their vote at the same time, so they are not influenced by others.

Table 6: Making Potential Misconceptions Explicit

Key Teacher and Student Dialogue and Actions	**Points to Consider During Implementation**
Part III. Why Someone Might Think $0.999\ldots \neq 1$, approximately 5–7 minutes	
Student1: It seems like the zero point nine, nine, nine, repeating number looks so different from the simple 1 on the right-hand side. That maybe if I "rounded" the left-hand side number up I would get 1, but that means they were not equal to begin with. So, that is why I said, "no." *Teacher*: Who wants to build on what Student1 shared? *Student2*: Yeah, it's like how we talked about there being infinitely many numbers on the real number line between zero and one. So, this decimal on the left-hand side of Thea's question is some number super close to 1, but there is always a little bit of space between them, hence why you would have to round if you wanted to say they were approximately equal to one another but not like equal-equal. *Teacher*: So, what I hear you two saying is that there is some space between the decimal number $0.999\ldots$ and the decimal number $1.000\ldots$, is that what you are saying? *Students*: Yes/Yup. *Teacher*: Let's continue this idea by trying to draw it on a number line together right now. Teacher draws the partial sums zooming in on number lines by hand or using technology for this representation if prepared ahead of time. (see right hand column of this Table 6 for more information) *Teacher*: So, based on these number line representations, if we interpret Thea's question to be about the sequence of partial sums as we have drawn here, then there is always a finite number of nines from the decimal on the left-hand side, which leaves some positive distance, or the "space" that Students 2 and 1 mentioned. But that would mean that there has to be some "last" nine, like the infinite decimal that is supposed to repeat forever stops repeating at some point. So, if we interpret Thea's question to be about the infinite decimal and not a finite sequence of partial sums, then this space cannot be there because we have to take this sequence of finite sums to infinity. Who remembers how we do that? Talk to people near you to see if your classmates know what that is called. *Students*: We have to take the *limit*.	I often hear students making comments like what is represented in this example dialogue. Specifically, students seem to focus on the notion that we are "rounding" one side of the proposed equality. Rather than directly saying "no we are not rounding" right off the bat, since we are exploring why someone might think the answer to Thea's question is "no," I intentionally leave the "rounding" issue alone for a bit. Once they notice the difference between the partial sums being finite and the number Thea asked about being infinite that usually starts to address the "rounding" confusion. But being sure to revisit that confusion during the closure is a good idea to see if it persists. Typically, the idea that $0.999\ldots$ has infinitely many nines is a big stumbling block. Students may think there is some last nine as if the decimal stops somewhere. This perspective would make them think that there is some space on the number line between $.999\ldots$ and 1. To help them visualize what question they are trying to answer, I draw a number line and mark the partial sums on it. Specifically, 0.9 and 1.0. Then, I zoom in on that portion of the number line, draw a new number line with 0.9 on the left and 1.0 on the right so I can mark where 0.99 is and I note that $1.00 = 1.0$. I zoom in again till I get to at least 0.9999. Thinking this way is closely connected to the process inherent in the sequence of partial sums, there is always some positive distance on the number line between the nth partial sum and 1. This number line visualization could help students who said $0.999\ldots = 1$ understand why some classmates voted for $0.999\ldots \neq 1$. Then, as a class, we can discuss how to consider $0.999\ldots$ as a decimal with an infinite string of nines. In the last comment from the teacher in the first column of this Table, I am emphasizing the connection, and difference, between a sequence of partial sums (as discussed in Section 5.1.3) and the meaning of an infinite series as the *limit* of partial sums. Focusing on a sequence of partial sums contributes to reasoning with a process that always results in a number falling short of equaling 1. Considering, instead, the limit of a sequence of partial sums can help students generate and refine arguments about why this specific series has a limit that equals 1.

Table 7: Proving the Equality

Key Teacher and Student Dialogue and Actions	Points to Consider During Implementation
Part IV. Proving Why $0.999\ldots = 1$, 15–20 minutes *Teacher*: Okay! So, we realized we need to take the *limit* of this sequence of partial sums as n goes to infinity to explore reasons why the answer to Thea's question is "yes". So, feel free to explore this question algebraically. See if you can come up with a way to actually prove that the left-hand side equals the right-hand side. I want you to spend a few minutes thinking about it for yourself before you discuss it with others. *Students*: (focused on their own approaches) *Teacher*: (after noticing that some students have made some progress or others are hitting stumbling blocks and need more input, typically 4 minutes, invite students to work together if desired) Please get together in groups and see if you can come up with at least one way to prove this equation is true algebraically. *Students*: (work in groups of 3–4 people) *Teacher*: (asks specific small groups to start writing their proofs on the [physical or virtual] board when noticing they are using different approaches than other groups) Let me have your attention, please. Now that a few groups have written some proofs up on the board, let's all take an opportunity to write these examples down while someone from the authoring group explains why each step in their proof makes mathematical sense. As always, while the proof is being explained, the goal is to see if you understand the entire proof or if something needs to be clarified. So the presenter can tag team with their group to help answer questions and the audience members should ask respectful but critical questions to make sure the proof is clear to everyone. *Students*: (groups self-select their presenter, who stands at the board near the written proof and explains each step. If questions arise the student groups work to clarify) *Teacher*: (only chimes in if something is glaringly mathematically incorrect and not ultimately fixed by the class community.) Now that we all have these proofs written in our notes, let's jot down a few comparisons and contrasts in the methods we have explored so far. (Class discusses specific approaches that are similar and different from one another)	If you want a less direct prompt than "how would you prove it," I have also used a more general, "how would you explain in your own words why $0.999\ldots = 1$." Depending how open to reasoning and proving your class of students typically are, phrasing the question in the latter way might help them just start getting their thoughts on paper, and worry about organizing them into a justification (more or less formal) later. In the example dialogue, the focus is on algebraic proofs since we have already discussed strategies for converting decimal representations to fractional representations algebraically and the sequences and series connection is also often explored algebraically. I did not always use the word "algebraic" in my prompts, so it can be left out. I have also never seen students prove this equation is true using any picture or geometric proofs, so I started focusing them on algebraic approaches in hopes that some of them might generate arguments similar to Tables 1 or 2 on their own. Usually the student-generated proofs are more of the Tables 3 or 4 style, which is perfectly fine. I typically keep this portion of the lesson focused on having the groups come up with one way to prove the equation is true so there is some time for them to explore the guided notes, but if the class really gets on a roll here, generating and sharing a number of types of proofs, the guided notes (described in Table 8) aspect of the lesson can be eliminated or focused on just one or two more ways to prove it rather than all four. If an instructor wants students to focus on particular styles of proving this equation is true, thereby transitioning to the guided notes section faster, the instructor can skip the group work in this portion of the lesson. After students have worked individually for a couple minutes, there are usually a few who have generated the start of a proof like Tables 3 or 4. So if they even share their idea verbally others in the class can help finish that proof and we can have at least one (or more) proofs of that style shared. Then, they can get into groups and work through the guided notes. This approach might keep this portion of the lesson to approximately 10 minutes.

Table 8: Guided Notes for Proving Equality

Key Teacher and Student Dialogue and Actions	Points to Consider During Implementation
Part V. Guided Notes for Explaining Why $0.999\ldots = 1$, 10–15 minutes	
Teacher: What I am handing out to your small groups now is a collection of four proofs for why the answer to Thea's question is "yes." You will notice some are similar (or the same) as what we already discussed. So let's just focus on the ones that are different. Work with your group members to fill in what is happening in each step of the provided proof and why that step is mathematically correct. Let's have the groups near the windows focus on Argument 1 first and the groups near the door focus on Argument 2 first. If your group finishes before we discuss, then start working on the opposite argument. *Students and Teacher*: (discuss together what the reasoning is for each step in the proofs that need to be unpacked)	I bring a handout for each student with only the left-column of Tables 1–4 filled in as guided notes, but depending how the individual and small group portion of class went, this guided-notes section can be adjusted on the fly. Typically, we only focus on Tables 1 and 2, and just briefly mention how Tables 3 and 4 are similar to what we already wrote down in our notes from the board work. If the individual and small group portion is shortened or only focused on individual work time without presentations, an instructor might choose to group students up with the four guided-notes arguments and have them work longer on this guided-notes portion and shorter on Table 7. The strategy in having different groups start by focusing on different arguments is for time management. Then, as a class, the intended arguments have been completed by at least some of the groups, even though not everyone in class might have enough time to complete all the proofs in their groups. During the whole-class discussion (after students have explored the guided notes proofs themselves), I will sometimes use a projector or SMART board to display Table 1 with all the columns filled in to reveal reasoning I prepared ahead of time for students to compare and contrast their wording with. We can then discuss what questions they have about my proof explanations and theirs. As explained in Section 5.1.3, we don't always use two- or three-column proofs in this course. But I decided to provide the algebraic argument to help evaluate two things, (1) can the students tell me what happened mathematically to proceed from one line of the proof to the next (procedural) and (2) can they also explain mathematically why that move is accurate or how they know that is true (conceptual)? This third column is important especially for prospective teachers where they need to make connections between high school content and collegiate content more explicit. Although they might not typically explain the ideas in the third column of the proof directly to secondary students in the future, they need that knowledge to guide students' learning and not reinforce misconceptions.

Table 9: Other Applications

<table>
<tr><th>Key Teacher and Student Dialogue and Actions</th><th>Points to Consider During Implementation</th></tr>
<tr><td>

VI. Application Problems about Uniqueness of Decimal Representation, *optional*, if time

Teacher: So, we just proved in more than one way that the answer to Thea's question is, "yes." And we have some time to apply this idea to a few application problems. Let's consider three decimal representations of rational numbers, and try to figure out if there is an equivalent decimal to each given number, like how $0.999\ldots = 1$ or if they are unique. If you think the given decimal is not unique, try to use the idea that $0.999\ldots = 1$ to rewrite the given decimal and convince us that you are correct.

Students: (Work individually or in small groups on the problems posed, as detailed in Table 9.)

</td><td>

Optional if there is time: It is possible for there to be time in a 50-minute class period to address these ideas if the initial review section (Part I, Table 5) is shortened and the total proving time (Tables 7 and 8) is efficient.

When implementing this task, I ask students the question as depicted in the example dialogue (this Table 9) using decimal representations such as the following three:

$14.56 = 14.55999\ldots$ $7.3333\ldots$ (unique decimal representation) $0.678999\ldots = 0.679$

If there is a lot of time left in class, one could include an irrational number and change the instructions so they are about real numbers. Since the focus in responding to Thea's question was rational numbers, I tend to keep the focus there.

If there is only enough time to discuss one of these examples, I rephrase the question to be more specific. I would focus on converting a terminal decimal to a repeating decimal representation since most of the proofs started with the repeating decimal and converted it to the terminal in the earlier work in this LAM. I would ask, "Can we write 14.56 as a non-terminal decimal using today's ideas?"

</td></tr>
</table>

Table 10: Closure for the Lesson

Key Teacher and Student Dialogue and Actions	**Points to Consider During Implementation**
Part VII. Closure, 5 minutes *Teacher*: We had a thorough investigation into ways to answer Thea's question. So let's revisit the poll we took when I first showed you Thea's question: Suppose a hypothetical student, Thea, asks you the following question, "Is it really true that $0.999\ldots = 1$?" We are all going to vote again, put your fist out and say thumb up for "yes", thumb down for "no", and thumb to the side for "I don't know". Ready, one, two, three, vote. (record students' votes on the board, if desired) *Teacher*: Okay—we made some progress unpacking the mathematical connections related to Thea's question today! Let's use the final minutes of class time to have you put in your own words what you learned today on the exit question for today. *Students*: (Write their anonymous response to the exit ticket question on the board and hand it in before leaving.)	I like to leave at least 5 minutes to revisit the quick poll and have students answer an exit ticket, but just doing one of those things is likely sufficient. Especially if a lot of students voted for "No" or "I don't know" in the beginning of the class period, it is helpful to give them a chance to express their perspective in some way through a formative assessment at the end. To describe exit tickets in more detail, I typically write a couple specific questions on the board for students to answer anonymously. These responses give me general feedback about how well the students understand major concepts from the class. In the context of this class period I might ask one of the following questions depending on the nature of class discussions and student work I observed, and below each question I include my reasoning for why this question could be appropriate to gauge student understanding at the end of this lesson. Although I list multiple questions here, I typically select only one, or at most two, of these questions for an exit ticket to keep the assessment focused and not overwhelm the students: • In 2–3 sentences, explain in your own words why $0.999\ldots = 1$? ∘ Especially if everyone in class voted "yes" to the closure quick poll, it is sometimes helpful to read how they would explain why the answer is yes in an informal way to see if there is any lingering confusion. • What was a major mathematical connection that makes more sense to you after today's class discussions? Please elaborate. ∘ This question is fairly broad because students could focus on different connections. Some possible things students might focus on include connections across different proof approaches, connections explored within a particular proof, or cycling back to the notions of sequences that could underlie some of the reasons why students voted "no" to Thea's question • Based on the ideas we discussed today, do you think every rational number can be written as a repeating decimal? Briefly explain your thinking. ∘ Typically, I only consider this question if students spent some class time exploring any examples in optional Part VI Table 9. Then, this question could provide an opportunity for them to articulate how those examples illuminate this relationship between rational numbers and their decimal representations. Other questions more specifically about uniqueness of representation could be used here, instead.

5.3 Post-Lesson Considerations

Overall, I found when implementing this lesson that students made positive progress toward accomplishing the first two learning objectives (about refining definitions of rational numbers and explaining why $0.999\ldots = 1$). The evidence I use for this evaluation includes the way students apply their understanding of rational numbers throughout the lesson discussions and how they successfully create and refine multiple arguments that prove the conjecture is true. I often structure the time to focus on the creation and refinement of mathematical justifications and save the discussions and content related to the third learning objective (i.e., Part VI Table 9) for the next class period. But if less time is needed for the creation and refinement of justifications or the class period is longer than a 50 minute block, then the material and discussions from the first five parts of the lesson set the stage nicely to continue the discussion into application problems in a longer class period or in subsequent lessons and homework assignments.

By using this instructional approach (perturbing students' misconceptions with a hypothetical student's idea that incorporates opportunities for students to explain connections between different mathematical content areas) I have gathered evidence that students are making progress on my overarching instructional challenge. I often hear my students saying (or read it on exit ticket submissions or informal course evaluations) that even if they do not plan to show future high school students a proof for why a mathematical procedure or idea is true, the fact that they understand the underlying mathematical reasons gives them more confidence in their abilities to communicate high school content accurately. Based on these outcomes, I use this approach and address this instructional challenge in other content areas in this Capstone course. Furthermore, I use this approach (and others could, as well) in other mathematics courses where prospective secondary mathematics teachers are part of the population of students. For instance, Abstract Algebra or Capstone courses could provide opportunities for students to explore questions such as (a) what kinds of number systems do we need to be working in to solve linear equations? And (b) if we want to solve polynomial equations of higher degree, they may not always have solutions, but if they do, then how do we find them? Answering those questions can give rise to two algebraic structures: fields and polynomial rings. Moreover, solving equations in Z_n can motivate a proof of when Z_n is an integral domain. For more information and explicit materials related to addressing this instructional challenge, the reader is referred to ongoing research projects exploring this topic (e.g., NSF funded MODULE(S^2) project [4]; & MAA's META Math project [3]). The teams of researchers on these two projects have developed resources other instructors could use to address this instructional challenge within a variety of collegiate mathematics courses even where the population may include, but not be exclusively prospective secondary teachers, including Abstract Algebra, Calculus, Discrete Mathematics, Geometry, and Statistics.

In general, integrated into this LAM is the intentional use of formative assessments and incorporating hypothetical student questions to address potential misconceptions among undergraduate students. If a reader is interested in learning more about implementing formative assessments or active-learning strategies in collegiate mathematics classrooms, they can refer to resources such as *Classroom Assessment Techniques* in [1]. And readers interested in exploring resources from which hypothetical student questions can be gleaned may consider exploring collections of such questions as compiled by Richard Crouse and colleagues [2].

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6

Introducing Qualitative and Graphical Analyses of Systems of Differential Equations

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Topic of lesson	The three lessons in this chapter focus on solutions to a system of two autonomous differential equations. Topics include a 3D representation of a solution, projections of this 3D curve into two dimensions, phase planes, vector fields, nullclines, and equilibrium analyses.
Course Context	The lessons take place in an undergraduate differential equations course taken by mathematics majors and other STEM majors, including future high school teachers.
Instructional Challenge	Most curricula do not motivate the need for the phase plane and associated vector field and simply "drop it in" on students. We therefore take on the challenge of motivating the value of and need for the phase plane for students and of enhancing students' ability to visualize the relationships between multiple, connected graphical representations (phase plane, time series graphs, 3D graph).
Brief overview of instructional approach	We take an inquiry-oriented approach, which embraces these pillars of inquiry-oriented mathematics education [5]: Students engage in an extended sequence of coherent and challenging tasks; Students collaboratively build ideas; Instructors explicitly attend to issues of equitable and inclusive practices.
Keywords	Solutions systems of differential equations, phase plane, inquiry-oriented instruction, active learning, equilibrium solution

6.1 Background Information

6.1.1 Instructional Needs Addressed

Most curricula do not motivate the need for the phase plane and associated vector field and simply "drop it in" on students. We therefore take on the challenge of motivating the value of and need for the phase plane for students. We also attempt to support students' ability to visualize the relationships between multiple, connected graphical representations (phase plane, time series graphs, 3D graph). Another instructional need that we seek to address is focused on promoting time-based reasoning and how related quantities unfold in relation to each other and in relation to time. Such reasoning is a critical mathematical perspective in a differential equations course that takes a dynamical systems point of view [3]. Finally, a complex and often underdeveloped skill related to such reasoning that we seek to develop through student reinvention (see next section) is the phase plane and associated vector field, as well as identifying patterns in the vector field (including nullclines) that can reveal how time-based quantities evolve. These instructional needs are at the heart of dynamical systems, but many first courses in differential equations never get to these beautiful graphical and qualitative approaches.

6.1.2 Learning Goals for this Unit

Early in the unit, we invite students to adapt Euler's method (which is something that they reinvented in a much earlier unit) and then we use their work as the entry point for introducing graphical approaches for qualitatively analyzing solutions to systems of two autonomous differential equations. The specific learning goals of this unit are to:

1. Empower students to be creators of mathematics and to feel the joy of discovery.
2. Visualize solutions for systems in three ways: as a 3D graph, as a pair of time series graphs, and as a parameterized curve in the phase plane.
3. Coordinate and make connections across all three ways to visualize solutions and develop an appreciation for the phase plane as a compact and informative way to represent solutions.
4. Discover and classify new types of equilibrium solutions.
5. Create vector fields in the phase plane to tell the story of how two interrelated quantities unfold and change over time.
6. Identify nullclines in vector fields and use nullclines to determine equilibrium solutions and to tell the story of how two interrelated quantities unfold over time.

6.1.3 Context and Curriculum

This set of lessons is part of a semester-long inquiry-oriented differential equations curriculum (https://iode.sdsu.edu/). The lessons follow several previous lessons which developed key concepts and skills for a single ordinary differential (rate of change) equation. However, these lessons introduce a new topic, understanding and finding solutions to systems of differential equations which can stand on their own and/or be adapted to other curricula. Nonetheless, there is essential prior knowledge for these lessons, including being able to carry out Euler's method, appreciating that solutions are functions, thinking of differential equations as rate of change equations that describe how quantities increase, decrease, or remain constant, and an appreciation of the multiple ways to represent solutions, including phase lines [6].

This is the first of four sets of lessons on systems of differential equations. This particular sequence typically requires 3–4, 75-minute class periods. Students will use a free, specially designed app for graphing solutions in multiple, connected representations (https://ggbm.at/U3U6MsyA; https://ggbm.at/DYaWxvaw). The use of similar, specially designed apps for learning occurs throughout the curriculum. We note that this set of lessons is available online at (https://iode.sdsu.edu/).

6.1.4 Instructors' Background

Both of us are undergraduate mathematics education researchers who regularly teach mathematics courses, both for math majors and those intending to be secondary school math teachers, as well as graduate courses in mathematics and science education. We are both authors of the Inquiry-oriented Differential Equations curriculum that contains the unit on systems of differential equations that we share and reflect on in this chapter [7].

As undergraduate mathematics teachers, we come to our classes with high expectations for students to work collaboratively on challenging problems, to share their reasoning with their classmates (and us), and to engage with their peers' mathematical reasoning. For us this means that we are not only keeping track of the mathematical progress of our students, but also paying attention to the social dynamics. To use an analogy from a system of differential equations that models cooperative species, we think of the growth of student ideas as positively benefiting from interaction with their peers. This requires us to consciously create classrooms where students feel safe to express their reasoning, however tentative, and constructively engage with and make sense of their peer's mathematical reasoning. We are increasingly aware of the fact that in classrooms that are highly interactive and contain a lot of student talk, it may be (and often is) the case that the White men in the class take more opportunities to explain their reasoning compared to women and students of color. This is troubling to us and while neither of us were trained in pedagogical techniques that can address such inequities, we both are actively retooling ourselves so that we can create more inclusive and equitable classroom spaces.

We also bring to our teaching a strong value for student thinking and reasoning and try our best to use this reasoning to further develop the mathematics. We strive to engage students in creating and sharing explanations of their work because we see that doing so is an opportunity for students to develop their ideas, for us to help students see connections to the work of others in the class, and for us to connect their intuitive and informal ideas to conventional and more formal mathematics. Achieving these goals is an instructional challenge, but in our work as curriculum designers and as mathematics teachers, we have been profoundly influenced by the instructional design theory of Realistic Mathematics Education (RME) [2] [4]. In the next section we reflect on the role of RME both in our broader view of curriculum design and in its role in the design of the unit highlighted in this chapter.

6.1.5 Rationale for Instructional Design

RME originated decades ago and has been a driving force in the creation of curricula at the primary and secondary school levels in the Netherlands and elsewhere across the globe (https://www.icrme.net/). Over the past several years we have joined many others who have been inspired by RME to develop undergraduate curricula for a wide range of courses, including differential equations, linear algebra, abstract algebra, combinatorics, and analysis. In broad terms, RME takes the perspective that students should learn mathematics by doing mathematics, as opposed to learning mathematics by reproducing demonstrated ideas and procedures. The ways in which students do mathematics often reflects how we as mathematicians go about doing mathematics. That is, learning occurs through the participation in authentic mathematical practices. To do this, students need an **extended sequence** of challenging and engaging problems that coalesce in important and significant mathematics. For this reason, we focus on not just one lesson, but a sequence of three lessons.

One of the primary heuristics of RME is referred to as *Guided Reinvention* [1]. The goal of the instructional designer is to identify contexts in which students can immediately and informally engage and that have the potential to lead to student reinvention of the intended mathematics. Student reinvention is "guided" both in terms of the curricular materials and in terms of the teacher's actions and interventions. We have chosen to show three consecutive lessons in this chapter as together they may provide insight into this ongoing process. Guided reinvention allows students to build their collaborative ideas together and describing it over three lessons may provide a window into the process for the students in our inquiry-oriented classrooms.

6.2 Lesson Implementation

We note that as far as time to do this goes, it will vary significantly depending on the class where it is taught. However, we provide approximate times in minutes for the three Lessons.

Lesson 1: 120–180 minutes

Lesson 2: 70–100 minutes

Lesson 3: 90–200 minutes

6.2.1 Lesson 1: Introducing and Visualizing Solutions to Systems of Differential Equations

Rationale: Instead of starting with systems that can be solved analytically using eigen theory, we decided to focus on providing students the opportunity to develop an understanding of the solution to a system of differential equations as a graphical mathematical object. The first tasks support and guide students to understand that the solution can be represented as a 3-dimensional curve that is dependent on time. This is done because we want to focus on supporting students' view of differential equations from a dynamic systems view. Also, we want to encourage "dynamic reasoning" [3].

In this lesson, we begin by introducing students to the idea of thinking about a solution to a system of two differential equations that are linked by common dependent variables. The concept of "solution" has been developed in the set of tasks connected to first order differential equations, so this task ties together the new conceptions with earlier work. Additionally, students reason about a solution in a real-world context and start discussing how a solution might be thought about in that context. We chose to use Rabbits and Foxes as it is a good example of a predator-prey population model often used in differential equations. Additionally, the Realistic Mathematics Education instructional theory encourages the use of contexts that are experientially real to the students, and students can think about predator-prey from a life science context and build ideas of mathematics on it. In the second problem, since Euler's method is used extensively in the tasks from the first half of the course, we ask students' to participate in an activity to introduce the notion of solution numerically. It is here where there are actually some calculations by hand to draw on students' mathematical thinking about what Euler's methods might mean in this situation.

This lesson consists of 3 tasks which look somewhat different but all tie together:

- Task 1 uses the predator-prey model and has the students consider that content. The students look at an actual system of differential equations and think of the content and the meaning of the terms.
- Task 2 asks the students to modify the Euler's method that was developed to solve single differential equations so that it works for a system of two connected DEs.
- Task 3 requires the students to investigate what three-dimensional solution curves might look like by "playing with" a dot on an airplane's propeller.

Task 1.

Most species live in interaction with other species. For example, perhaps one species preys on another species, like foxes and rabbits. Below is a **system of rate of change equations** intended to predict future populations of rabbits and foxes over time, where R is the population (in hundreds or thousands, for example) of rabbits at any time t and F is the population of foxes at any time t (in years).

$$\frac{dR}{dt} = 3R - 1.4RF$$

$$\frac{dF}{dt} = -F + 0.8RF$$

a) In earlier work with the rate of change equation we assumed that there was only one species, that the resources were unlimited, and that the species reproduced continuously. Which, if any, of these assumptions is modified and how is this modification reflected in the above system of differential equations?

b) Interpret the meaning of each term in the rate of change equations (e.g., how do you interpret or make sense of the $-1.4RF$ term and what are the implications of this term on the future predicted populations? Similarly, for $3R$, $-F$, and $0.8RF$).

Implementation Notes We begin this lesson by engaging the students. We begin a conversation about predator-prey models for population in biology and have students think about what they might expect from how the population size may ebb and flow in the predator- prey relationship. We also might have them do a short discussion and share out about what kinds of predator-prey they can come up with (this would be before we mention Rabbits and Foxes). This may engage them in the lesson, as well as allow them to bring their own contextual background to the activity. After the "engagement", the Task above is given to the student groups.

One of the primary ways that new differential equations are created is by modifying earlier assumptions and the DEs associated with these assumptions. In IODE, we try to bring up some of the ways that DEs arise. (We talk to the students about modifying and two other ways that we get DEs are from modeling data from the real world and from scientific laws). For the given DEs in the task, the only assumption that has changed from the single DE implementation is the introduction of a second species. Continuous reproduction is still assumed as are unlimited resources for the rabbits. This latter conclusion can be observed by students when considering what the DEs predict will happen when there are no foxes. Thus, the $3R$ in the dR/dt equation is expected to be interpreted as unlimited growth. Students should also be able to interpret the $-F$ term in dF/dt as exponential decay. In most classes, there will be at least one group who would come up with this and using "share out" as part of the class, it can be introduced in a whole class discussion.

We also encourage the students to talk about the $-1.4RF$ and $+0.8RF$ terms. Often students interpret them as "interaction terms" that make an expected impact on the respective rate of change. For example, students may justify why it makes sense that it is $-1.4RF$ instead of $+1.4RF$ in terms of the meaning of dR/dt, R, F. If a student has seen the predator-prey model before, they may be able to articulate this. Often, though, we scaffold the ideas by asking questions of the students. In subsequent problems, students will see this system and other similar systems and work numerically and graphically with the solutions; it is important for students to understand that the DEs be interpreted in terms of the context.

As faculty, we may not know exactly what to expect from this tasks so would be prepared to guide the conversation wherever the students' reasonings lead. One path of reasoning that we have seen is that students often use the words "change in population" when technically they mean "rate of change" (for some students this may actually be a conceptual problem rather than being careful with their language, and it is up to the instructor to help the students address this). We suggest that instructors try to help students be precise in their talk, thereby helping them become more explicitly aware that the rate of change in a quantity is not the same as the quantity itself or the change in the quantity.

Task 2.

(a) Scientists studying a rabbit-fox population estimate that the current number of rabbits is 1(scaled appropriately) and that the scaled number of foxes is 1. Use two steps of Euler's method with step size of 0.5 to get numerical estimates for the future number of rabbits and foxes as predicted by the differential equations.

t	R	F
0	1	1
0.5		
1.0		

(b) What are some different two-dimensional and three-dimensional ways to graphically depict your (t, R, F) values?

Implementation Notes This problem provides an opportunity to adapt Euler's method developed to find solutions to one differential equation to numerically solve systems of two DEs. Students do not seem to need much support on this question. We have seen in our classes that this might be a good chance to reinforce to the students how much they are learning; students may have struggled with reinventing Euler's method several units ago, and as a result of that struggle, they can generalize the method fairly easily to numerically approximate solutions to a pair of differential equations. That early struggle paid off!

There are many ways to explore the graphical representations of the (t, R, F) data. Students will not likely use a 3-dimensional display, so the teacher will have to encourage students to try this. One particular concept that our years of teaching has shown is that students may choose to think about the tables in terms of R and F, and miss the concept that these two functions are intimately linked and that each point cannot be found without using both of the functions for each value. As far as student thinking, it often happens that some students will consider the three-dimensional view but not have a way to think about representing it. This is okay as they will be spending more time on this in the next sections.

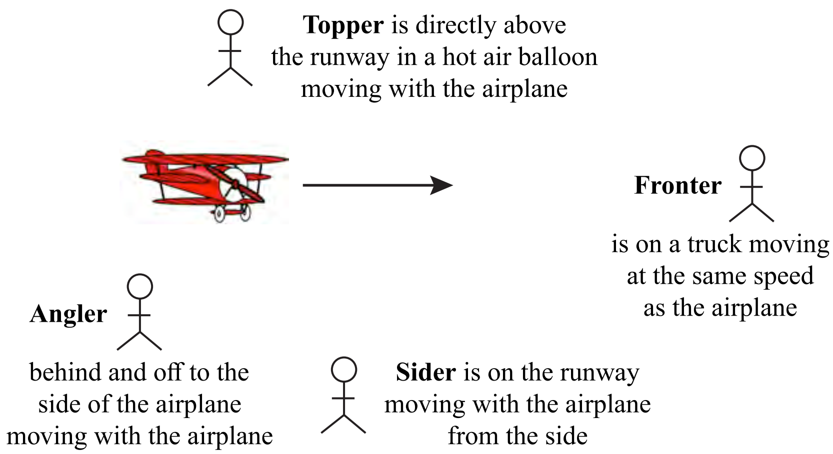
Rationale for Task 3 After the students reinvent Euler's method for finding approximate solutions to a system, the students continue the journey into understanding solutions by participating in an intuitive group activity that promotes deep conceptual thinking about what it means for solutions to exist as a three-dimensional curve, and how that curve may look from different perspectives. The students work in groups to investigate visually what an imaginary trace of a dot on a propeller will look like from different perspectives. This has been revised many times over the instantiation of the lesson, as the idea of a trace on a propeller has proven to be so powerful; primarily we have tweaked the details to make the task more user friendly. The four perspectives to the trace lead to four different graphical representations of one solution to a system. The four different representations connect loosely with (in terms of R and F being functions from the activity before), the $R(t)$, $F(t)$, the three-dimensional view, and the phase plane view of the system. Students will have this visual (and conceptual) notion of a solution that will then connect to the continuing mathematical work.

Task 3.

Three-Dimensional Visualization

A crop duster with a two blade propeller is rolling along a runway. On the end of one of the propeller blades, which are rotating clockwise at a slow constant speed, is a noticeable red paint mark. Imagine that for the first several rotations of the propeller blades the red mark leaves a "trace" in the air as the plane makes its way down the runway.

Simulate this scenario over time with a pipe cleaner. On appropriate combinations of the x, y, and t axes, sketch what Angler, Sider, Fronter, and Topper would ideally see assuming that they could always see the red mark. What view do you think is best and why?



Implementation Notes This task usually takes about 30 minutes. We tried not to push the "answers" as there are several deep ideas and the reinvention of this conception of a solution to a system can develop as students work in groups or share out to the class. Additionally, we find that students will return to the thinking they did in this lesson to develop new conceptual understandings in later lessons. We wrote this part of the lesson to provide an opportunity for the students to build intuition about the representation of a solution to a system of two autonomous differential equations. It is a little unusual to ask students to participate in physical tasks, but hands-on activities can contribute to students' reinvention of mathematical concepts and processes. The philosophy is that even if it is not formal mathematical development, it requires students to think about the ideas and use their informal thinking to come up with ideas that will be more formal later in the course. Experience tells us that this is one of the most powerful activities as the students almost always refer back to it as the course continues. As instructors, we believe that activities that allow students to discuss with each other something they are "seeing" is a powerful strategy.

This problem may be started by briefly revisiting the need to represent three bits of information, such as (t, R, F) from the previous problem (Euler's method in 3-D). The task is followed by several additional questions to encourage students to probe the ideas of a three-dimensional curve as the solution. Pipe cleaners are excellent physical representations of the trace the airplane leaves. We usually give each student a pipe cleaner and they use it to explore what is going on. They can rotate their pipe cleaner in space to see the different perspectives. If there is a document camera, we encourage the students to use the pipe cleaners to illustrate the trace, and the shadow allows for the different perspectives to come to life as well. We often supported the students' participation by mentioning that you can imagine a flare attached to the blade which leaves a red trail of smoke that you are observing as the "trace".

Here is one student's verbalization of his reasoning about this activity.

> Jerry: Angler sees the three-dimensional view of it where Fronter sees a two-dimensional projection of that three-dimensional view on that plane, Topper sees it from another plane, Sider sees it from a third and Angler sees the combinations of those three planes to a three-dimensional model.

Jerry's discussion is with another student, Adam, and is based on visualization. Together, they visualize the situation and then change perspectives to look at the imagined curve from different directions. We also find that it helps to suggest to students that in seeking the "ideal" perspective, we are assuming that we can see through the plane to see the mark at all times. Another place that perspective can be discussed is to think about how things look smaller in the distance. Perspective is going to come up, but it is helpful to students for the instructor to suggest that this be ignored to help understanding the situation and for future activities. Students pick up on the phase plane view (Fronter) quite quickly, after many first call it a spiral (because of the perspective issue). The other three views are not as clear for students to conceptualize, as they do not translate quite as well from the intuition of the task to what is hoped for (the 3 projections). In one of our classes, a student referred to the $R - F - t$ view (the view that is not seen from any perpendicular perspective) as the "mother curve." We find that nice as it helps see that that is the actual curve in 3-D and the others are representations that can be visualized as a "projection" on the $R - F$, $R - t$, or $F - t$ coordinate planes in a 3-D graphing space

Students usually make sense of the four views (Angler, Topper, Fronter, Sider). As instructors, we may need to guide them to make the connection to Tasks 1 and 2. Additionally, there will be a lot of technology involved in the next section, which will provide more detail for the students as they think about this; the connections among the three tasks in this lesson will then be more explicit in Lesson 2.

6.2.2 Lesson 2: Revisiting Rabbits and Foxes and Visualizing Solutions

Rationale The next lesson revisits and extends the ideas from Lesson 1. In Lesson 1, students participated in a physical task to engage their visualization skills to build a conceptualization of a solution. In Lesson 2, our students engage with technology to continue to make this vision more concrete and dynamic; we have developed a Geogebra applet that specifically provides the technology for the tasks. We also believe that the notion of equilibrium solution (which comes in Task 6) is a very important notion in the qualitative interpretation of solutions to systems, so we chose to introduce the idea early, soon after the students invest time in building some time-based reasoning and visualization skills.

Finally, we want our students to continue to think about the 3-D curve as the representation of the solution; however, we know that the phase plane is a very necessary representation for students to interpret solutions to systems as it is one of the primary tools in the mathematical community. Phase planes are one of the primary tools used by those solving differential equations using qualitative tools. The last problem in this lesson provides support for students to communicate their thinking about this and hopefully begin the process of reinventing a phase plane.

One of the primary trajectories for this lesson is to build a time-based model of a solution. We have found that one reason our materials support this is that the applet we use is not just "showing" the solution, but the students can actually watch the solution being constructed on the screen.

This lesson uses technology and the students' understanding from the airplane task to support students building the notion of the 3-D solutions:

- Task 4-5 uses the DE Explorer applet and asks the student to create and study the 3-D solutions to the system.
- Task 6 asks the students to think about concepts again from single DE solutions, like equilibrium solution and connect it and make sense of it in the 3-D setting.
- Task 7 introduces the notion of seeing the solutions from different perspectives and how important understand the phase plane perspective is. It also shows how two solution graphs together can represent the 3-D solution.

Tasks 4–5.

For the same system of differential equations from problem 1, use the DE Explorer https://ggbm.at/U3U6MsyA (select Runge Kutta with step size 0.01) to generate predictions for the future number of rabbits and foxes if at time 0 we initially have 1 rabbit and 3 foxes (scaled appropriately). Generate and reproduce below the perspectives of Topper, Fronter, Angler and Sider from the crop duster problem.

$$\frac{dR}{dt} = 3R - 1.4RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

Implementation Notes This first question is followed by several additional problems in Task 4 and 5 which allow students to investigate the perspectives with different initial conditions.

We continued to build on the notions from Lesson 1, but include more of the traditional concepts of equilibrium solution in this lesson. In Tasks 4–5, students investigate the solutions to a system using a physical representation, then a GeoGebra Applet, then more analytic techniques. Additionally, students use the ideas from the airplane problem and connect to the Rabbit/Fox system; they use the GeoGebra applet for the Rabbit and Fox system. As instructors, we did not often provide a lot of scaffolding on this section. The applet was enough instructional support for students to reinvent the more formal idea of a solution to a system. We encouraged the students to spend time just "playing" with the applet. There are many things that often are found by the students as they work on these tasks. For example, we had one student run the app many times to try to figure out by just trying numbers if there is a place where the "line" is straight. Spending time on this before beginning Problem 4b helps with the three-dimensional perspective of the solutions.

Student Thinking and Reasoning Returning to Adam and Jerry from Lesson 1, we provide a short vignette as they did this work to illustrate the trajectory of the students' thinking (taken from Keene's dissertation):

> When they began using the applet, Adam immediately connected the computer-generated graphs that the students were creating using the computer program to the previous day's airplane task.
>
> Adam: That looks something like a pipe cleaner: No look at this, it gets smaller in the back, so this thing does have perspective?

In this quote, Adam was referring specifically to the airplane task where they used pipe cleaners to represent the trace of the red line in the air. Here Adam makes connections; he connected the ideas from the airplane task and the applet task. At this point, Adam had a way to reason about the solution represented as a curve in three-space and its various projections using his understanding of time, and visualization skills. The use of the computer program here was particularly important in Adam's early conceptualizations. Dynamic computer visualization contributed to understanding of systems of differential equations. This visualization was not grounded in an understanding of systems of differential equations or rates of change, but in thinking about the picture that the computer created. The applet actually created the picture as you watched, and it created the curve slowly so that students could watch as time passed. By watching this, time was made explicit for the students, and time-based reasoning was supported.

Sometimes we ask students to draw solutions from the app on the board and compare and contrast them. Another teacher encouragement we have used is if students do not already experiment with the initial condition near an axis or on an axis, or even at or near equilibrium, we might guide them to do so.

Task 6.

(a) What would it mean for the rabbit-fox system to be in equilibrium? Are there any equilibrium solutions to this system of rate of change equations? If so, determine all equilibrium solutions and generate the 3D and other views for each equilibrium solution.

(b) For single differential equations, we classified equilibrium solutions as attractors, repellers, and nodes. For each of the equilibrium solutions in the previous problem, create your own terms to classify the equilibrium solutions in 6a and briefly explain your reasons behind your choice of terms.

Implementation Notes The goal we have for this problem is for students to begin to integrate the meaning of equilibrium solution to first order differential equations (constant function that satisfies the differential equation) into solutions to systems. An equilibrium solution to a system of differential equations is a pair of constant functions that satisfy the differential equations. Graphically, an equilibrium solution is a straight line in 3-space, a pair of horizontal straight lines in the $R-t$, $F-t$ planes, or a point in the $R-F$ plane. The students use the applet in conjunction with algebra to figure out the exact equilibrium functions in this task.

One question that we often address from students is whether or not both differential equations have to be zero in order to have an equilibrium solution. We note that the answer is that both differential equations have to equal 0 all the time. That is, if there is a point where both differential equations have a value of 0, but it is only at that point, the values you are considering do not give an equilibrium solution. Also, if one of the differential equations is constantly 0, but the other is not, that is not an equilibrium solution.

One of the points of inquiry-oriented instruction is for students to reinvent the mathematics, but we also want them to "play in the mathematician's sandbox". We intend for the second half of the task to connect students' imagery to conventional terms of saddle and center (the expectation is that they will make a conceptual distinction between the different equilibrium solutions, the conventional terms can then be labels for their distinction). Over the years, we have done this problem many times and we note that this problem may be left out without loss of continuity. Many students are not really "into" the renaming, particularly since there are standard names already created.

Task 7.
A group of scientists wants to graphically display the predictions for many different non-negative initial conditions (this includes 0 values for R and F, but not negative values) to the rabbit-fox system of differential equations and they want to do so using only one set of axes. What one single set of axes would you recommend that they use ($R-F-t$ axes, $t-R$ axes, $t-F$ axes, or $R-F$ axes)? Explain.

Implementation Notes In Task 7, the students are asked to return to some of their thinking when they first generated the solution using Euler's method. But now through the visualization tasks, they might begin to see the value in the $R-F$ view. We find that this is an important "conclusion" to this lesson and time should be allowed for students to work on this. This is a nice place for having groups share their thinking to the whole class. Particularly, students who are in other science majors may find this is a place where they can connect ideas here to Physics, Engineering and other disciplines. Additionally, we note that this is part of the trajectory we are helping the students to participate in as they reinvent some of the mathematics. It leads into Lesson 3.

6.2.3 Lesson 3: The Phase Plane, Vector Fields, and Nullclines (oh my!)

Rationale In this lesson we seek to have students build on their earlier work that motivated the phase plane to develop the ideas and techniques related to vector fields and nullclines. We think that it is important for students to initially develop these ideas without the support of technology so that the underpinnings of how vectors are plotted does not become a black box. In later lessons we use Geogebra apps to offload some of the more pedestrian work once the core ideas have been developed. As in previous lessons, we are inspired by the theory of RME and continue to leverage the predator-prey context to ground and drive forward the ideas. There are three main tasks to the lesson and the overall flow of ideas is as follows:

- Task 8: With guidance from their instructor, students reinvent vector field notation. Nullclines are also foreshadowed and then formally introduced in the next task.
- Tasks 9–10: Vector fields and nullclines are introduced and then used to sketch solutions in the phase plane for a decontextualized system. We use a decontextualized system for algebraic ease. This is followed up with a task that returns to a context with two species.
- Task 11: Students return to the predator-prey system and use nullclines to sketch in solutions in the phase plane and to identify and classify equilibrium solutions.

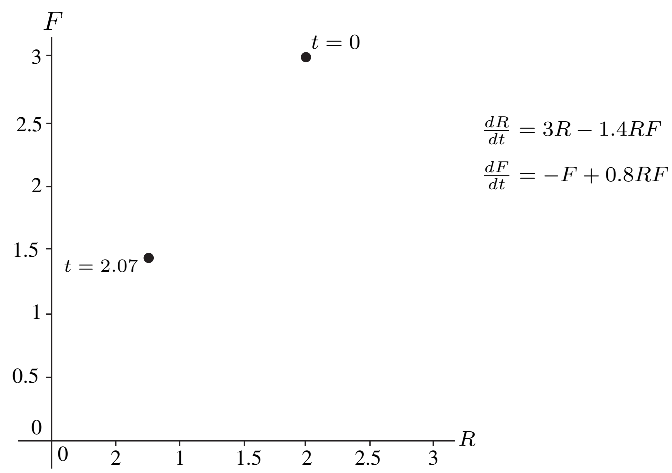
Implementation Notes We begin this lesson by presenting students with what we have found to be a puzzling, genuine question for them that provides an entry point to vector notation. Plotting vectors for our students is something that they have done either in high school or in physics, but doing so with a system of rate of change equations is something that is new for most of them.

Task 8.

8. One view of solutions for studying solutions to systems of autonomous differential equations is the $x - y$ plane, called the **phase plane**. The phase plane, which is Fronter's view from the crop duster problem, is the analog to the phase line for a single autonomous differential equation.

   (a) Consider the rabbit-fox system of differential equations and a solution graph, as viewed in the phase plane (that is, the $R - F$ plane), and the two points in the table below. These two points are on the same solution curve. Recall that the solutions we've seen in the past are closed curves, but notice that the solution could be moving clockwise/counterclockwise. Fill in the following table and decide which way the solution should be moving, and explain your reasoning.

$\boldsymbol{t}$	$\boldsymbol{R}$	$\boldsymbol{F}$	$\boldsymbol{dR/dt}$	$\boldsymbol{dF/dt}$	$\boldsymbol{dF/dR}$
0	2	3			
2.07	0.756	1.431			



When students share with the class how they plotted the vector and why they decided that the vector at $t = 0$ points in the direction it does, we talk with them about the options of plotting the resultant vector with its actual length and direction or plot a unitized vector that preserves only the direction. Students are quick to make connections to what was done with slope fields, and so we agree as a class not to plot the actual length. While this decision is consistent with most all textbooks, it does present a challenge when using nullclines to graphically determine equilibrium solutions. More on this is in Task 9.

We then follow up with the following task, which is designed to elicit student insight into which vectors are easier to plot than others. In particular, it is easier to plot a horizontal vector than one with a non-zero slope. The task below invites students to consider several vectors that are this type.

On the same set of axes from part (a) plot additional vectors at the following points and state what is unique about these vectors.

R	F	dR/dt	dR/dt	dF/dt
1.25	0			
1.25	0			
1.25	0			

Implementation Notes While at first glance this task seems relatively straight forward, we have found that the following are useful (and necessary) points to discuss:

- What does it mean if the slope of the vector dF/dt equals zero?
- How do you know if the arrow points to the left or to the right?
- Other than horizontal vectors, vectors with what other slope might be relatively easy to identify? Go ahead and plot a few of these vectors.
- What name would you give to the horizontal vectors that point either left or right?

After students have reinvented, with guidance, the way to plot vectors and glean information about the direction of solutions in the phase plane, they get a chance to practice plotting a vector field for an algebraically simpler system and then use the vector field to sketch in graphs of solutions in the phase plane.

Task 9.

Slope fields are a convenient way to visualize solutions to a single differential equation. For systems of autonomous differential equations the equivalent representation is a **vector field**. Similar to a slope field, a vector field shows a selection of vectors with the correct slope but with a normalized length. In the previous problem you plotted a few such vectors but typically more vectors are needed to be able to visualize the solution in the phase plane.

9. On a grid where x and y both range from -3 to 3, plot by hand a vector field for the system of differential equations

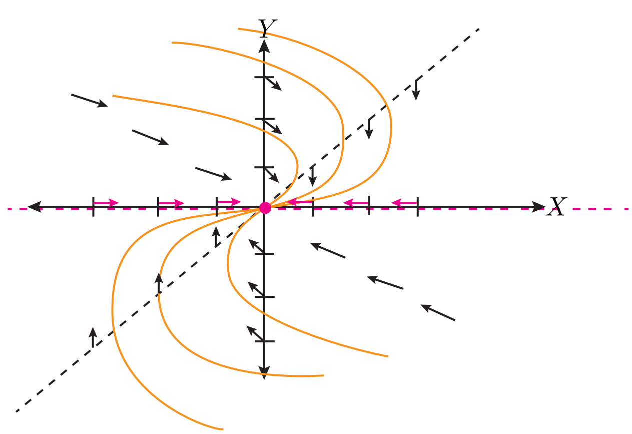
$$\frac{dx}{dt} = y - x$$
$$\frac{dy}{dt} = -y$$

and sketch in several solution graphs in the phase plane.

Implementation Notes While we are tempted to suggest that groups divide and conquer the calculations and then combine results, doing so is less likely to promote pattern seeking and discovery of shortcuts. We therefore intentionally discourage the divide and conquer approach and instead invite students to look for patterns and shortcuts to plot vectors. The payoff for these framing of the task is illustrated in the following example of student work, which was submitted as part of their end of the semester portfolio assignment, which asked students to select several problems from the semester that they were most proud or found most interesting, include their work on the problem, and to write a rationale statement that explains why they selected this particular problem. As a side note, we have found that these portfolios are excellent ways for students to prepare for the exam and they give us insights into students' mathematical progress and identity development that we would otherwise not have access to.

Student Thinking and Reasoning

Example of Portfolio Submission



Student Statement about his thinking

> The previous problem had us plot specific vectors you chose. Doing so we noticed that everywhere along the line you chose the vectors had the same slope. This problem we were tasked with creating a true vector field and first we set off to replicate the table method with more entries. However, my group had a light bulb moment thinking back to the pattern we saw in the previous problem. Could we find similar patterns that would make this process easier while still giving us a good sense for the entire vector field? What we found were three lines in which we saw patterns of similar slopes along a single line through the plane. We were later told through this process of self-discovery we taught ourselves how to find isoclines and nullclines! Learning is so much more satisfying when you discover things like this for yourself, as if you were walking in the same footsteps of the mathematicians who first did it!

When students discover patterns in the vector field, like the student above did, we take this opportunity to formally introduce the terms x-nullcline, y-nullcline, and isocline. Of course, not every time we teach the class do we receive such innovative student discovery as exemplified here and hence we follow up with the following task. Our goal here is to formally introduce the notion of isoclines and nullclines, to give students a chance to practice finding them, and to seed the potential for students to discover a graphical way to locate equilibrium solutions.

Task 10.

10. (a) You may have noticed in problem 9 that along $x = 0$ all the vectors have the same slope. Similarly, for vectors along the $y = x$. Any line or curve along which vectors all have the same slope is called an **isocline**. An isocline where $dx/dt = 0$ is called an $\boldsymbol{x}$**-nullcline** because the horizontal component to the vector is zero and hence the vector points straight up or down. An isocline where $dy/dt = 0$ is called a $\boldsymbol{y}$**-nullcline** because the vertical component of the vector is zero and hence the vector points left or right. On a grid from -4 to 4 for both axes, plot all nullclines for the following system:

$$\frac{dx}{dt} = 3x - 1.4xy$$
$$\frac{dy}{dt} = -y + 0.8xy$$

Implementation Notes The above nullcline problem inevitably leads to a discussion about how to plot vectors when the nullclines intersect. We often find that students do not immediately associate these intersection points with equilibrium solutions, perhaps because they plot vectors with a unitized length and hence at the point where the nullclines intersect there is a non-zero horizontal vector and a non-zero vertical vector. This gives rise to some interesting discussion about what is really going on. Below is an excerpt from a student's rationale statement for why he included his work on this problem in his portfolio.

> **Student Rationale Statement**
>
> This problem also stood out to me because I really enjoyed learning about nullclines, it was a concept that I found very fun and interesting how you can relate finding equilibrium solutions and then use what we know about differential equations to help plot the vectors. Once again, this problem was an example of how we build off concepts we learn throughout the year.

We also use the nullclines that students plot to demonstrate how just these alone allow one to make decent sketches of graphs of solutions in phase plane. This leads into the final task which provides students with the nullclines for a system of DEs that models an unfamiliar interaction of rabbits and sheep grazing in the same field (but the actual DEs are not provided so that students focus on the graphical interpretation). Students are provided with the sketch of nullclines (see Figure 6.1) and asked to graphically predict what will happen to the Rabbit-Sheep population for a wide range of initial conditions.

Task 11.

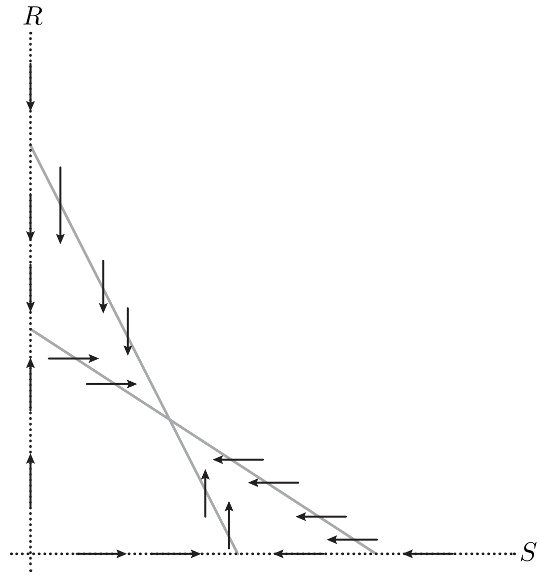


Figure 6.1: Nullclines for a Rabbit-Sheep population model

A certain system of differential equations for the variables R and S described the interaction of rabbits and sheep grazing in the same field. On the phase plane below, dashed lines show the R and S nullclines along with their corresponding vectors.

a) Identify the R nullclines and explain how you know.

b) Identify the S nullclines and explain how you know.

c) Identify all equilibrium points.

d) Notice that the nullclines carve out 4 different regions of the first quadrant of the RS plane. In each of these 4 regions, add a prototypical vector that represents the vectors in that region. That is, if you think that both R and S are increasing in a certain region then, draw a vector pointing up and to the right for that region.

e) What does this system seem to predict will happen to toe rabbits and sheep in this field?

Implementation Notes We use student work to have discussions on issues such as if the phase plane reveals if either population has limited resources (and why), whether the system is consistent with a cooperative or competitive model (and why), what are the equilibrium solutions, how to interpret them in terms of the two species, and how to classify each equilibrium solution. We intentionally do not include the actual system of differential equations because we want to foster and promote qualitative and graphical analysis. One particular connection that we strive to bring out in discussions with students is how to see in the phase plane two logistic growth phase lines, harkening back to their earlier work with a single autonomous differential equation. If students do not notice this connection on their own, we have the following prompt to be productive: "Last semester Valeria noticed that if there are no sheep, then the rabbit population will eventually level off at some non-zero equilibrium value. What do you think of Valeria's observation? Is something similar true for the rabbits if there are no sheep?"

6.3 Post-Lesson Considerations and Additional Implementation Notes

The guided reinvention of mathematical ideas is the driver for our instructional design choices, but the reinvention of ideas is an *intended* outcome. To achieve the goals for our students, we need to create a classroom environment in which students routinely make public their thinking, however tentative, engage with the mathematical reasoning of their peers, and collaboratively build ideas. At the beginning of this chapter we stressed the importance of this kind of safe and brave space. Creating such learning environments, however, is a non-trivial task and requires the instructor's explicit and intentional attention. Over the years we have honed some teacher talk moves, that is, language that an instructor might use in the facilitation of the discussions, that go a long way to make the intended social environment a reality. While what follows is not a basis for all that an instructor does, it offers examples of the kind of teacher talk moves that have, in our and our colleagues' experience, fostered a classroom environment where students share their thinking and orient to and engage with others' thinking.

In the IODE class, students do more than practice or apply previously worked out examples. They actually *do* mathematics by working on problems (often with their peers) that are novel and/or challenging to them. Similarly, instructors do more than check for correct answers to such problems. They strive to bring forth student ideas and make them public even if the answers are incorrect or the students are tentative in their thinking, and use these ideas to further the mathematical agenda. The following teacher talk moves have become a staple in our instructional repertoire:

- Dave, I know you haven't finished the problem, but tell us your initial thinking.
- Take your time, we're not in a rush.
- Can you say more about that?
- That's an important point. Keisha, can you say that again so that everyone can hear?
- Learning from mistakes is an important part of doing mathematics. Who can share with us an initial approach that didn't work out?

A critical characterization of these types of teacher talk moves is that there is not a known answer. Instead, these kinds of questions reflect genuine curiosity on our part to learn how our students are thinking and reasoning. Students become comfortable sharing their thinking when what follows from us is non-evaluative. Nothing shuts down students' explaining their thinking quicker than being corrected, evaluated, or funneled to say something different. Moreover, students' draft thinking is often muddled or difficult to interpret.

So, what do we tend to do with such draft thinking if we do not correct it or ask a series of narrow follow up questions to funnel students to an expected (and known response)? One teacher talk move that we have found productive is to repeat back what the student said, with or without an interpretation or rephrasing, and then check back with the speaker for accuracy. This type of talk move is referred to as *revoicing*, which can function to assign intellectual ownership (key for student confidence) and allow others a second opportunity to engage with the ideas.

A second teacher talk move is focused on promoting cross talk among students where they attend to and make sense of other students' thinking. Instructor prompts that can help achieve this goal often make use of questions that we do not normally use in everyday conversation. For example, it is rare to sit around the dinner table and ask, "Jose, can you say in your own words what your sister just said?" While such questions are not part of our everyday speech patterns, the following teacher talk moves can be quite productive for establishing a classroom environment where students actually listen to their peers and attempt to make sense of others' thinking.

- Who can repeat in their own words what Juan just said?
- Do you agree with what Darrel just said and why or why not?
- Can someone rephrase Diane's explanation in their own words?
- So Ramona, is Minsoo saying that...?
- Jesika, what do you think about what Ding just said?

These types of talk moves essentially serve the function of encouraging other students to revoice the thinking of their classmates. There are several productive consequences that often come out of student revoicing. Students are often better than we (as instructors) are at seeing the point hidden by draft thinking and thus they can help clarify what their peer is saying for the rest of the class (and for us). Sometimes student revoicing leads to a different conclusion or justification, and thus creates an opportunity for a broader debate among multiple students. A student misinterpretation may come to the surface, encouraging the original speaker to use more precise mathematical language or explain their idea in a different way. All of these outcomes are mathematically productive.

Creating a classroom environment in which students regularly share their thinking and engage with the thinking of their peers is foundational for the joint creation of meaning and the guided reinvention of mathematical ideas. While such classroom environments are empowering, we are increasingly aware of the fact that they are not automatically equally empowering for everyone. Who gets a chance to share their reasoning? What kinds of questions (recall, explanation, justification, etc.) are asked of which students? Whose voice is valued? In whole class discussion and small group work whose voice gets talked over or ignored? These and related questions center on issues of equity and inclusive practices. Creating classroom environments that are not only safe and brave spaces, but are also spaces that are equitable and inclusive does not come for free. Just as we have, over the years, developed a repertoire of talk moves that center student thinking, we are in the early stages of developing strategies that explicitly and intentionally centers issues of equity and inclusion. We look forward to working with others to further these ideas and learn new ones.

In conclusion, we chose to use three lessons because so much of the implementation of this sequence of tasks is a type of learning progression and we wanted the reader to see how it all builds together. The lessons you have just read have been through many changes, and so every time we (or the many other instructors involved) taught it, we made tweaks and changes that have hopefully improved the materials. We have also continued to update the technology (as far as the visualization software) so it is free and most useful.

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7

Developing Taylor Polynomials and Power Series

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Brinley Stevens, *Brigham Young University*

Topic of lesson	To develop the idea of Taylor Polynomials, and later power series, by extending students' understanding of linear approximations.
Course context	This lesson has been taught in a Calculus 2 course in a large, private university, both during TA recitation sections for a large lecture course, as well as in smaller sections taught by the professor.
Instructional Challenge	Textbooks often give no or weak motivation for introducing or studying power series. How can power series be developed in a natural way based on what students already understand up to that point?
Brief overview of instructional approach	The lesson begins with using tangent lines as linear approximations for a curve near a point. Students then use derivatives to find higher degree polynomials that provide a better approximation of the curve. This leads to the development of Taylor Polynomials. Students specifically work with the graphs of $\sin(x)$, $\cos(x)$, and e^x. These functions have Taylor polynomials (centered at $x = 0$) that give identifiable patterns that make them easy to extend to arbitrarily higher powers. This can be used to motivate the need for studying and understanding power series. For example, one natural question is: Does $\sin(x)$ really equal that series for all x?
Keywords	Taylor Polynomials; Power Series; Linear Approximation; Convergence

Author note: Although there are two authors to this chapter, it is written in the voice of the first author, who initially develped the lesson.

7.1 Background Information

7.1.1 Instructional Challenge

I have struggled to help my students gain the depth and flexibility of the power series in my Calculus 2 class similar to other topics in the class. As I reflected on why this was so difficult, I realized (with the help of a colleague) that the sequence of topics in the textbook we were using [1] did not help students build up to the idea of power series and Taylor polynomials in a natural, motivated way. After working with series and learning many tests for convergence, Stewart [1] introduces a power series with no motivation and no development. The first statement in the section on power series is:

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \cdots \tag{7.1}$$

Then the book continues to give vocabulary (coefficients) and explains that a series may converge for some values and not for others. Nothing in this introductory section seemed to help significantly ground the idea of power series, or the related idea of Taylor polynomials, to ideas students were comfortable with or that were meaningful to students.

I attempted to develop a progression of ideas that would help students see power series as extensions of Taylor polynomials and that Taylor polynomials are a natural extension of an idea that they had previously learned, linear approximations. I wanted to have a progression so natural that students would be able to write down power series for such functions as e^x, $\sin(x)$, and $\cos(x)$ without ever explaining the idea of power series or Taylor polynomials to the students. In a way, this was a standard I could use to evaluate how effective my approach would be. If students could find and extend patterns to construct the power series from a very natural question (or set of questions), then I would consider it a success.

Before I explain the progression of ideas in this lesson, I explain some related mathematics and the background of my students.

7.1.2 Students and Instructional Context

I teach at a large, private university that attracts academically successful students. Students can take calculus in a large lecture section (about 150–200 students) with three lectures a week and recitation sections (about 30–40 students) twice a week with a TA, or students can take a small section of calculus taught by a professor (about 30–40 students). I teach this particular lesson in a small section or have my TAs do it in recitation sections in a large lecture course. It is typical in our university that more than half of the students that take Calculus 1 have had some form of calculus in high school. Many students in the Calculus 2 class took AP calculus (either AB, or BC) in high school and most of these retake calculus (either 1 or 2) in college.

Students in first semester calculus are taught about linear approximations using tangent lines to functions. This lesson builds on this content, but many students who took Calculus I in high school have either not seen it before or do not remember, so we do a short introduction to it before the lesson.

7.1.3 Mathematics and Mathematical Thinking of Students

There are at least two ways for second semester calculus students to think about power series and Taylor polynomials, as an extension of linear approximation and as a family of series. (For the remainder of the manuscript I will just use the term *power series*, although it should be clear from the context when I am using power series as a short label for power series and *Taylor polynomials*).

Extension of Linear Approximation. Power series can be thought of as an infinite extension of linear approximation. It is reasonable (although NOT obvious to students) to think that a linear approximation is a good approximation at a differentiable point $x = a$ on a curve $f(x)$ because 1) the approximating line has the same value at $x = a$ as $f(x)$, and 2) the approximating line has the same first derivative at $x = a$, so it changes at the same rate (and in the same direction) as $f(x)$. Of course the rate at which it changes will probably not be a constant, and in that case, the curve will stop having the same values as the approximating line at values not equal to $x = a$ (except for "chance" crossings).

One advantage of tangent lines is that they can easily be generalized. The reasons that that tangent lines make good approximations can be extended to include higher order polynomials. A good-fitting second order polynomial would not only have the same function value and rate of change as $f(x)$ at $x = a$, but also have the same acceleration, or match the second derivatives with $f(x)$ as well as first and "zeroth" derivatives. If $f(x)$ is infinitely differentiable, then this thinking can be extended further and further. Fortunately, simple patterns emerge for many functions that allow students to first conjecture about the form of next terms, and then justify and reason through why such patterns should hold.

For the second derivatives to be the same at zero, the coefficient of x^2 in the Taylor polynomial should be the same as the second derivative – almost. By the time you have taken the first derivative, you have multiplied by 2, so you need to divide the coefficient by 2 in order to counter that doubling. This thinking can be extended (or this pattern can be discovered) so students can recognize that matching the nth derivative with a polynomial requires the nth degree term of the polynomial to be the nth derivative of the function at $x = 0$ divide by n-factorial. The pattern can be further generalized to center values different than zero, which then gives the coeffiecients of x^n as the nth derivative of the function at $x = a$ divide by n-factorial.

A Family of Series. Students would also have the capability of thinking about power series in a different way than an extension of linear approximation. Series can be thought of as a family of related functions, or polynomials in the finite case. Students are often actually exposed to this idea of a family of functions when working with series. Stewart discusses the general case of the geometric series with the following expression:

$$\sum_{n=0}^{\infty} ar^{n-1} = a + ar^1 + ar^2 + \cdots \tag{7.2}$$

This expression represents a family of series, namely, all geometric series. It is not developed in the book to help students think about it as a family of series, however. The thinking required for a family of series would be connected to functional thinking. Functions can be thought of as a device that takes an input and produces an output. Each real number input value of r in (7.2) or x in (7.1) produces a specific output, each of which is a particular series. This would be just one way to think of a series (with process, object, etc. type thinking following). Each of the particular values of r (or x) produces a series that could then be thought of as a particular series that could then be investigated (does it converge? If so, can we find the value to which it converges?)

Since students have been working with series for a couple of weeks (or so) before being introduced to power series, one possible way to develop the idea of a power series is to consider related series that they have seen before, such as geometric series, or harmonic/alternating harmonic, and begin to think about the group of series (really the family of series). For example, a few geometric series (see (7.3)) could be seen as having a similar form (see (7.4)).

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots \tag{7.3}$$
$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} \cdots$$
$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} \cdots$$

These can be generalized as:

$$\sum_{n=0}^{\infty} (x)^n = 1 + x + x^2 + x^3 \cdots . \tag{7.4}$$

And now there is a representation of an entire family of series. The family (7.5) series could be explored now without having to explore all of the particular series individually. Another example could be connecting the harmonic and alternating harmonic series. Seeing a pattern that produces a power series is not as obvious in this case, but once the general power series for (7.4) is developed, students might be able to see a way to connect these two series, like in (7.5) below.

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) (1)^{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots \tag{7.5}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right) (-1)^{n-1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

Now the series in (7.5) can be viewed as special cases and can become an object of study.

Summary. The linear approximation development is built from an effort to find better approximating polynomials to a curve. This development, in essence, goes through Taylor polynomials to get to power series. In the family-of-series development, power series are seen as a way to group and study series. It capitalizes on the power of algebra to represent many series at one time, this could make mathematical work connected to series more powerful. No natural connection to Taylor polynomials, approximations, or even graphs of curves emerge in the family-of-series development. It does provide an advantage of being a local development, however, since it builds off of work on series which students have been engaged in. The linear approximation development does jump back to material from a previous semester and does not seem connected to the topic they have been working on.

7.2 Instructional Approach

In this section I give an overview of the instructional approach and some reasons for my instructional choices. I give a detailed account of the instruction later in the manuscript.

7.2.1 Why Extension of Linear Approximation?

I chose to develop a lesson based off of the linear approximation approach. I think the meaning of power series developed in this way is more powerful, and more interesting, than the idea developed by thinking of families of series. I hope that my students can develop both ways of thinking about power series by the end of the unit, however, for initial development I like the idea of getting better and better approximations, and the joy of finding a clear pattern in the series. In the past, many of my students have really gotten excited when they see the pattern and being able to write something like (7.6) on the board. One nice thing that tends to happen in my classes is that some students begin to ask questions about things like (7.6) when it first goes up on the board: Is the right hand really equal to $\sin(x)$? For any x or just some values? Or is it just an approximation (never really equal)? Where does the pattern come from?

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots . \tag{7.6}$$

I planned to have students work in groups of four and give students the following task: Find a quadratic function that will do better at approximating a function than a linear approximation by finding a quadratic that has the same second derivative at $x = 0$ as well as the first derivative and function value. If you find a good quadratic, can you find even a better cubic? Quartic? Etc.

However, I have found that how I launch the task makes a big difference in how students might respond so I discuss the launch next.

7.2.2 The Launch

The first couple of times I tried this out, I gave a very open task. I like to let students do a bit of problem solving in my classes, so I thought that I would not give them any direction in trying to find a good fitting parabola. I hypothesized that many would think of the concepts associated with linear approximations from the previous semester and try to build a parabola with the same ideas as described above (a quadratic with the same derivatives as the function). With students in groups of 4, I thought most groups would have someone in that would think of this strategy. I was wrong.

Without some direction very few students tried to use derivatives to get an approximating parabola. The two most common strategies were 1) picking three points on the curve and finding a parabola that would go through these points, and 2) using a graphing facility to test coefficients to get a parabola whose graph was “close” to the curve for the largest interval. Although these solutions have some merit, they do not help to generalize to higher-order polynomials very well, and no clear pattern may emerge for students to extend their polynomials to infinite degree. Because the particular goal for the lesson is to develop better approximating polynomials to a curve, build an infinite series, and connect it to derivatives, the other solutions were not helpful for the class’s thinking. When I picked up one particular strategy to highlight and run with as a class, the students that had tried other strategies did not feel like their work was valued. The group work also took a long time, since more time was spent trying to figure out a strategy and finding a solution than

I had anticipated. This left little time to discuss the ideas at the end of class. For these reasons I changed the launch of the task.

I have had more success if I start with an introduction to linear approximation (for those that haven't had the opportunity to learn it) or a review of linear approximation (for those that have). The key point is to discuss why the process of finding the equation of the tangent line would give a good approximation. This allows the two key ideas to emerge: both the curve and the line intersect at the same value at $x = a$, and also both the curve and the line are changing at the same rate. From this I ask the class about how we might use the ideas that make a good approximating line, to make a good approximating quadratic. As a class we can come up with the fact that a good approximating quadratic would also have the same acceleration (or 2nd derivative) as the curve we are trying to approximate. We then extend this idea as a class to higher orders.

It is only after this discussion that I give the students the task to work and have them solve it in groups—to find an actual approximating quadratic, cubic, quartic, etc. to the highest order they can in the time allotted for a specific function. Including the class discussion about why the linear approximation makes a good approximation before turning them loose to work on the task solves the problems I was having before, but still gives space for some problem solving and exploration, since they don't know how to equate derivatives of a polynomial and a function. They also are surprised when they see patterns emerging from their work once they start solving for coefficients. These patterns (such as dividing by $n!$ and the nth derivative being part of the coefficient of x^n) and seeing how they come to be by repeated differentiation allow us to make sense, as a class, to generalize the process to the Taylor polynomial formula (although the generalization usually comes in a subsequent class).

7.2.3 Which Functions Should I Have Students Approximate?

I have tried different functions for students to approximate during different versions of this lesson. I started with students approximating e^x centered at $x = 0$. I have also tried $\sin(x)$ and $\cos(x)$ both centered at zero. One advantage of each of these is that the derivatives are fairly easy (there is a simple pattern for the derivatives at $x = 0$ for each of them). They are also power series they will be using frequently, and each has a nice pattern to the terms of power series that are easy for students to see. There are disadvantages, however. With $\sin(x)$ and $\cos(x)$, every other term is zero. This can be difficult for students as they are looking for coefficients of the next term (x^2, for example, in $\sin(x)$) but they get a coefficient that should be zero. Some students will think that they have done something wrong.

The function e^x does not have that problem, no terms of the power series have a zero coefficient. However, every time I have used e^x as the function to approximate there has been at least one group of students that uses a different, but effective, strategy for developing their higher-order approximating polynomials. The strategy is to find the antiderivative of each term and then add 1 to the end. Students keep doing this until they see a pattern and can write down subsequent terms without finding anti-derivatives. This is a very interesting strategy, but in this lesson it has caused me some difficulties. First, the strategy does not emerge from reasoning based on solid understanding of the properties of e^x. As I have questioned students in class about why they are using this strategy, they are just "trying something", and they find a pattern. They don't reason about why the constant term should always be 1, but "put a 1 there" because there was a 1 there before they took the anti-derivative. Second, this is not a generalizable strategy for finding coefficients of Taylor polynomials/power series in general. A goal of this lesson is to have students develop a strategy that they can use to start building approximating polynomials, and this strategy doesn't fit well with this goal. In a subsequent lesson we will connect the properties of the power series and the functions. (For example, the derivative of e^x is itself, so the derivative of the power series of e^x should also be itself. Students get a kick out of seeing the derivative of the terms producing the same series of terms). However, I do not have time in this lesson to have that conversation. In the moment, if this strategy comes up in a group, I ask questions to understand if they are generating it from a reasoned approach based on mathematical understanding, or if it just a shot in the dark strategy. If it is a reasoned approach, I would consider having them share it as a group if there was time. But up until now students could not reason through why this strategy is working. I applaud them for what they have done and tell them the they have actually bumped into something that is very interesting that we will be able to talk about in a subsequent lesson, but for now, I ask them to work on a strategy that would work for other functions, not just e^x. (Another possibility would be to give them another function on which to test their strategy, where they can find out for themselves that their technique does not generalize.)

I have found a lot of success by using all three of the functions: e^x, $\sin(x)$, and $\cos(x)$. We divide up into groups of four and I assign groups to extend approximating polynomials to the highest degree possible for one of these. I try to assign an approximately equal number of groups to each of the three functions. There are several reasons that I have students do all three functions. First, as mentioned earlier, these are all power series that students will be working with extensively and the patterns in the coefficients are easy to see. Second, they all have strong connections to the generalized Taylor series formula, where as a function like $\frac{1}{1-x}$ does not have such obvious connections because of the canceling out of the nth derivative in the numerator and $n!$ in the denominator. Third, students can see that the same process can be used to generate approximating polynomials for a variety of functions. Near the end of class I have three students come to the board and write down how they found the coefficients of the higher order polynomials for their function. This allows us to discuss the similarities of the process across different functions and will help to build toward a generalized formula of Taylor Polynomials. Fourth, students may feel like their work is valued more if they are not all working on the same problem, but instead, working on something different as part of a smaller group of students that they will then share with the rest of the class.

7.2.4 Equitable and Inclusive Instructional Practices

Although any teacher action (or lack of action) can impact students' access and mathematical identity there are a few points in this lesson that I would like to highlight where issues of fairness and equity may emerge. First, I choose to give only a brief introduction to the idea of linear approximation because a lot of the students have been in classes where that topic is explicitly taught (Calculus 1 at my institution used to cover this topic but has recently cut it from the curriculum). Students that have not had the opportunity to think about linear approximations could be at a distinct disadvantage. If you are teaching a class where linear approximations were not in a prerequisite course, then a quick introduction here may not be the right choice. Ensuring all students have the opportunity to access the mathematics of the lesson and not feel marginalized may mean that more than a brief introduction is needed, perhaps a more significant portion of another lesson dedicated to this idea or outside resources for students to study and learn from before class (as long as the resources/technology is available for all students).

Second, selecting which group and which individual in the group to share (or having the entire group share) allows me to reflect on the interactions that have been happening in class and use the choice advantageously. At my institution we only have a few Hispanics in our calculus classes, so I like to highlight and encourage them as much as possible. Most of them come from Central or South America and are learning English as a second language (which doesn't make learning Calculus any easier). When possible I have students that have not presented much, especially if they are from a marginalized population, make the presentations to the class.

7.2.5 Technology Use

There is one place in the lesson where I use a graphing facility to project the graph of a function and the graph of a tangent line on that function so we can discuss some key attributes of tangent lines and issues of approximation.

7.3 Lesson Implementation

Of course, you are free to adapt this lesson in any way you see fit for your circumstances and students. I have included some specific questions, responses, and explanations all in a particular progression and flow. This section is structured in a back-and-forth pattern, with a section first showing what we did as a class, followed by commentary for instructors about the instruction in the previous section. These two sections are called ”Key Instructional Activities” and ”Points to Consider During Instruction.” Some short comments are included in the Key Instructional Activities section and are indcated by the use of brackets.

7.3.1 Reviewing Linear Approximation

Key Instructional Activities: Part 1. (About 3 minutes).

T: I would like to take a few minutes to review something that you may have learned in Calc 1. Some of you may not have seen it yet because not all courses of Calc 1 include it. The topic is a way to approximate a function on a small interval by using a tangent line. Let's begin by finding the equation of the tangent line at $x = 0$ for the function

$f(x) = e^x$. You can work by yourself or with a neighbor but let's spend a couple minutes on this and then we will come back to class.

[Students tend to come up with $y = x + 1$ fairly quickly because finding equations of tangent is a technique students tend to remember well from Calculus I.]

T: [Put equation of tangent line on the board and graph the line and e^x on the board or projected on a graphing facility for all to see.] Here we can see the graph of e^x and our tangent line. As you can see, around $x = 0$, the graphs of the two are almost indistinguishable. For many applications, where we only care about values close to zero, we can throw away the complicated function e^x, and only use the very simple $y = x + 1$ to generate values. The question I would like you to think about is: Why does the process of finding the equation of the tangent line make a line that seems to approximate the function well around this point? [Let students think for themselves for about 30 seconds, then share with a neighbor their thoughts]

As you walk around you might hear things like:

S1: The line is the best one to hug the graph.

S2: It is touching the graph at $x = 0$, and it is going the same direction as the function.

S3: The tangent line has the same value as the function at $(0, 1)$.

S4: It has the same derivative as the e^x.

S5: It is changing at the same rate as e^x.

T: I heard some very good thinking and interesting thoughts as I listened to your conversations. Many of you said that it was a good process because we end up with a line that hugs the graph – [S1 -student name] could you tell the class what you were explaining to your neighbor?

Points to Consider During Instruction. The goal of this first section is to make sure students understand why a tangent line is a good choice for approximating a function on a small interval around $x = a$:

1. The tangent line has the same value as the function at $x = a$, and

2. The tangent line is changing at the same rate as the function at that point.

Don't spend too much time getting the equation of the tangent line to e^x at $x = 0$. The point is to discuss why the tangent line would be good at approximating the function and to get to the two points above.

The specifics that you have students share will depend on what your students talked about. If you feel like you need to highlight more student thinking than that which gets close to the two points above, do so, especially if you can connect the earlier points to the two key points that you want to make sure students understand. My example gives an example of how this might be done.

Key Instructional Activities: Part 2. (about 3 minutes)

S1: Well, I was thinking about all of the possible lines that you could have that touched the graph at that point [pointing to the point (0,1) on the graph of e^x]. I imagined a line going through that point, but being able to spin around. Where would I want it to stop so it fit the function best? It would be the one that stayed closest for the longest, and so I would want it to lay right along the curve, and that is what it looks like the tangent line does.

T: Thank you so much S1. You have started us off with some key ideas. First, you wanted the line to go through this point, (0,1). I heard other pairs talking about this. S3, can you tell the class what you were explaining to your neighbor?

S3: The graph and the tangent line both have the same point (0,1). If you plug 0 into the equation of the line and into the function you would get 1.

T: Great, so we know that they share that point in common. S5, you shared another thing they have in common. Would you please explain that a bit more?

S5: Yeah, they have the same rate of change. If you plug 0 into the derivative of the function, you get the same slope as the tangent line.

T: So, let me summarize some of your ideas that seemed really important to me. First, if we want to approximate the function around a certain point, our approximating line should go through that point – so at least they have the exact value near the center of the x-values that we care about. Second, matching the slope of the approximating line and the derivative of the function ensures that both the line and the curve are changing at the same rate at that point. Some of you said this colloquially as ‘they are both heading in the same direction.’ It is both of these things that make the tangent line a good approximating line for the function, it has the same value at that point and the same first derivative at that point.

Points to Consider During Instruction. Clarifying what S3 said about the line and function having the same value is good because the student might mistake “value” for rate of change. If that’s the case, other students should be able to notice the shared point value. Both the value of the function and the rate of change of the function being the same as the tangent line are critical features to highlight.

7.3.2 Extending the Use of Derivatives for Approximating Functions

Key Instructional Activities: Part 3. (About 5 minutes)

T: The tangent line does a good job approximating, but only for a small interval around $x = 0$. This may not be good enough for some applications. What could we do to try to get good approximations for more x-values around the point x=0 than we could get with a tangent line?

S1: Use multiple tangent lines.

S2: Move our tangent line.

S3: Use a curve instead of a line.

S4: Find a good parabola that hugs the graph.

S5: Use a polynomial with lots of terms to fit the graph.

T: Interesting ideas. S1 and S2, you two are getting to something that will be very important later in the unit, the idea of changing what we might call our center, or finding different centers. We will need to do more work first to really build up to those ideas. S3, S4 and S5 had ideas of using a curve, S4 suggested specifically a parabola. I think that is something we could try in class. We will need to find out how we could find a good parabola that hugs the curve. What properties did a tangent line have that made it a good fitting line? [This is a leading question to focus their attention back on the properties of a tangent line that make it a good approximation to the function.]

S2: It might have the same properties as the tangent line, but also the same second derivative.

T: Interesting, so a parabola with those properties would have the exact same value as e^x at $x = 0$, it would have the same rate of change as e^x at $x = 0$, but it would also have the same acceleration as e^x at $x = 0$. This might help a lot. Our tangent line had to keep the same rate of change for all values of x, but e^x doesn’t have the same rate of change. A parabola also doesn’t have the same rate of change. We could try to match the rate at which the derivative is changing of e^x and the parabola (at $x = 0$) and hope that it will hug the curve better.

We could start with $ax^2 + x + 1$ and try to find the value of $\boldsymbol{a}$ that would give the same second derivative of this quadratic at $x = 0$ that e^x has at $x = 0$.

Points to Consider During Instruction. This could be a more open discussion if you would like to have the class brainstorm ideas about how to generate a good fitting parabola. A more teacher-centered review in section 1 could free up time here for a more open discussion.

7.3.3 Student Work and Teacher Support for the First Task

Key Instructional Activities: Part 4. (About 8 minutes)

[T: Present or Write the Task on the Board]

Task: Find a quadratic function that has the same value, first derivative, and second derivative at $x = 0$ as your assigned function: Either $f(x) = e^x$, $g(x) = sin(x)$, or $h(x) = cos(x)$. Graph it along with your assigned function to check to see if it makes a good fit.

T: Work in groups of four on this Task for about 5 minutes.

Group 1: $f(x) = e^x$. First find derivatives at $x = 0$. Since the derivatives are all e^x, then at $x = 0$ all derivatives (and the function value) are $e^0 = 1$. Next, find second derivatives of $y = ax^2 + x + 1$. $y' = 2ax + 1$, $y'' = 2a$. So $2a$ needs to equal 1, so $a = \frac{1}{2}$.

Group 2: $g(x) = \sin(x)$. First find derivatives at $x = 0$. $g'(x) = cos(x)$, $g''(x) = -sin(x)$. Evaluated at $x = 0$ we get $g'(0) = 1$ and $g''(0) = 0$. Find the equation of the tangent line, $y = x + 0$. Next, find second derivatives of $y = ax^2 + x + 0$. $y' = 2ax + 1$, $y'' = 2a$. So $2a$ needs to equal 0, so $a = 0$.

Group 3: $h(x) = \cos\cos(x)$. First find derivatives at $x = 0$. Since $g'(x) = -\sin(x)$, $g''(x) = -\cos(x)$. The derivatives at $x = 0$ then are $g'(0) = 0$, $g''(0) = -1$. Find the equation of the tangent line, which is $y = 0x + 1$. Next, find second derivatives of $y = ax^2 + 1$. $y' = 2ax$, $y'' = 2a$. So $2a$ needs to equal -1, so $a = -\frac{1}{2}$.

T: [After all groups have found $\boldsymbol{a}$ and are comfortable with the process] Great job. Here are the parabolas that each of you came up with for your assigned function. [Write them on the board]. Let's quickly look at graphs of these. [If possible, project graphs onto the board to test to see if the parabolas are better than the line. Verify that they are, except in the case of the sine, where no parabola would do better than the line.]

Points to Consider During Instruction.

Supports/Extensions

–If groups don't know where to start, you can suggest trying to find the second derivative of their assigned functions.

–If they need more help, consider rephrasing the task as: what would the coefficient of x^2(or a) need to be in $y = ax^2 + x + 1$ so that y would have the same second derivative as your function.

–As groups finish, ask them to make sure everyone in the group understands the process and reasoning behind the process (why it makes sense).

If a group finishes early you can ask:

–You now added another term. Didn't this change your value and first derivative at $x = 0$? Why or Why not?

–You found a good fitting quadratic, could you find something better? How? [the idea here is to get them to look for higher order polynomials and match derivatives, something that will be brought up as a whole class in a few minutes]

Students may question why there is only one coefficient to solve for in the quadratic. If they need more justification why the $x + 1$ for the linear approximation can remain for e^x, you can use the general form $ax^2 + bx + c$. Solving for each coefficient a, b, and c, students can find the same answer and begin to justify why the coefficients b and c match the linear approximation. Since the derivatives need to match, these coefficients cannot change as you increase the power of the polynomial. Then when extending to higher degrees, they won't have to find each coefficient over again.

Perhaps discuss the sine, with no x^2 term, explain that that is OK, the approximating parabola might not have all of the terms, sine doesn't have a constant and cosine has no x term. Or instead of explaining, ask the students to think about why it might make sense that there is a zero coefficient for the x^2 term.

You can decide how much time should be spent here depending on how much you discussed the jump from linear approximations to using quadratics, as it's a similar concept.

Key Instructional Activities: Part 5. (About 12 minutes)

T: Ok, We made some progress. We have quadratics that fit better than lines. Could we do anything to generate curves that could even be better than these quadratics?

S1: Find a cubic that fits well.

S2: Use more terms.

S3: Higher orders/higher degree polynomial.

T: Why would having a higher degree polynomial give a better fit than the quadratic?

S1: Because more derivatives will match.

S2: It will hug the curve longer.

S3: They will have the same rates of rates of change.

S4: Since $\sin(x)$ and $\cos(x)$ will have multiple bumps, we want a polynomial with lots of bumps.

T: Let's try finding a polynomial like that, in groups see if you can find a good fitting cubic using the same strategy that we did to find a good fitting quadratic – matching up derivatives. A cubic would need to also have the same 3rd derivative as the function. If you can find a cubic, see if you can find a 4th degree polynomial (quartic), or 5th. See how far you can go in the next 10 minutes.

Points to Consider During Instruction. It is interesting to discuss why we would want more terms to approximate e^x since it is not periodic like $\sin(x)$ and $\cos(x)$ and does not have multiple "bumps" or local extrema. However, I wouldn't bring this up myself if a student did not address it. Later when finding a cubic or higher degree polynomial, graphing them alongside e^x will show how it still gives a better fit, despite being a much different shape than the trigonometric functions.

7.3.4 Extending to Higher Degrees

Key Instructional Activities: Part 6. (About 8 minutes) In groups, students can use the strategies they've used previously to find a cubic function with the same 3rd derivative as e^x, $\sin(x)$, or $\cos(x)$. As students find the cubic functions, encourage them to graph them alongside the function they are approximating to see how well they match near $x = 0$. The teacher selects students or groups to present their strategy at the board.

Group 1: $f(x) = e^x$. First find derivatives at $x = 0$. Since the derivatives are all e^x, then at $x = 0$ all derivatives (and the function value) are $e^0 = 1$. Next, find third derivative of $y = ax^3 + \frac{1}{2}x^2 + x + 1$. $y' = 3ax^2 + x + 1$, $y'' = 6ax + 1$, $y''' = 6a$. So, $6\boldsymbol{a}$ needs to equal 1, so $a = \frac{1}{6}$.

Group 2: $g(x) = \sin(x)$. First find derivatives at $x = 0$. $g'(x) = \cos(x)$, $g''(x) = (x)$, $g'''(x) = (x)$. Evaluated at $x = 0$ we get $g'(0) = 1$, $g''(0) = 0$, and $g'''(0) = -1$. If the polynomial will have the form $y = ax^3 + 0x^2 + x + 0$, $y' = 3ax^2 + 1$, $y'' = 6ax$, $y''' = 6a$. So, $6\boldsymbol{a}$ needs to equal -1, so $a = -\frac{1}{6}$.

Points to Consider During Instruction. After two groups have shared their strategies, and before the third group shares, you can begin asking them if they notice any patterns as the power of the polynomial increases. The goal is that eventually they can predict what the next piece would be without having to solve for the coefficients. Asking this question now has them primed to see if their pattern continues as they watch the third group present.

Students should be able to quickly see they made a mistake if the graph of their polynomial does not reasonably match the graph of the function. This is a good way for them to check their work as they go.

Many groups will have a tendency to multiply their constants together and may find it difficult to find a pattern. You may suggest that they keep them separate so they can more easily see what is happening. That way they can get $4 * 3 * 2 * 1$ instead of 24 and more easily see that it is $4!$

Have groups place their polynomials underneath each other to highlight similarities between the three different equations.

Key Instructional Activities: Part 7. (About 10 minutes)

Group 3: $h(x) = \cos(x)$. First find derivatives at $x = 0$. Since $h'(x) = -\sin(x)$, $h''(x) = -\cos(x)$, $h'''(x) = -\sin(x)$. The derivatives at $x = 0$ then are $h'(0) = 0$, $h''(0) = -1$, $h'''(0) = 0$. If the polynomial will have the form $y = ax^3 - \frac{1}{2}x^2 + 0x + 1$, $y' = 3ax^2 - x$, $y'' = 6ax - 1$, $y''' = 6a$. So, $6\boldsymbol{a}$ needs to equal 0, so $a = 0$.

[Continuing this pattern, students should get the following polynomials. They may not have time to get to the fifth power, depending on the amount of time remaining. You may rewrite the denominators on the board in a factored form, or using factorial notation to help make it easier for students to make the pattern explicit.]

$$e^x \approx \frac{1}{120}x^5 + \frac{1}{24}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$$

$$\sin(x) \approx \frac{1}{120}x^5 - \frac{1}{6}x^3 + x$$

$$\cos(x) \approx \frac{1}{24}x^4 - \frac{1}{2}x^2 + 1$$

T: What do you notice about the three different equations you've found?

S1: $\sin(x)$ and $\cos(x)$ skip some powers. $\sin(x)$ only has the odd powers of x and $\cos(x)$ only has the even powers of x.
S2: e^x does not skip any powers of x. The power always increases by 1.
S3: $\sin(x)$ and $\cos(x)$ switch between positive and negative coefficients.
S4: The coefficients are a fraction of 1 over the number of the power of x with a factorial

T: Why is it a factorial in the denominator? How do you know this will continue if we kept adding more terms?
S1: Each time we take the derivative, the power of x is multiplied by the coefficient, so we will multiply $n * (n-1) * (n-2) * \ldots * 3 * 2 * 1$ for the nth power. But since the derivative either equals 1 (or -1 or 0 for the trig functions) we have to divide by the factorial.

T: Why do $\sin(x)$ and $\cos(x)$ skip certain powers of x? Or why are some of the terms negative and some are positive?
S1: The derivatives of $\sin(x)$ and $\cos(x)$ follow a pattern that repeats when you plug in $x = 0$. If you keep taking the derivative, it will repeat. They alternate between $1, 0, -1$, and back to 0 for $\cos(x)$ or $0, 1, 0, -1$ for $\sin(x)$.

T: I want to end with this question–Are the polynomials you found really equal to the functions? We've been writing $e^x = \frac{1}{120}x^5 + \frac{1}{24}x^4 + \cdots$, $\sin(x) = \ldots$, and $\cos(x) = \cdots$, but are these equal? How could we tell?
S1: We could graph them.
S2: Plug the same value of x into both sides.

Points to Consider During Instruction. Students may use equal signs instead of approximation signs when they write down "equations" like the ones above. I don't correct them there, but hit it below as a key question to have them think about whether both sides are equal, even if we extend the terms infinitely.

You may choose to focus on explaining one specific piece of the polynomials as time starts running out. The discussion of this can be picked up in the next lesson as well. The suggestions listed here may not all make it into the class discussion but include what students should be noticing from the equations they found.

Students should remember factorials from their previous work with convergence tests. The trickiest part is for them to recognize the pattern. But as said above, writing out the multiplications rather than simplifying as they go make the factorial more apparent.

Students should be able to justify why they are seeing specific patterns in the terms to help them justify their predictions for the next term.

This is where the 50-minute class would typically end. When plugging in values, students see that the polynomial and the function are very close near $x = 0$.

The key idea to leave students with is the idea of convergence–are these polynomials going to converge to the functions? Will this always be the case? While this would not be fully explored in the remaining time, it's an excellent idea to think about and pick up in the next lesson. The other thing students will need to do is find the general formula for finding Taylor polynomials.

7.4 Post-Lesson Considerations

Obviously, there is still much more to explore in this topic. The concept of Taylor polynomials, Taylor series, and Power Series extends far beyond the lesson here. This lesson is intended as an introduction to the unit to help students begin to develop a strategy for finding Taylor series. What is written here is where the lesson has ended based on previous experience. However, you may choose to spend more time in some areas and not make it all the way through the suggested discussions. Any missing pieces can be picked up in the following lesson.

The big questions left at the end of this lesson is, "Will the infinite polynomial converge to the function?" and "If so, for what values of x will it converge?" One activity in the following lesson could be to use a graphing program to show what happens as more terms are added to the polynomials. For the three functions students have worked with, more terms will cause the polynomials to become increasingly better fits. This could lead students to think that adding more terms will always give a better estimate (and eventually get close to any given point on the graph of the function).

However, you could then explore functions with a small radius of convergence and what happens as the number of terms increases (such as $\ln(x)$ centered at $x = 1$, or $\tan(x)$ centered at $x = \frac{\pi}{4}$).

In the next lesson, a good place to start is to have students write out a general formula for finding the Taylor polynomial for any function. They should notice the similarities in the process for each of the three functions they found polynomials for previously. The Taylor series for e^x, $\sin(x)$, and $\cos(x)$ should be as following:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots .$$

Placing all three series on the board helps in highlighting the similarities among the series. The factorial pattern remains in all three series, and there is also an element of finding the nth derivative and plugging in 0 (for series centered at $x = 0$). A difficult part for students may be seeing that the power of x always increases by one, despite $\sin(x)$ and $\cos(x)$ skipping powers of x. If they write out each term, even those with a coefficient of 0, the pattern may become more apparent. It becomes more difficult for students as the center is changed, so begin first with a center of 0. Then you can have students experiment with what happens when the center is changed.

You can also implement the use of a graphing system, such as this one:

https://www.geogebra.org/m/C4S6CEdm

This program allows the user to change the function, the center, and the number of terms being graphed in the Taylor expansion. Experimenting with this can help students see how the center affects the Taylor expansion, or observe differing radii of convergence among functions. Teachers can draw attention to specific functions, like the three functions students previously worked on, as well as $\ln(x)$ or $\tan(x)$ which have limited radii of convergence. Students can also drag the point of the center to a new location and see how the written Taylor expansion changes.

In the end, students should arrive at the general formula for Taylor polynomials centered at a value $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x-a)^n = f(0) + f'(a)(x-a) + \frac{f''(a)}{2} (x-a)^2 + \frac{f'''(a)}{6} (x-a)^3 + \cdots .$$

One project that I have had students do in the past related to this lesson, is to create their own sine calculator in a spreadsheet such as Excel or Google Sheets. I have instructions on how to use some basic commands in the spreadsheet, such as a MOD() function, so any angle can be handled by evaluating sine between -180 and 180 degrees. They are responsible for figuring out how many terms they need to reach $\sin(x)$ within a certain accuracy. Student's can use the Alternating Series Estimation Theroem to justify why their calculator will always produce a value within a certain error of the true value of $\sin(x)$. I usually have students do this project outside of class, but it could be done in class as a lab activity as well. Many students are amazed to learn that the sine function programmed into the spreadsheet does something very similar.

7.5 Bibliography

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8

Congruence and the Nature of Geometric/Mathematical Knowledge and Truth

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Topic of lesson	This module uses the development of congruence to highlight key issues with definitions, axioms, and proofs.
Course context	Proof-based Geometry course for future secondary teachers.
Instructional Challenge	Math majors, especially future teachers, need to critique the disciplinary methods of mathematics. In order to start engaging these critiques, students need to participate in choosing definitions and axioms and in evaluating possible approaches to proving.
Brief overview of instructional approach	This module uses a short string of straight-edge and compass constructions and investigations of congruence, which is familiar to the students from secondary school. Rather than focusing on the results of these tasks, the discussions are facilitated to focus on the structure of our definitions, assumptions, and justifications. The class separates what they think is true from what they have justified, makes warrants (general principles used to support claims) explicit, identifies gaps in the construction axioms and modifies them, starts working with diagrams as representatives of arbitrary objects, distinguishes between objects that are given and those that are constructed, contrasts scientific and mathematical ways of knowing, replaces an explicit definition of congruence by choosing an undefined term and axioms, and negotiates the form of justifications for the rest of the course.
Keywords	geometry, congruence, epistemology, inquiry

8.1 An Instructional Challenge

As part of a panel in July 2019 at the Park City Mathematics Institute, Rochelle Gutiérrez asked the room: who has had a dehumanizing experience in mathematics? [5] Within the error bounds enforced by my vision, I saw everyone in the room raise their hand. And this was a group of adults who had chosen to spend their summer at math camp. Similarly, when strangers learn that I teach mathematics, they usually respond with a math confessional: telling the story of how they loved math until their brain stopped working, loved math until they had a terrible teacher, or cheated.

All of these stories are about what counts as mathematics, who gets to do it, and who gets to decide these first two questions. I believe that mathematics contributes to profound systemic dehumanization and epistemic violence when we allow our subject to be about knowledge without making the associated ways of knowing central to our teaching. Going even further, I believe that teachers who cannot critique mathematics as a disciplinary way of knowing are only equipped to allow this system to continue. This chapter is about starting teachers on a path to leading classrooms of their own that approach mathematics as a human way of knowing.

Epistemology is not the only way to think about dehumanizing mathematics experiences. For example, as a student and a professor, I have been asked to hide or ignore my queer identity in spaces where various forms of toxic, white hetero-masculinity are treated as useful, neutral, and invisible (to a majority of participants). However, bringing my favorite epistemological and ontological sensitivities to these experiences, I see that power in mathematics can be constructed unequally among people from the assertion that our mathematics is independent of us as people.

Given these beliefs, I think it is critical for college and university mathematics programs to build in experiences for students that allow them to consider the discipline consciously and to critique its methods, especially for future teachers. I think that intro-to-proofs should function like the disciplinary research methods courses housed in many other disciplinary programs, but it can be hard to critique a tool as you are learning to use it, especially an abstract tool. I see a college geometry course as a productive companion context in which students can look back on mathematics and participate in critiquing it and their experiences. First, the content of this course is more familiar to students than much of the other proof-based mathematics they have engaged in college, so they can focus on questions of epistemology rather than being overwhelmed by the goal of understanding. Second, this course is often required for future teachers and the concepts are directly connected to their future teaching; this connection helps avoid the issue that these new ways of thinking are only salient in collegiate mathematics and hence do not or should not impact secondary school teaching. Third, many students view geometry as the domain that requires proof, though I think we should make this thinking more visible in algebra in high schools for the reasons above. And fourth, geometry is at the heart of historical debates about our disciplinary methods. Moreover, geometry admits multiple, incompatible models; this incompatibility helps resist the temptation to collapse disciplinary methods into noticing truth because the truths cannot be universal in the same way.

In this chapter, I will attempt to make explicit the moves I make to build a geometry course that works toward the values above by describing the first week of the course in detail. This kind of critical inquiry into epistemologies is not sufficient for us to rehumanize the discipline and our classrooms, but I do believe that it is necessary.

8.2 Goals

The previous section lays out how I see some of the large themes in an undergraduate curriculum; this section makes these themes specific to the context of this course and articulates the instructional challenges engaged by this chapter. In this section, I will try to be explicit about my goals for three audiences: education researcher readers, practitioner readers, and students in the course.

8.2.1 Goals: Education Researcher Readers

I believe that this chapter will illuminate some of the challenges of articulating a written instructional product focused on abstract goals. I believe it will also emphasize coherence across the learning episodes and between the episodes and the larger course.

8.2.2 Goals: Practitioner Readers

This course will use inquiry as a central pedagogical tool. I think that there are two significant pedagogical challenges to starting up an inquiry course: (1) How will students learn what it means to do mathematics in this classroom, especially if they have only experienced didactic methods previously? (2) How will students come to value the work of the course sufficiently for their learning that they engage in it productively, especially when it can feel harder at first than other approaches?

These two questions are bound up in each other. How can students learn to value something if they cannot imagine it, and how can they come to participate in something cognitive if they believe it has no value? I will attempt to demonstrate how I develop the intellectual need for the course learning in ways that address both of these themes together.

8.2.3 Goals: Students

I have extensive goals for this instructional activity because this module opens the course. With this chapter, I am essentially asserting that the larger goals below need to follow from the smaller ones and need to be present in the opening of the course. I have organized the goals for students as *conceptual*, *process*, *beliefs*, and *meta-conceptual*. I use the grammatical structure "Students will. . ." for goals to operationalize them as (changes in students') actions.

Conceptual. The core conceptual goal for this first portion of the course is for students to develop intuition for congruence while we establish it as an equivalence relation based on an undefined term about which we will assume axioms. Students will use definitions and axioms related to congruence while being careful about implicit quantifiers. Similarly, students will make sense of straight-edge and compass construction tools while critically attending to the ways these statements are quantified.

Students will grapple with the challenge of demonstrating a proof for a general claim using a picture of only one instance of that claim by beginning to attend to only using general properties and by imagining the instance as part of a dynamic collection of instances (using a tool like GeoGebra).

Students will make meaning of the difference between proofs that require them to construct an object and statements that allow them to begin a construction given such an object.

Process. Students will agree on the form of an argument in geometry as a sequence of constructions and associated claims warranted by straight-edge and compass constructions, axioms, and previous results accepted by the class community. Here, "agree" means that the students will negotiate an explicit agreement about this process, but it is also a meta-conceptual course goal that the students make individual meaning of this process.

Students will condense the use of physical straight-edges and compasses into more abstract processes that can be represented more abstractly.

Students will develop basic fluency with dynamic geometry software such as GeoGebra for constructions. The students will also start developing skill with mathematics communication tools such as a wiki and LaTeX.

Beliefs. Students will start to view axioms and definitions as human choices. Students will start to identify learning that they need to do in geometry, as opposed to seeing the course as being about content they already know and are prepared to teach; students will connect the course work with this identified learning.

Students will begin to problematize their beliefs about truth and knowledge in mathematics by starting to notice, name, and question formerly implicit ideas about mathematical ways of knowing.

Meta-Conceptual. Students will begin to reify proof, transforming it from a process to an object that has structure that can be studied.

Students will participate in pedagogies like those they are being asked to use while attending to how they work and how they feel as students.

Students will begin to reorganize their understanding of geometry. They will consider contrasting options for the foundations of geometry and some affordances of transformational and axiomatic foundations, both of which are consistent with the Common Core State Standards - Mathematics (CCSSM), http://www.corestandards.org/Math/.

8.3 Background Information and Contexts

I have taught this course at a small, predominantly white, private liberal arts college and a regional, public, Hispanic-serving university.

At the college, the class had between five and thirteen students. The course was required for pre-service secondary mathematics teachers, and almost all students in the course were considering teaching mathematics in middle or high school. The students were almost all juniors who had taken an introduction-to-proofs course and proof-based Linear Algebra, Abstract Algebra, and Real Analysis courses. They were generally also taking a secondary mathematics teaching methods course during the same term, and I collaborated with this methods course instructor informally on this experience. Through this other course, the students were generally doing clinical observations of secondary teaching in preparation for a student-teaching experience. The students were generally well-acquainted with each other, and as a result they had formed a supportive social group before taking this course. Most years, a small number of these students had taken a previous course with me, usually Calculus I or II.

I taught this course at the university as part of my sabbatical; I was at the time and am still less familiar with the institutional factors for this context. At the university, the course had approximately thirty students. The course was primarily intended for pre-service secondary teachers, but there were students taking it as one of their math major electives for other reasons. The students ranged in their class standing, but I believe that the majority were sophomores or juniors who had taken an introduction-to-proofs experience in some format. Some of the students had taken Linear Algebra, Abstract Algebra, and Real Analysis while others were taking one or more of these or planned to take them in the future. Some of the students knew peers from other course work, but most knew few or none of their peers in the course well when we started.

In all contexts, the students were familiar with some of the concepts of Euclidean geometry from secondary school course work. Beyond the requirement to engage Euclidean and non-Euclidean geometries, I had significant flexibility with the course, and at the college my version of the course was designated as part of the core curriculum options for "perspectives on human values and existence" because of the emphasis on epistemology. In both contexts, this course met for approximately thirty 75-minute, synchronous, in-person sessions. I do not think that this class length is critical, though it certainly impacts how many of the activities below happen in a given session. Students had access to computer labs outside of class, though many had other access to laptops; when a session required technology, class met in a lab or I was able to bring a set of computers for the groups.

I selected "Euclidean Geometry: A Guided Inquiry Approach" by David Clark [2] as the core text for this course. This text has been used with a range of courses and students, including 9th Grade Geometry and courses like mine, which function as undergraduate capstone courses for pre-service teachers. Leveraging this text is core to the ideas below, so I will discuss the pieces of it that are directly salient to the instructional approach below. More generally, this text is a carefully constructed sequence (series?) of questions and tasks to be explored and statements to be proved. The text supports a careful axiomatic approach to justifying geometric claims. Clark positions the text as follows:

> In contrast to Hilbert, we will begin with a foundational base large enough to reach our content goals in a single course. In contrast to Euclid, our foundational base will be sufficient to yield our content goals through tight rigorous proofs. Our foundational base will add to a standard naive understanding of logic and set theory three foundational principles, ten axioms and three unproven limit theorems.

Three features of this approach are particularly important here.

First, each axiom is added when all possible approaches to justifying appear to have intractable gaps, and it is assumed at that point in the flow of the tasks, not at the start of the text. For example, my students regularly say that the most memorable day in the course was the one in which they attempt to construct a square. Over the years, students have generated many beautiful constructions, but in each case they cannot justify that their construction produces a square, either because they cannot support the existence of a point of intersection or because they cannot prove that certain angles are congruent in service of showing that the object they constructed is a square. This struggle is the result of not yet having assumed a version of the parallel postulate, meaning they are working in neutral geometry, where squares do not necessarily exist. Curiously, Clark's text is explicit that they will not be able to succeed in this construction, but that hint means nothing to the students until they have tried to construct a square for nearly a whole class meeting. Once they begin to suspect that the task is impossible, they can quickly move to convincing themselves and each other that we need another tool, and the axioms suddenly become artifacts of human choices and agency. It takes some support from me to help them analyze the previous axioms through this new lens, but the subsequent ones are approached from this perspective by students.

Second, I disagree with Clark's claim about "tight rigorous proofs." However, unlike many sets of geometric curricular materials that do not have a clear set of assumptions or that focus on conceptual intuition without attending to the differences between conjectures and theorems, this text is consistent with a careful, axiomatic development. In particular, we need to make assumptions about the plane related to Hilbert's betweenness axioms that are not stated in the text. I think it is completely appropriate to assume these facts quietly/implicitly and move through the text as it appears (and this is what I would do with younger students), but with this population, I push them hard to have warrants for their assertions. They cannot in general warrant the existence claims for many of their intersections, leading to identified gaps in our proofs. As the course goes on, we build a list of these gaps and then try to craft a (minimal) set of new axioms to fill these gaps. This push from me is, I believe, necessary, for the anecdote above about constructing a square to work out as it does. We generally end up with two more axioms written by the students, one about when two circles intersect and one about when a circle and a line intersect. The existence of intersections between lines comes from our parallel postulate.

And third, the ordering of the concepts feels unfamiliar to the students. For example, we wait until the seventh axiom, which comes in Chapter 6, to measure angles. As a result, students hold strong beliefs about conjectures (especially that all right angles are congruent), but the simple proofs they might have given in high school are unwarranted or flawed. They do prove, without measuring angles, that all right angles are congruent, which is another moment of significant pay-off.

Taken together, these aspects of Clark's text allow me to frame our discussions around how we, as a class community and as members of humanity, know the things we think we know about geometry. The other two texts for the course are "Incompleteness: The Proof and Paradox of Kurt Gödel" by Rebecca Goldstein [4] and "Logicomix: An Epic Search for Truth" by Apostolos Doxiadis and Christos Papadimitriou [3]. With the experience of attending carefully to our knowledge through our work with Clark's book, we use these other texts to look directly at questions of epistemology and the nature of mathematics.

H. Wu [9] has argued that undergraduate programs for future high school mathematics teachers need sections of the advanced courses that focus on connections to their high school teaching. I certainly agree that we need to help future teachers connect their college learning to their future teaching, but I disagree with Wu's assertion that this can only be done by separating the future teachers from the rest of the mathematics majors. Every student can be challenged to do meaningful, deep mathematics by critiquing and extending their past thinking from high school. I do, however, welcome the context of a course for future teachers because these students are more enthusiastic about the pedagogical dimensions of our discussions.

8.4 Instructional Arc

The focus of this instructional arc will be the second and third days of the course, but I will also connect the arc to what happens before and after. This discussion intermingles the tasks, the likely student thinking, and my rationale for choices; I have chosen to combine these threads because they depend on each other at each step, so separating them would require significant repetition.

8.4.1 Before the arc

When the context has allowed it, I have asked students to read "Logicomix" [3] in preparation for the first day of class. This graphic novel helps them build some historical context for the subtle questions of the course and sensitizes them a little to the epistemological questions we will explore. The actual ideas about mathematical knowledge and truth generally do not make sense to students, but it is useful that they know that there are questions to be asked.

The first activity in the course is a concept mapping task. I give the students a set of terms related to mathematical truth and ask them to build a concept map with those and a few of their own additional terms. I have written elsewhere about these maps [6], but in summary, the pre-course maps make almost no sense to mathematicians. Students also regularly treat definitions, theorems, and axioms as interchangeable or at least sisters in their maps; mathematicians often find this worrisome because some of these are assumptions or choices and others are conclusions.

The Logicomix reading and the concept mapping task are not leveraged directly in the arc I will describe below. They do prime the students to notice when I draw our discussion toward the nature of our knowledge rather than the content of our claims, but both tasks are really the first half of tasks that will play out later in the course once students have used our work with geometry to become consciously aware of their own thinking about mathematical truth and knowledge. I think a little of this priming might impact the main arc below, but it could be possible to do that in an introductory email with a description of the course: "This course considers familiar concepts of geometry from an advanced perspective. We will ask how we know the things we claim to be true in geometry and consider whether they are actually universally true.".

The other task on the first day of the course involves discussing the course objectives. I want to motivate both the need to consider geometry from an advanced perspective for future teachers and the pedagogical approaches of the course. I like to show a particular video of a high school teacher [1] running a class about parallelograms (as tangrams of triangles) that also includes some reflection to camera by the teacher after the class. Sadly, Annenberg Learner has removed this video since the last time I taught the cours, but I expect to find a similar video before the next time I teach the course. The students and I agree that the classroom was positive and productive. I ask the students to watch the video and consider what knowledge, skills, and beliefs the teacher needs in order to be able to run the session we watched. My students point out skills and beliefs related to teaching with inquiry, including that she needs to have skills with listening rather than just talking and she needs to be able to make sense of messy mathematics in progress; I use this to point out that we will practice this so that they get to experience these skills as students before they try to teach. My students also point out that the teacher needs to be able to make sense of student thinking and in particular that she needs to be able to clarify the assumptions made by a particular student speaker and reorganize her own approach from that thinking as a starting point; I use this to motivate our focus on why we believe what we believe, exploring geometry that is organized differently that their previous learning of it, and being very careful about which statements have been accepted as true by the class community.

If there is time, I will say a few of my own thoughts about mathematical truth or foreshadow the course structure and likely challenging moments. After this first day, students read the syllabus and explore the course website thoroughly.

8.4.2 The main arc: Lesson Implementation Day 2

The first activity about congruence asks students to attempt to complete some straight-edge and compass constructions. The handout [Appendix 8.6.1] states four allowable construction moves: (C1) given two points, drawing a line/segment/ray through them, (C2) given a point and a segment, drawing a circle with that center and radii congruent to that segment, (C3) picking (an arbitrary) point on a line or circle, and (C4) locating the intersection of lines and circles, assuming those intersections exist.

The handout then asks students to work on the construction tasks, pointing out that some may not be possible given the tools. We generally discuss the first five tasks:

T1: Given a segment AB and a ray $\overrightarrow{CD}$, find a point E on $\overrightarrow{CD}$ such that $AB \cong CE$.

T2: Construct an equilateral triangle. Given a segment AB, construct an equilateral triangle whose sides are all congruent to AB.

T3: Given three segments AB, CD, and EF, construct a triangle ΔXYZ so that $XY \cong AB$, $YZ \cong CD$, and $ZX \cong EF$.

T4: Given $\triangle ABC$, construct a different triangle, $\triangle XYZ$, such that $\triangle ABC \cong \triangle XYZ$.

T5: Given an angle $\angle ABC$ and a ray $\overrightarrow{DE}$ (and a side of $\overleftrightarrow{DE}$), construct a point F (on the selected side of $\overleftrightarrow{DE}$) so that $\angle FDE \cong \angle ABC$.

I give students twine and whiteboard markers. With multiple people, the twine can be held as a straight-edge and wrapped around a marker while the string is held in place at a point to be a compass. I like this version of a compass and straight-edge for several reasons. First, these materials travel easily and I can provide them at minimal cost, so I can give them to students and ask them to bring them back to class. Second, these tools are much better for the collaborative whiteboard than paper, and they require coordination within groups. Third, they are messy; students are immediately suspicious that their pictures are imperfect, which supports them moving to an abstracted version of the plane and their constructions that is pointed to by the diagrams rather than treating the diagrams as the objects themselves. And finally, they are a bit annoying to use, so students quickly decide for themselves that they want to move on to more powerful and abstract tools.

The students work in groups on these first tasks while I circulate to support and challenge them. T1 gets students to read C1–C4 carefully and make physical meaning of them in terms of a compass and straight-edge. Some students will initially assert that T1 is trivial, saying "You just copy it.". In this situation, I ask them which of C1–C4 allows them to do that, and they start to use these tools explicitly. Generally, this will cause a smaller portion of the students to swing the other direction, noticing that C4 only works "assuming those intersections exist". If they get stuck on this point, I say that today it is OK to assume that intersections that appear to exist do in fact exist.

T2 builds core construction techniques of drawing circles with congruent radii at two different centers or drawing circles at the different ends of a segment. These techniques are critical for many of our early constructions. T3 is an extension of T2 with less symmetry, but students approach it similarly. Generally they view T4 as trivial, and to conserve discussion time I usually tell them to skip to T5. T5 is a more interesting extension of T3. Students generally realize that they have no tools for copying angles (sometimes I ask this explicitly to draw their attention); once they think to copy the triangles and get that the angles are congruent, they easily use the construction technique from the previous tasks to complete this new one.

When all groups are ready for discussion, meaning that they are ready to contribute meaningfully and understand a discussion of these five tasks, not that they have written up solutions for each task, we move on to a full-class discussion and presentation of the constructions. In this case, this is both about the group's animus toward the ideas and whether they have generated a reasonable approach to T3. I will generally ask a group that has not progressed fully into T5 to go first to make sure they get feedback and get to contribute. Generally, while they are presenting, I mimic their construction on GeoGebra, which is projected in the room. At the end, I move one of the initial objects (usually a point) in the GeoGebra construction, which I think quickly cues a model in students' minds for doing the construction in all cases simultaneously by only using properties that are independent of this dynamic change. All of the groups make the transition into using C1–C4 carefully during the group working time, so this presentation is generally quite slick and focused, though I will also ask how they thought differently about it at first and why they moved to what they presented. I use this discussion to point out that I want the uses of constructions C1–C4 to be explicit in our class, at least for now. Sometimes a peer will ask how we know that the ray and circle in the construction intersect, allowing the presenters to use C4. Sometimes a student (who is usually the one most attached to empirical proof schemes) will assert that it does because how could it not, looking at it, or the line would have to be a spiral. If the students try to get me to weigh in here, I will generally playfully deflect the question to turn it back to them while validating it as a concern, but if they get stuck unproductively I will allow that for today we can accept that intersections that appear to exist do exist, as I said to the groups. But they should be cautious about this.

I ask a different group to present their work on T2. If their group is confident or has a tech-comfortable student, I may ask a non-speaking group member to rebuild the construction live on GeoGebra, though given time it is usually easier for me to do it or to have them present by telling me what to do in the software. All groups are able to generate the idea of using multiple, intersecting circles, and their constructions are essentially identical, so this presentation is also quick, and students find the dynamic GeoGebra result compelling that the construction works in general.

After we have completed this presentation, I ask each student to count the number of times that we called on C1–C4. After time to think, I ask the students to say their numbers out loud. There is always significant variation, so I ask them to discuss in groups. There are questions about whether drawing two circles with radius congruent to the same segment is calling on C2 twice or once (similarly with C1), but once we agree on an interpretation for this count, there is still disagreement, and the students numbers are generally higher than mine. If needed, we list all of the claimed calls. I ask students why my number/list is smaller than theirs. Looking back carefully at their calls to the constructions (warrants), a few students will notice that they were presenting a version of T2 that starts with "Given a segment AB", and they were counting the constructions for constructing AB.

Then I ask a different group to present their construction for T3, pre-praising them for incorporating the feedback from the two previous groups about how to be careful and convincing in a presentation by citing their calls to the axioms explicitly. I start them by drawing the segments on which to apply their construction. The students generally start by commenting that they are given, not constructing, these segments, and then they proceed to construct a triangle carefully. The construction requires intersecting two circles (hence using C4 to find the intersection). Either the presenters or other students will tag that as a gap. After we have agreed on this construction, I ask how many of the students think I'm worried over nothing about the "assuming they exist" point and how many think it's a real concern. Usually, almost everyone thinks it's nothing. So I ask the group that just presented to repeat their construction on three new segments; I have chosen these segments so that the lengths of the two shorter segments sum to less than the length of the longest and as a result the circles do not intersect. This realization happens quickly, often with presenters or an audience member jumping right to the problematic step in the construction. I ask again if we need to worry about intersections, and the students agree. I put a pin in this discussion, saying that we want to note the places in which we are concerned about gaps, but we will return to this issue when we have better tools to support our justification.

We do not present T4, though if we have time, we have a quick discussion about strategies; students usually generate two different strategies. In one strategy, students point out that they had to pick one of two circle intersections in their construction, and picking the other produces a different triangle (on the other side of the first line segment). Alternately, the construction could start by picking a point other than an endpoint of the first segment and constructing a new first side of the triangle. If it hasn't happened before, this will cause students to point out that C3 actually only allows us pick points on lines and circles, not points in the plane. We agree that this needs to be revised, and we often agree to revise it into being able to pick a point in a non-empty subset of the plane along with some claims that our plane/points/lines are infinite and hence the complement of any finite subset will be non-empty.

I've made sure to reserve a group to present T5 that had completed this task. All students are comfortable by this point in the discussion that we should draw some circles, but there are choices to make about which circles to draw and when/where, so once the group starts asserting that they will draw a circle, I ask them why they chose that circle (or those circles). If they cannot answer this question, I let them finish the presentation and then ask again why they drew the circles they drew or what roles each circle plays. The class as a whole is able to articulate that one circle finds the second endpoint of the edge that will end up on the ray while the other two circles' intersection(s) determine the location of the third point. Once this point is located, the construction of the ray and the observation about the congruent angle is direct.

At some point, a student will likely ask whether they could just "copy the segment onto the ray using T1". I'll ask the room, and we generally agree that this is logically equivalent to what was done. I will comment that, in general, citing a previous construction task is equivalent to reconstructing it, and so when you cite a task you should quickly recall how it was built, but also this simplifies the diagrams. It's an important point about how our knowledge is built to which we will return later (as hyperlinking in the course wiki).

This is the end of our second day of class. After this class, the students do a little learning of the wiki technology [7] by writing up some of our arguments from the day, but the purpose is to get some experience with the wiki, LaTeX, and GeoGebra. This work helps them to reflect on our discussion (both the constructions and the subtleties that came up) as well as to see the knowledge in the course as co-constructed and co-owned. This work is aligned with Clark's Section 1.3, so I will ask them to skim this section now.

8.4.3 The Main Arc: Lesson Implementation Day 3

In preparation for Day 3 of class, the students read Sections 1.1 and 1.2 of Clark's text and attempt Problems 1–12, which I will describe below. These materials define congruence in terms of being able to move a copy of one object on top of another and have them match perfectly. In other words, at this point, this is a transformational approach to geometry, consistent with the beginnings of CCSSM [8]. From there, we need to build to the familiar axioms, such as "Side-Angle-Side" (SAS), as well as the other theorems of Euclidean geometry. My approach is to use this transformational context to establish these familiar axioms as axioms rather than as theorems based on transformational axioms in part because these materials are not sufficient for developing transformations deeply and in part because this move is interesting epistemologically, as you will see below.

Problems 1–5 ask students to consider six figures called "cyber-frogs" (Figure 8.1 labelled A–F) that look like frogs if you had to draw them in Graph Theory. In particular, Problem 1 asks students to find two congruent cyber-frog and to explain how to move one to coincide with the other; Problem 2 asks students to find two cyber-frogs that are not congruent and to explain how they know they are not congruent; Problem 3 asks the students to write down the motions needed to show congruence for every pair of congruent cyber-frogs; Problem 4 asks students to describe a feature that shares only with the figures to which it is congruent; and after a definition of congruence classes Problem 5 asks students to draw out the congruence classes from this set of cyber-frogs. We will re-engage generalized versions of these tasks in this third class meeting, so I will give versions of the broader prompts below. Problems 6–11 repeat similar explorations with other objects, and I do not engage them as deeply in class. Problem 12 is a very interesting task in which students notice (in the style of Galois theory) that equivalence relations with the same objects but fewer relations (in this case the subgroup of only translations rather than all transformations) produce more but smaller equivalence classes, but the larger equivalence classes of the original relation are made of disjoint unions of these smaller classes. Problem 12 could take a full class session of exploration for students to abstract from the particular equivalence relation and the details of transformations and congruence into this larger observation. As a result, when I have engaged this Problem, I have done so by coming back to it later in the term. Problem 13 asks students to apply the definition of betweenness to a set of points; Problem 14 asks students to draw the endpoints of congruent segments all with the same initial endpoint, thereby reinventing a definition for circles; Problem 15 asks students to compute unions and intersections of segments and rays; Problem 16 asks students to describe transformations that will show that two triangles are congruent; and Problem 17 asks students which of SSS, AAA, SAS, SSA, ASA, and AAS is enough to guarantee congruence.

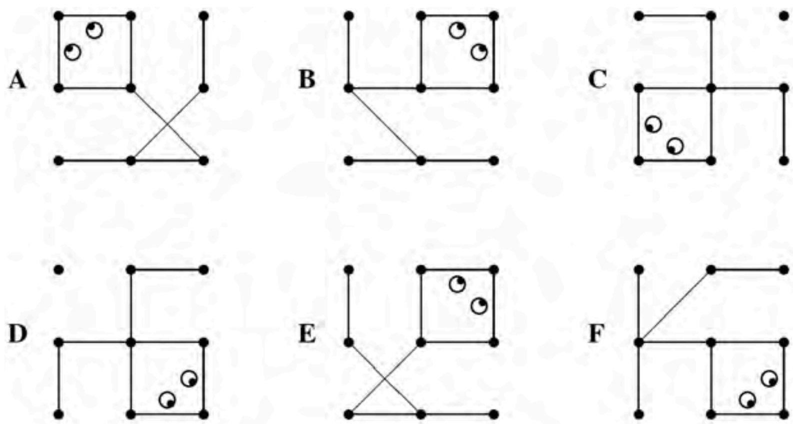

Figure 8.1: Six Cyberfrogs

Returning to the core of the day's tasks (Problems 1–5), notice that the notion of a transformation is not carefully defined, and the types of transformations (translations, reflections, rotations) are not named until after Problem 5.

While most days of this course involve students discussing in small groups what they did in preparation for the day, at this point the students see the tasks as not having much depth, so I choose to go straight into having individuals share and the full-class discuss. Here is how I implement this day of class.

Problem 1: Find two cyber-frogs A to F that are congruent. Explain in words how you would move one to make it coincide with the other.

I am careful to select a student who will respond well to having their ideas challenged, meaning that they will both remain confident in their ability to make sense of ideas and that they will incorporate the new ideas as they go. In my experience, this is a gendered phenomenon in several ways, but it is especially important that this first presentation of thinking not be dismissive of the subtleties from a student who appears to be from privileged identities. In general, I select a student who has had a course with me before if possible so that I can be more confident in this assessment and so that I can frame this choice in relation to our past familiarity. If possible, I inform this student about this role in the minutes leading up to the start of class. Once I have selected a student, I ask them to go to the document camera and share their thinking. We usually present from group writing at the boards, but most of this discussion references these images in the text, so being able to gesture at them directly feels more natural and comfortable to the students.

Cyber-frogs C and D are congruent, as are cyber-frogs A and E, but B and F are not congruent to other cyber-frogs. The first student presenter usually selects C and D for this task. I believe this is because the transformation does not require a reflection. The presenter's transformation amounts to saying that they would rotate one of the cyber-frogs. This makes intuitive sense to everyone, but I want us to be more careful, so I generally have to prompt the class to consider this explanation.

First, there is significant opportunity to connect to prior geometry understanding. I ask the room what it takes to describe a rotation. As a group they usually quickly articulate that a rotation needs a center of rotation and an angle of rotation. They generally do not articulate that the angle needs an orientation. If that point doesn't emerge, I can ask them to describe a rotation that I will execute, and I willfully misinterpret the direction of the rotation. Once this point is salient, student seem to quickly agree that direction is just a convention. Now that the rotation has been described precisely, I ask one of the students to read aloud the working definition of congruence from the text. At this point the group realizes that the rotation puts the two cyber-frogs in matching orientations, but the first does not coincide with the second without a translation. The presenter is usually able to articulate that a translation is needed and finish the task. Sometimes they can articulate a precise translation, but they often need to pull in the full class. Given the context of students with an advanced Linear Algebra course as a prerequisite, the group will either say that they translate by a direction and distance or by a vector, which are equivalent in my reading. I then ask them what it takes to describe a vector, and students are able to articulate that it takes either two points or a direction and length. I will comment that both are useful but throw it back to the presenter to give the whole presentation again now carefully. The student will generally articulate it like this: "Take a copy of cyber-frog C, rotate it about its center point by 90 degrees counter-clockwise and then translate by the vector defined from the center of C to the center of D.".

At this point, I need to make a choice based on my assessment of the presenter's stamina. If they are going strong, I will finish by asking if the group is certain that the copy coincides perfectly with the second object. This is intended to stun the group, and it works. Once I've asked the question though, students are easily able to assert that they could never check that two figures match perfectly because of the challenges of perception. Some students will assert that it's close enough. I respond to all of this by commenting that this means that if we want to be able to write proofs, we are going to need some more abstract knowledge; for example, we would need to know in advance that certain angles and lengths match, and even then we do not have a good sense if we have described all of the salient features, making this approach challenging. This discussion is about contrasting mathematics and science as ways of knowing; I will tag this observation explicitly. If there is a student in the room who identifies as a science student with a second major in math, I will ask them about the differences in justification between their math and science courses; if not, I will reflect out loud on my own undergraduate degree in chemistry. A big goal of the rest of this class session is to resolve this issue of how we can know for certain that two objects are congruent if perception can't be trusted.

If the first presenter is running out of confidence or stamina, the discussion above can happen after the second presentation for this task. A second student presenter will select to show that cyber-frogs A and E are congruent. They are able to be careful with the translation, but sometimes they struggle to bring the reflection up to the level of specificity of the other transformations. Whether they are able or not, I will ask the full group what it takes to describe a reflection, and they are able to articulate that it takes a line across which to reflect. With this having been made explicit, the second presenter is quickly able to give a careful transformation from one of these cyber-frogs to the other. This pair admits multiple reasonable sequences of simple transformations, including some that do and some that do not use a rotation, and I like to get quick summaries from each student of their personal sequences if the group is small. If there is time, we could discuss their preferences and evaluations of these, which usually rest on either efficiency or intuition (often relying on the transformations being simple in the implied rectangular grid students have imposed on the page based on the orientation of the edges and text).

At some point in the discussion of a rotation or translation, a student will say "until" in the sense of "rotating/translating until they match". I will often say that I have issues with the word "until" and ask them to be careful there. Students generally replace "until" with similarly problematic phrases like "so that", which assume that a process terminates or has a solution when that existence is the goal of their argument. If a peer is not able to articulate why this term also concerns them, I offer metaphors as needed. For example, what could go wrong in a computer program if we had a loop that added 1 to an integer and ran "until" it got to the biggest number?

Problem 2 is flexible; it can be discussed here or after Problem 3. Given the connections between Problems 1 and 3, I will discuss Problem 3 in this development first. Problem 3 asks students to write down ALL of the congruences in the cyber-frogs A–F. I ask the students how many congruences that requires. After the discussion above, students will sometimes start with two: $\{C \cong D, A \cong E\}$. I say that I have more, and several of them add $\{D \cong C, E \cong A\}$ to their lists. I ask someone who hasn't already presented a congruence to walk us through their transformation for one of these. I'm trying to generate intellectual need for a more general approach here. When this is done, I say that their lists are still smaller than mine. They will often double-check that B and F are not congruent (which is the reason to do Problem 2 first or here, so that we have established a technique for proving non-congruence). If no one generates any more, I will point out that I have 10 congruences. A few of them will notice that I have 6 more than they do and will connect that to the 6 cyber-frogs and will add $\{A \cong A, B \cong B, C \cong C, D \cong D, E \cong E, F \cong F\}$ to the list. Again, I ask a new speaker to give a transformation showing one of these new congruences. There is usually a discussion of whether "do nothing" is acceptable directly as a transformation or whether we need to view it as a rotation by 360 degrees to think of it as a transformation, but the students accept this congruence.

Now I ask them if they want to do this "the hard way" every time or if they have some general conjectures that they would like to articulate and prove. The students easily articulate reflexivity and symmetry. Writing these out as quantified statements is often enough to get a student to remember that this is a third property (transitivity) that often appears with these, but if not I will ask the students to name these properties and that cues the prior knowledge, at which point we reframe our conjecture as the claim that (the transformational definition of) congruence is an equivalence relation. Transitivity is hard to see from the pairs in general, but it is especially hard to see when the largest equivalence class does not have at least three elements. We then prove each piece. Reflexivity is proved (by a student) by saying that we already accepted "do nothing" as a transformation, so any figure is congruent to itself. Generally transitivity comes next when a student observes that we can just do one string of transformations then the other. Sometimes a student will ask if transformations need to be a single move or not, which is an interesting connection to both closure and associativity (which I might tag after it has been articulated by a student), but the students quickly resolve this by pointing out that we already accepted transformations that were combinations of our simpler transformations without requiring them to be described as a single "motion" for which we have a name. Symmetry is the most interesting. I will frame our task as: given a transformation from figure X to Y, called T, what transformation do we know exists from Y to X. After the discussion related to transitivity about complex transformations built from the simpler ones, a student or I will frame T as made from simpler transformations. Students often get stuck at "undo" here, so I move to a socks-and-shoes metaphor. If after a shower, I put on underwear, a shirt, pants, socks, and shoes to go to work, how should I go from work to get ready to take my next shower? Now students are able to access prior learning about inverse functions to point out that we should reverse each move and reverse the order. In other words, if $T = T_n \circ \cdots \circ T_2 \circ T_1$, then $T^{-1} = T_1^{-1} \circ T_2^{-1} \circ \cdots \circ T_n^{-1}$ if these inverses exist (which should be familiar for students who have studied Abstract Algebra), so we need to check that each basic transformation can be undone. The questions and student ideas in this paragraph are almost exactly what gets said out loud or written on the board, and in the same order. Note that Clark's notes suggests that rotations, translations, and reflections are sufficient to describe all transformations, and I allow this assumption to be accepted by the class often without comment. At the same level of precision above, students are able to articulate the inverse transformations of each of our basic transformations, so our proof that congruence is an equivalence relation is complete.

Now we return to Problem 2. Clark's text asserts that corresponding parts of congruent figures are congruent (CPCFC) and asks students to use that to show that two cyber-frogs are not congruent. Two important ideas will arise. First, we do not have a careful definition of a transformation, but the text asserts that they are isometries. As a group, we have to accept some properties of objects that will not change when transformed by an isometry. Given the particular objects, degrees of vertices are clean and compelling to students, as are crossings in the images and being on the outsides/center of a figure. Once we accept these as invariants under transformations, students are able to justify that their desired pair of cyber-frogs are not congruent. However, when a student presents their argument, they will implicitly use the contrapositive of CPCFC. This is subtle because it's not just that if corresponding parts are not congruent, the figures are not congruent because the problem could be that these are not the parts that should be considered corresponding! Looking back on their argument of non-congruence, the students are able to reframe their claim as something like "cyber-frog F has an external vertex of degree 4, and no vertex in the other cyber-frogs has that property, so cyber-frog F is not congruent to any of the other frogs". If it did not arise in the discussion, I will ask if there were other ways that we could have proved non-congruence, and the students point out that none of our definitions/results at this point conclude with "are not-congruent", so we did in fact *need* to use a contrapositive. This is the beginning of a significant pattern in the course of considering whether our complete body of tools could possibly conclude a desired result. At this point, we observe that our discussions about congruence classes have addressed Problems 4–5 as well.

The approach above is intense, for the students and likely for the reader. I will ask the students if they want to discuss Problems 6–11, and they always agree that our ideas so far are sufficient but that they don't want to talk through those details. This observation is what I hoped would happen. I believe that Clark has included this repetition so that the text can be more accessible for users (including 9th grade classes) that do not have prior experience with congruence, but I do not need to leverage it in class here beyond the students considering these tasks as a warm-up before class.

If we have time, I will ask the students to talk through Problems 13–17 in small groups. Most of these problems introduce some of the language and concepts of points, lines, and angles, but the work is not critical at this moment or repeats what we have done. Problem 17 asks students to consider which of the six possible scenarios of knowing three pieces of information about a pair of triangles should be enough to know they are congruent: SSS, SAS, SSA, ASA, AAS, AAA. If we are short on time, I would only ask them to consider Problem 17 in groups. Students generally recall the Euclidean responses from previous course work without justification.

At this point, we have developed significant meaning from the concept of congruence, but we have also identified that we will never be able to prove that two physical figures are congruent without more abstract knowledge. It would be appropriate to prove Side-Side-Side and Side-Angle-Side congruence theorems as a consequence of transformational assumptions, but I choose a different path for this class. Instead, I explicitly ask the students to take congruence as an undefined term. Drawing their attention to CPCFC is a useful lever here because to them it feels circular: congruent means that parts are congruent!?! Here comes the hard part of teaching the session: pulling the philosophy of mathematics and geometry out of students' nascent thinking in ways that are consistent with the rest of the coming development. Notice that up to this point, our discussion has been about congruent figures. So I ask the students, what are the simplest geometric figures? They will identify points, line segments, and circles. Now I ask what assumptions we would need in order to know that two of these objects are congruent. Students will generally agree that all points should already be congruent without extra assumptions, but line segments should be congruent when they are the same length. We will pause here and read Axiom 1 in Clark's text, which asserts that congruence is equivalent to length for segments as well as two other assumptions that make length function as measurement. If the students have explored Problem 14 (which asks students to reinvent the definition of a circle by asking them to find all points X such that OX is congruent to OP), I will ask one of them to summarize; if not, we will look at it live. From this, the students will assert that circles are congruent when their radii are congruent, and we already have a notion of congruence for line segments from Axiom 1. Now I ask if this feels like enough to be able to discuss more complex figures, and if not, what is the next most simple figure? Students will identify triangles; if other polygons are also suggested, a student or I will suggest that polygons can be broken into triangles, so perhaps triangles are enough. So I ask, based on what we've already assumed, how we might hope to know that triangles are congruent. At least one student will point out that SSS is like extending our beliefs about length (an assertion that the 1-dimensional length of the sides is sufficient to know the congruence of these 2-dimensional segments). Usually someone will point out that we have no angles in the discussion, and in particular that nothing in our tools would allow us to use angles to prove congruence, so we probably need something else that concludes triangle congruence based on at least one angle. CPCFC allows us to conclude that angles are congruent, however. At the end of this discussion, we have loosely agreed to set aside our definition (and intuitions, at least in justifications) of congruence and to accept (i) length as equivalent to congruence for segments (Axiom 1), (ii) both SSS and SAS as tools that allow us to consider 2-dimensional figures, which we locate as Axioms 2 and 3 in the text, and (iii) that congruence is an equivalence relation and satisfies CPCFC (but is otherwise undefined).

The class ends with me summarizing for the students that the work they will do for class for the next few weeks is to try to prove the upcoming theorems and to explore the more open-ended tasks in Clark's text, justifying their claims using the constructions C1–C4, Axioms 1–3, and the assumption that congruence is an equivalence relation satisfying CPCFC while taking "congruence" as an otherwise undefined term. This concludes our work in Chapter 1, but the Theorems at the start of Chapter 2 are familiar statements that the students are now able to prove carefully.

8.4.4 Post-Lesson Considerations: After the Arc

The fourth day of class feels like settling into a routine, finally. The students prepare proofs of statements, some of which are familiar from our constructions on Day 2, but now they feel like they have a clear set of accepted tools they can use, so their proofs and discussions are very focused. This pattern continues for many days, with the next major hurdle being that we want all right angles to be congruent but do not yet have tools to prove it.

As discussed above, almost everything in the course is set up by the themes in the two days of this arc. I have written about this course elsewhere [6, 7], so I will limit myself to three connections across the course.

We will continue to think carefully about axioms and definitions in relation to our collective mathematical activity by asking why the axioms appear where they do and by considering the structure of each axiom. For example, while I don't mention it on Day 3, we have two other measurement axioms coming, one for area and one for angles. As we encounter these new axioms, I ask students to compare and contrast these measurement axioms to notice patterns in their structures. By the time we get to the third one, they are building a concept of a measurement axiom from these structures as requiring a choice of scale and assumptions about containment/additivity.

The habits of stepping outside our knowledge to see what we could possibly conclude and to decide when and how to make new assumptions will continue. Similarly, contrasting mathematical knowledge to knowledge in other disciplines, especially physical sciences and computer science, will continue. The students are writing a textbook based on the work inside class in the form of a hyperlinked wiki [7], and later in the term they will build a visual representation of this hyperlinking structure to consider their system of knowledge as an object. The last third of the course will shift to explicit discussions of the philosophy of math using the other course texts, but geometry plays key roles in that discussion. Historically, Hilbert was proving claims about the nature of geometry, and attempts to prove Euclid's parallel postulate ended up generating incompatible geometries. But the ability to look at what we built as a body of knowledge is also key in these reflective discussions.

The students from this class regularly contact me while they are student-teaching asking if we can work on adapting one of the explorations for their context, suggesting that the pedagogy and content both feel connected to their future work. Moreover, we spend some time asking about the structure of Clark's text in class. What is the big question/goal of each chapter, and why does it come before and after its neighbors? Reflecting on the structure of a text, seeing that as purposeful but malleable, is an important shift for the students, and it all starts with us deciding how to make sense of the fuzzy beginnings.

8.5 Discussion

Let's return to the Goals from above. For education researchers, I hope that I have provided an example of an instructor making their thinking explicit about both lesson planning and in-the-moment teacher moves. In particular, I think this chapter provides an example in which all of the choices are driven by a coherent stance toward the course goals. Shifting how the ideas are introduced into the class discussion, how the concepts are defined, or how we decide as a group if we accept a justification would create misalignment. In particular, the notion of whether we trust our senses to determine congruence needs to be handled consistently across these domains.

For practitioners, I hope that I have offered an example of the main levers I use at the start of a course to establish the kind of classroom discourse around justification that I am seeking. The intensity of this approach is high because of the particular focus on epistemology in this course, but I would say that I use these same levers and use them in similar ways in most of my courses. In terms of student learning goals, I hope I have illuminated the details of a lesson that brings ideas about congruence and about how we prove in mathematics to the surface where they are accessible to students. The first week or two of my courses tend to be quite intense, but then we establish these things as rhythms rather than a new patterns and our class discussions become efficiently focused on the challenging questions in our work.

My working definition of teaching with inquiry is that the classroom discourse is driven by questions that are being asked from the learners' perspectives. Many of the questions built into collegiate mathematics curricula are asked from expert perspectives in ways that are not accessible to learners. I see this lesson analysis manuscript as an example of my effort to get students to start asking the question: how DO we know the things we think we know about geometry/mathematics? I think that this portion of the course accomplishes this by generating the kinds of experiences that make this question salient, helping students examine past ideas and experiences that may not have been considered critically, and drawing attention to tensions that may not have been explored before. In particular, sometimes the way we talk about course materials focuses on smoothing out and removing the rougher patches; I think we should do the exact opposite by identifying the ways that tasks create some kind of cognitive imbalance, which can be leveraged for learning. These moments require students (and instructors) to step outside of our own thinking and practices to consider why we think as we do; as such, the goal is always this critical stance toward inquiry as part of the inquiry. In addition to this critical stance toward the mathematics, it is important to me that students participate in a critical conversation about course design and pedagogy: how is learning supposed to happen here, and why is the learning environment structured the way it is.

This kind of critical stance toward mathematics and mathematical ways of knowing is not sufficient to rehumanize mathematics classrooms, but I do think it makes a significant contribution. I believe that teachers who adopt this critical stance have a starting place from which to resist the ways mathematics is treated as neutral and objective and yet powerful in our societies and schools. Furthermore, I think these teachers also have the kernel of a worldview and pedagogical perspective that is incompatible with the banking models and authoritarian teaching methods sometimes found in mathematics classrooms, models and methods that assert that students are empty vessels into which knowledge flows from experts and that I think assume that mathematics can only be about universal and unquestionable truth. In other words, I think the course experiences discussed in this chapter help students critique aspects of mathematics teaching and learning that were dehumanizing for them or their peers but may have seemed natural, as a step on their journey to designing rehumanizing experiences for their future students.

Stepping back a little, other aspects of how I think about teaching are illuminated in this chapter. The previous paragraphs focus on individual psychology, but I think the sociological elements of my perspective on teaching also speak loudly in this chapter. The identities of students matter in the classroom, especially how they are positioned as knowers, question askers, and decision makers. Ideas are generated by small groups through discussion and negotiation of meaning. At a more structural level, my courses are organized around a collective narrative or repository; we each have individual perspectives, but our work together is deciding what gets into this communal record and how it gets there, including both things like proofs of statements and normative expectations for how we expect each other to engage the community. In particular, we are always having a conversation about whether an idea offered by one of our members should be added to the shared record, and full-class discussions are the times when we have the legislative power to add to that record. I certainly have an agenda in these meetings; in the lesson above you can see me pushing particularly hard to make sure that the foundational entries in this shared record meet the needs of our coming tasks and that they are entered through a particular critical conversation. This lesson also shows two particularly powerful aspects of this kind of structure. First, we are able to step outside of the record and ask what is in it. We can ask questions such as: could any of our accepted results possibly conclude that two figures are not congruent? Second, this record is both the accepted ideas and rules of engagement; in this context, the students participate in the conversation about changing the rules of engagement, and much of the course learning is really about the uptake and exploration of these proto-norms. The most compelling evidence I see of learning is the rapid ways in which students take on and internalize these changes in the rules within each class meeting.

8.6 Bibliography

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8.6.1 Constructions

MATH510 Euclidean Geometry - Constructions

The two most perfect geometric objects are the line and the circle. As a result, we study the geometric objects that we could construct using only a straight-edge and compass. This simple description belies a world of subtlety, but for now this allows you to make the following (simple) constructions.

1. Given two distinct points in the plane, A and B, you may use the straight-edge to build the line segment AB, the ray $\overrightarrow{AB}$, or the line $\overleftrightarrow{AB}$
2. Given a line segment AB and a point C, you may use the compass to build a circle centered at C with radius having the same length as AB.
3. Given a line or a circle, you may choose an arbitrary point on it. When using this property, you may not assume that the selected point has any special properties, which is what "arbitrary" means.
4. Given two lines, two circles, or a line and a circle, you may locate the intersection point(s) of this pair of objects, assuming those objects do intersect.

A **construction** is a sequence of instructions for pictures drawn using only straight-edge and compass techniques that are applicable to any figure satisfying the hypotheses, arriving at the conclusion. Once we have some tools in our belts, we will also expect a construction to include a rigorous argument that the sequence produces the desired conclusion.

Note: Not all of these are possible. Also note the lack of a definition of "congruent".

Task 1. *Given a segment AB and a ray $\overrightarrow{CD}$, construct a point E on $\overrightarrow{CD}$ such that $AB \cong CE$.*

Definition. *A triangle is the union of the three line segments created by three non-collinear points; if these three points are A, B, and C, then the triangle is written ΔABC. An **equilateral** triangle is a triangle whose sides are all congruent.*

Task 2. *Construct an equilateral triangle. Given a segment AB, construct an equilateral triangle whose sides are all congruent to AB.*

Task 3. *Given three segments AB, CD, and EF, construct a triangle ΔXYZ so that $XY \cong AB$, $YZ \cong CD$, and $ZX \cong EF$.*

Task 4. *Given $\triangle ABC$, construct a different triangle, $\triangle XYZ$, such that $\triangle ABC \cong \triangle XYZ$.*

Definition. *An **angle** is the union of two rays emanating from the same point. If the two rays are $\overrightarrow{BA}$ and $\overrightarrow{BC}$, then the angle is written $\angle ABC$ (or $\angle CBA$, depending on the orientation).*

Task 5. *Given an angle $\angle ABC$ and a ray $\overrightarrow{DE}$ (and a side of $\overleftrightarrow{DE}$), construct a point F (on the selected side of $\overleftrightarrow{DE}$) so that $\angle FDE \cong \angle ABC$.*

Definition. *Let A, B, and C be collinear points with B between A and C, and let D be a point not collinear with the other three; then $\angle ABD$ and $\angle DBC$ are called **supplementary angles**. A **right angle** is an angle that is congruent to its supplement.*

Task 6. *Construct a right angle.*

Definition. *Given an angle $\angle ABC$, we say that a ray $\overleftrightarrow{BD}$ is a **bisector** of $\angle ABC$ if D is in the interior of $\angle ABC$ and $\angle ABC \cong \angle DBC$.*

Task 7. *Construct a bisector of a given angle.*

Definition. *Two intersecting lines are said to be **perpendicular** if their intersection makes right angles.*

Task 8. *Given a line ℓ and a point P not on ℓ, construct a line through P that is perpendicular to ℓ.*

Task 9. *Given a segment AB, construct a point M so that $AM \cong MB$. The point M is called the **midpoint** of AB.*

Task 10. *Given a line ℓ and a point Q on ℓ, construct a line through Q that is perpendicular to ℓ.*

Definition. *Two lines are called **parallel** if they do not intersect.*

Task 11. *Given a line ℓ, construct a line parallel to ℓ. Given a line m and a point P not on m, construct a line through P that is parallel to m.*

Definition. *A line is said to be **tangent** to a circle if their intersection is a single point.*

Task 12. *Given a circle C and a point P outside of C, construct a line through P that is tangent to C.*

Task 13. *Given a line ℓ and a point P not on ℓ, construct a circle centered at P to which ℓ is tangent.*

Task 14. *Given two circles, construct a line tangent to both circles.*

Task 15. *Given a triangle ΔABC, construct a circle whose circumference contains A, B, and C.*

Task 16. *Given a triangle ΔABC, construct a circle tangent to $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{CA}$*

Task 17. *Given a segment AB, construct a square, a regular pentagon, and a regular hexagon with sides congruent to AB.*

Task 18. *Given a circle, construct its center. Given a circle, construct its center using only a compass.*

Task 19. *Given three parallel lines, construct an equilateral triangle with one corner on each line.*

9

Addressing Ambiguity in the Language of Limits

William Hall, *Washington State University*

Topic of lesson	This lesson focuses on the language we use in the mathematics classroom and community surrounding limits.
Course context	This lesson is meant to be utilized to set the stage for a unit on limits in precalculus or calculus.
Instructional challenge	The language traditionally used to describe limiting behavior (e.g. "getting closer and closer" or "approaching without reaching") does not reflect the mathematical concept of *arbitrarily close*. Neither do introductory lessons help students develop an intuitive or embodied notion of *arbitrarily close*.
Brief overview of instructional approach	This lesson situates learners in a fictional environment where they analyze the workings of a strange bridge. Students are invited to participate in the activity to make real the sense of a sequence whose terms reduce by half each step. Through a series of guided questions and activities, students unpack what it means for the terms of an infinite sequence to get arbitrarily close to a limiting value without relying on the phrase "closer and closer" exclusively.
Keywords	Limit, sequence, language

9.1 Background Information

Janet: What do you mean when you say: "*The limit is 3*"?
Marri: The function's output gets closer and closer to 3 as the input goes to infinity. The curve will approach, but not touch, the limit.

This brief exchange is meant to demonstrate the common language we might hear in precalculus or calculus classrooms during a lesson on limits. Secondary students (traditionally those ages 14–18 years) in the United States frequently learn to sketch rational functions by hand in precalculus. They are asked to analyze the characteristics of the graphs and understand how horizontal asymptotes emerge when the function "gets closer and closer, without ever touching" some output value. This language and conception of limit aligns, in part, with our sense of the word itself: *limit*. To assign a limit may be to impose a boundary that should not or cannot be crossed, such as speed limits, weight limits, temperature limits, etc. some of which include sharp boundaries with dramatic consequences if passed.

We know from research on the teaching and learning of limits that the language we use to describe limits outside the mathematics classroom (e.g. as boundaries or restrictions) serve as helpful metaphors for certain aspects of limits while becoming potential cognitive obstacles in others (see [8], [4] for more reading). Referring to a limit as the value a sequence or function gets "closer and closer to" or "approaches but never reaches," may help learners understand limits at infinity as described above, but simultaneously prevent them from seeing a constant sequence/function as having a limit at all. This language does not allow learners to engage with the subtle mathematical ideas captured by *arbitrarily close*. To be arbitrarily (or infinitely) close is to say that the function/sequence can be brought to within an arbitrarily small distance of some fixed value. The activity described here was designed to solve the instructional challenge of enabling students to interact with limits in a way that supports the evolution of their understanding to include the nuance of *arbitrarily close*. In doing so, learners come to understand limits in introductory calculus in a way that supports formal uses of limit in analysis while not relying on students becoming fluent in formal proof language in calculus. The activity includes characters inspired from J.R.R. Tolkien's *Lord of the Rings,* a personal favorite book series of mine. Included in this Lesson Analysis Manuscript is discussion on utilizing your own characters and settings to make the activity relevant and meaningful to you and your students.

9.1.1 Background and Context

In this section I will give the history of the design of the activity along with my positionality as the author and primary task designer. I also include a brief description of the theoretical considerations used in creating and iterating the activity. My purpose in sharing this information is to situate the activity and my suggestions within relevant social and cultural settings. I hope you can use this information to better map the learning goals and pedagogical strategies to yours and your students' needs.

History. This activity was designed for a calculus course for U.S. undergraduate students majoring in elementary education. The full course design was an interdepartmental endeavor that saw collaboration between faculty from mathematics, mathematics education, and elementary education[1]. The preservice elementary program at the public research university where this activity was developed was considered STEM-focused - all students were required to take at least one semester of calculus. Therefore, the course materials that were designed, including the activity described here, were done so with the philosophy that calculus concepts should be connected to elementary mathematical content where appropriate and that contexts and settings for the activities should be interesting and/or meaningful for students who intended to teach elementary school-aged children in their future careers. The original context, a popular children's television program, has been adjusted to fit a less specific audience. Other alternate contexts and how instructors can personalize the activity are shared later. While the activity was designed for preservice elementary teachers, the fundamental ideas students interact with would benefit anyone seeking to understand limits more deeply. A full copy of the activity is included as an appendix.

Positionality. I wrote the first draft of this activity in 2011 as a graduate research assistant at North Carolina State University on the project described above. My identity as an educator and researcher guides my choices in task design and pedagogical strategies. This includes my philosophical beliefs regarding the teaching and learning of mathematics in addition to factors such as race and gender. While no collection of factors represents a whole human being, I present several that I feel appropriately situate myself within relevant existing social, cultural, and political structures. I am a White, male, first-generation college graduate who has lived exclusively within the United States, but across several distinct regions. I acknowledge the existence of systemic racism within the social structure of society, including the mathematics classroom. One of my guiding principles is my belief that everyone engages in sophisticated mathematical reasoning and that it is vital to our collective societal health that mathematics educators ensure students do so in a way that positions them as mathematical authorities and empowers them to assume a mathematical identity.

I take it as a personal goal to establish a positive, reflective learning environment that illustrates my joy for mathematics as a counterpoint to some of the damaging perspectives students hold about themselves and mathematics. I place a high value on challenging self-deprecating attitudes about mathematical aptitude. I begin my courses briefly addressing modern concepts surrounding neuroplasticity and the human brain's ability to change and grow, challenging traditional mindsets that some are simply gifted, and others not, in the ways of mathematics. I encourage a classroom

[1]This work is based upon work supported by the National Science Foundation (Project #0942843). PI: Dr. Karen Keene; co-PIs: Drs. Paola Sztajn, Alina Duca

culture that excludes, explicitly and regularly, the idea that only some are capable of achieving at the highest level. "I'm not a math person" statements are challenged and not accepted. We need to set high expectations for all students, help them believe that they are capable of meeting those expectations, and work openly and fairly with each student to help them achieve their goals. Our students need our positive reinforcement as much as our critical reflections on their work. These actions help build relationships with students and a more fair classroom experience for all.

Theoretical Framing. I believe learners can only engage in sophisticated mathematical reasoning when they are actively engaged in the learning process and that this process is embedded within a given social and cultural context. Therefore, social constructivism (e.g. [1] would most accurately describe my philosophical foundation. Additionally, I value the theoretical and experimental work done surrounding embodied cognition (e.g. [3]) the idea that gestures and physical experiences are meaningful, irreducible elements to learning. I acknowledge that certain aspects of this activity assume free movement among the classroom, a privilege not all may share. In the post-lesson considerations, I have included a discussion of how the activity can be modified to be less restrictive.

When designing this particular activity, I employed realistic mathematics education (RME) as a guiding framework [6]. Activities developed under an RME framework are designed to be *experientially real* to the learners, meaning instructors choose contexts for problems students can ascribe meaning to. RME does not stipulate that these be "real-world" contexts per se, just that learners can ascribe meaning to the constituent elements of the problem. Another component of RME important to understand for the present work is guided reinvention; learners are expected to "reinvent" mathematical concepts via interacting with the activity, their peers, and the instructor. This necessitates careful planning when writing activities and usually a commitment to an iterative refinement process. One suggestion from practitioners of RME is to look to the history of the subject for inspiration in hypothesizing how students might be guided to reinvent the given concept for themselves, which is what was done for this activity. See [5] for a more detailed discussion on guided reinvention in undergraduate mathematics.

9.1.2 Learning Objectives and Curricular Setting

Learning Objectives. The learning goal for this activity is that students will use the phrase 'arbitrarily close' in discussing limits of sequences and functions. That is, the activity will illustrate and help students understand the specific mathematical concept that a limit exists when the function/sequence can be found to be within an arbitrarily small margin of some fixed value. Meeting this goal engages learners in developing an intuitive foundation for the formal definition for limits. It is important to note that the goal of this activity is ***not*** for students to read and understand a formal definition for a limit, but that they engage with the mathematical concepts necessary to do so. Students are expected to know the terminology surrounding sequences, though the activity could be adapted to include an introduction to such language. Students are not required to have any experience with limits prior to this activity.

Curricular Setting. I have used this activity early on in various instantiations of calculus including Calculus for Life Scientists and AP calculus at the secondary level. I have also introduced limits to Precalculus students using this activity and had success with a group of secondary students in Upward Bound, a TRIO program[2]. Classroom norms for discussing mathematical ideas within small groups and the whole class are established prior to this activity. Students regularly work in groups of three or four in which roles are assigned randomly. The organizer keeps the group on task and ensures all voices are heard, the recorder takes notes in a shared space, and the reporter is responsible for sharing the group's work during whole class discussion. Used regularly, randomizing groups and roles within groups positions all students as equally intellectually capable of leading and participating in mathematical discussions.

Equitable Teaching Practices.

When utilizing group work in mathematics, I try to set norms that illustrate my belief and confidence in every student. Each group is asked to randomly assign four roles to group members: organizer, recorder, reporter, and time-keeper. Establishing these roles using randomness ensures all students know they are expected to fill any role, on any given day. See Appendix B for the group roles and responsibilities handout if you want more information on the roles and how to set up these norms in your classroom.

[2]"The Federal TRIO Programs (TRIO) are Federal outreach and student services programs designed to identify and provide services for individuals from disadvantaged backgrounds." (U.S. Department of Education)

9.2 Lesson Implementation

In this section I discuss the main parts of the activity, tying the specific tasks students are asked to complete back to the guiding principles outlined above. I also make note of past student responses and helpful pedagogical suggestions. I have found I can best support students when they work in groups of three, though I have had success with pairs and groups of four as well. The student activity is included in Appendix A. A description of group roles and responsibilities is included in Appendix B.

9.2.1 Rationale for Instructional Decisions

Realistic mathematics education (RME) suggests mathematical activities be experientially real to students. This activity begins with a short narrative piece about two fictional characters (hobbits Murry and Poppin) who come to a bridge. The bridge requires the hobbits to walk such that each step's length is exactly half of the remaining distance left to cross. The students are positioned as helping the hobbits figure out the bridge. The inspiration for this setting came from my own reading of Zeno's paradox of Achilles and the tortoise, a connection that is alluded to in the activity itself to hopefully inspire a connection to the human history of mathematics. The narrative structure is also a pedagogical tool in itself. An exploration of the limit of a sequence situated within a narrative structure that students can ascribe meaning to, implies they have the opportunity to make sense of the mathematical concepts running parallel with the story. Furthermore, they can use elements of the story to make sense of the emerging mathematical concepts. Giving all students the same foundational assumptions for the fictional world ensures all have an opportunity to make connections between the mathematical concepts and metaphors in the story without additional cultural assumptions.

9.2.2 Introduction

Lesson Flow.(3-5 minutes) The context of the version of the activity included here involves characters from a fictional universe that I find personally meaningful (J.R.R. Tolkien's Lord of the Rings). While the chosen narrative structure is not directly relevant to the mathematical concepts under consideration, it can help catch students' attention and once they are into the task, they will have begun contending with the mathematical concepts at hand. Another reason for selecting this particular setting is that it makes the activity more enjoyable to teach. I place a high value on challenging self-deprecating attitudes about mathematical aptitude and I have found when I am present with students and I can share some of the joy I have for both mathematics and popular culture, they respond positively. Therefore, I would recommend you take some time prior to using this activity to decide on a setting for yourself. This could be something personally interesting or better yet, something you know to be interesting to your students. Suggestions include fictional characters, popular local celebrities, university officials/faculty/staff, team mascots, etc. In practice, this section can be read aloud by the instructor, a student, a group of students taking turns, or quietly by oneself. A brief pause for any questions regarding the context is helpful before moving on to Part 1: Try it Out.

9.2.3 Part 1: Try it Out!

Lesson Flow. At this point, we step away from the context of the fictional characters and the bridge and ask students to utilize the physical space, their bodies, and movement to better understand the setting of the activity. This task was designed to ensure the activity is indeed experientially real for the students in an explicit way: they physically explore what it is like to traverse such a bridge as the one in the story. Asking students to enact these steps with their bodies provides an opportunity to develop a shared understanding of the context of the task—it is clear when a group works together if one person understands the context differently than the others. Thus, many clarifications are provided without the instructor having to provide any feedback. This gives students agency in the task.

Students are asked to find a space in the classroom, usually near a wall, where they can imagine a bridge they must cross. Students then move, or imagine moving, so that the remaining distance is half what it was prior and take notes on how many steps they can take. After walking their bridges, the students sit down and answer the following questions in their groups:

1. Did you ever cross your bridge? Why or why not?
2. How many steps did you take?

3. Could you have taken more steps? If yes, how many more steps could you have taken? If not, explain why your number of steps is the maximum.
4. So, do you think Murry and Poppin will ever be able to cross the bridge? Be prepared to provide a convincing argument with evidence from your answers in #1–3.

While students are working, I build rapport with the students and observe their interactions. I will ask how they are doing in addition to how many steps they could take. It is helpful to ask students why they had to stop taking steps and how someone might be able to take more steps, focusing their attention on the end behavior.

Student Thinking and Reasoning. Students commonly take 6–7 steps and answer that no, they could not cross the bridge. One interesting answer that comes up in Q3 is shoe size - students sometimes claim that if their feet were smaller (or just very small in general) then they could continue taking steps. This implies a sense of zooming in on the boundary (i.e. the wall) and shrinking to get closer, which is a very helpful image and physical experience to call back to when the mathematical content starts to mirror that process. In Q4, "So, do you think Murry and Poppin will ever be able to cross the bridge? Be prepared to provide a convincing argument with evidence from your answers in #1–3." students pull together their thoughts from the previous questions to make an initial conjecture about the situation, one they can reflect on at the end of the activity. Here you may find phrases like, "in reality the distance will just get so small they will cross over" or "in theory they can do it but not in real-life". During whole class discussion, I draw students' attention to this juxtaposition of "in reality" versus "theoretically" as it is a useful callback to make once students have interacted with the mathematical representations later on. Usually at least one group will make such an argument and it is useful to attend to that perspective last and use it to transition to Part 2.

9.2.4 Part 2: Back to Murry and Poppin (Q5–Q8)

Lesson Flow and Student Thinking and Reasoning. (15–20 minutes) Part 23 includes Q5–Q11. First, let us consider Q5–Q8.

5. If Murry and Poppin walk on the bridge for hundreds of years (hobbits really are amazing creatures), what total distance will they travel?
6. We can view the step sizes as elements of a sequence. Write out a few terms of this sequence and describe its behavior. Can you represent this sequence using an explicit formula?
7. What about the sequence for the distance remaining after each step? Can you represent this sequence using an explicit formula?
8. How small will the distance remaining get?

After the whole class discussion regarding Part 1, students return to work in their small groups on Part 2, Q5-Q8. First, students are asked to find the total distance traveled if we assume the characters travel for hundreds of years on the bridge. In the version included at the end of this chapter, there is no specific bridge length given in the problem set-up. Allowing for several different bridge lengths provides an opportunity to generalize the findings beyond whatever specific value one group happens to choose. After giving students a few minutes to discuss this question in their groups, usually a group will ask a question about the omission and I will pause to pull the class back together to say they can choose their own bridge length. This adds a level of independence to the work the students produce and allows for absurd (and generally silly) conversations about "realistic" bridge lengths and step sizes. Levity is a welcome ingredient in this lesson. Some groups, usually those who recall their experiences with sequences more readily, will assign a variable to the bridge length and proceed without a fixed length. I encourage groups that do this to push as far as they can symbolically but am prepared to redirect them to try a numerical value if they get stuck in an ensuing section. Allowing for several different bridge lengths provides an opportunity to generalize the findings beyond whatever specific value one group happens to choose.

Q6 reads, "We can view the step sizes as elements of a sequence. Write out a few terms of this sequence and describe its behavior. Can you represent this sequence using an explicit formula?" Here again is an opportunity for students to fine-tune their interpretation of the context, given they must now use their chosen bridge length to calculate what the first few step sizes will be in the given scenario. This serves as a moment for humor as well since some groups inevitably do not consider what a realistic step-size will be prior to choosing their bridge length and end up having to assume hobbits (or whichever character you have chosen) can make exceptionally large steps. You can choose whether or not to include the part asking students to write the sequence using an explicit formula, it is not necessary that students write the explicit form to complete the activity.

In Q7, students transition from the sequence of step sizes, to a sequence of the distance remaining left to cross after each step. By moving from physically walking toward the wall, to a sequence of step sizes for a fictional bridge, to a sequence of the distance remaining after a given step, students end up positioned to analyze a (now personally meaningful) case of a sequence with limit zero while preserving the connection back to the given setting and their intuition.

Q8 asks students, "Just how small will the distance remaining get?" This question is intended to focus the students' attention down to the diminishing step sizes, just like when they took very tiny steps when approaching the wall. Students have generally responded to this question with phrases like "infinitely small" or "very close to zero". In my work with secondary students in a regional TRIO program, one student wrote "very very very very small" where each "very" got smaller and smaller. This level of creativity and mathematical awareness is unavailable in a traditional lesson on limits.

After each group has answered Q5–Q8, I bring the class together to discuss their work. For Q5, I ask a group to share their bridge length and how far the hobbits will travel. You may see students say "a distance really close but not exactly" equal to the bridge length in addition to groups who claim the two are equivalent. This is an appropriate spot to return to the language of "in reality" versus "theoretically" and how they might relate to this situation. Q6 is usually a quick discussion as most students do not have issue with calculating and writing the terms and Q7 should be a contextual pivot point - note publicly how it is true that each group has equivalent numerical sequences, regardless of bridge length, for Q6 and Q7. When observing groups work on Q8, look to highlight the kinds of creative descriptions that provide a new perspective on the nature of *arbitrarily close* that you are building, instead of only revoicing the more traditional or formal descriptions and representations.

9.2.5 Part 2: Back to Murry and Poppin (Q9–Q11)

Lesson Flow and Student Thinking and Reasoning. (15–20 minutes) Now let's consider Q9–Q11 where students start to consider an argument that looks closer to a traditional approach to proving a sequence has some finite limit.

9. Clearly there is something very strange happening at the end of the bridge, let's look a little more closely at what happens there. Based on your analyses, will Murry and Poppin ever be less than 0.01 meters from the end of the bridge? If yes, how many steps will it take them?

10. Will they ever be within 0.001 meters from the end of the bridge? If yes, how many steps will it take? Try to use your sequence notation from earlier in your argument here.

11. Try to show that for any given distance, k meters, that there is some number of steps, n, such that after Murry and Poppin take those n steps on this bridge, they will be less than k meters from the end of the bridge.

Q9 reads, "Clearly there is something very strange happening at the end of the bridge. Let's look a little more closely at what happens there. Based on your analyses, will [the characters] ever be less than 0.01 meters from the end of the bridge? If yes, how many steps will it take?" Since many students will have just reasoned that the steps are getting smaller and smaller without end, they usually intuit that the answer is yes. To calculate the number of steps, I encourage students to use a calculator to write out the elements of the sequence of distances remaining until they get a value less than 0.01m (if they are using the formula they can calculate directly). On the final page of the activity Q10 pushes students to consider the same question but now within 0.01 meters, students are asked after what step number will the distance remaining to cross be smaller than several fixed values (first 0.01m in Q5, then 0.001m in Q6). These questions, supported by the previous work, establish a landscape where students explore and make sense of the idea that given some very small, fixed distance, there exists a spot in the sequence of distances left to cross beyond which all step sizes are smaller than the chosen fixed distance, no matter how small that initial distance.

The final question, Q11, reads, "Try to show that for any given distance, k meters, that there is some number of steps, n, such that after Murry and Poppin take those n steps on this bridge, they will be less than k meters from the end of the bridge. " Note that here we finish not by asking the students to make a computation but to feel convinced that this is statement is true, thus providing the instructor an opportunity for formative assessment regarding conceptual understanding at this point. Here again is an opportunity for groups to engage at their own current computational level (e.g., if they have been working with an equation, can they prove the limit of the sequence is zero?) but does not assume all will end up here at the lesson's end.

When selecting groups to share their work on these questions in whole class discussion, it is important to again build from concrete to formal. First ask a group that you saw had a productive conversation regarding Q9 who listed all the terms until they found where in their sequence the terms representing the distance remaining left to cross was less than 0.01m. For Q10, I choose a different approach to highlight, such as those who were able to compute the step number using an explicit or recursive equation but also indicating listing terms is an equally valid method for this question. In Q11, listing strategies no longer work and therefore students must entertain the more abstract method.

9.3 Post Lesson Considerations

To meet the goal of this activity is not to just get students to use the phrase "arbitrarily close" in quizzes and tests but then ascribe no real meaning to it. This entails a broader commitment from the instructor to talk about limits in more careful and constructive ways on a regular basis and ask students to do so as well. When discussing the limit of the slopes of secant lines or the limit of Riemann sums, using *arbitrarily close* in lieu of the more general "closer and closer" language is more accurate and supports students in developing a unified understanding of calculus. In addition, asking students to describe limits using accurate and productive language on formative and summative assessments instead of solely computational questions places value on the conceptual ideas. Appendix C includes my initial attempts at re-conceptualizing the activity without the language of "steps" and instead relies on passing through gates. This may be one method of utilizing the activity in a large-lecture setting where students cannot move around quickly/easily. I encourage mathematics educators who utilize this task to contact me with implementation notes and ideas should you be willing to share.

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Appendices

A

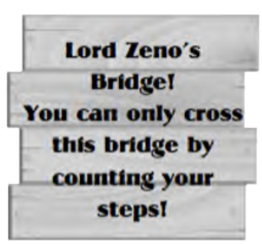
There and Back Again...

An Introductory Activity for Limit of a Sequence

Image credit: "Gothic Bridge - Central Park, New York City, New York" by Trodel is licensed under CC BY-NC-SA 2.0

Introduction

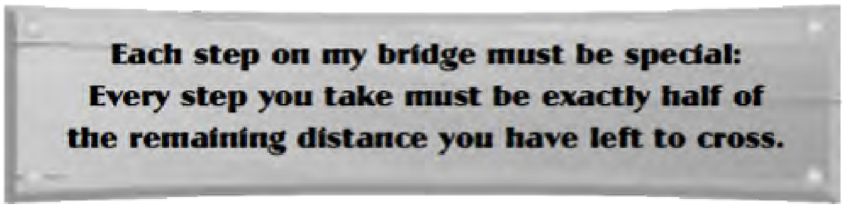
One bright day two hobbits named Murry and Poppin were traveling back to the Shire after a lovely second breakfast when they come to a strange bridge. Just before the bridge there was a sign that read, "Lord Zeno's[1] Bridge! You can only cross this bridge by counting your steps!"



[1]Zeno of Elea was a 5th century Greek philosopher. You can read more about him here: http://www-history.mcs.st-andrews.ac.uk/Biographies/Zeno_of_Elea.html

"Murry, I've never seen this bridge before! What do they mean we have to count our steps? This bridge is ridiculous!" Poppin said indignantly.

Unfortunately, when they get closer, Poppin notices another sign, it says "Each step on my bridge must be special: Every step you take must be exactly half of the remaining distance you have left to cross."



"Lord Zeno must be one odd fellow; this is a strange bridge indeed! Are we ever going to get across this horrible bridge and how many of these silly steps are we going to have to take!?" moaned Popping.

Murry and Poppin are clearly in a bind, we should help them out.

Part 1: Try it out

To help Murry and Poppin, we first need to understand how this bridge works. With your group, outline a distance on the floor to model the length of this bridge and try to follow the directions on the sign: every step you take should be half the distance that is left between you and the wall. Keep taking steps until you can't take any more. Once you are done, sit down with your group and consider the following questions. The person with the **recorder** role should take notes while the **reporter** prepares to summarize the work of the group for the whole class if called upon.

1. Did you ever cross your bridge? Why or why not?
2. How many steps did you take?
3. Could you have taken more steps? If yes, how many more steps could you have taken? If not, explain why your number of steps is the maximum.
4. So, do you think Murry and Poppen will ever be able to cross the bridge? Be prepared to provide a convincing argument with evidence from your answers in #1–3.

Part 2: Back to Murry and Poppin

Let's help Murry and Poppin figure out just how this bridge works. In your groups, using any tools (e.g., laptop, calculator, chalk board, etc.) you think will be appropriate, analyze how this bridge works by answering the following questions.

5. If Murry and Poppin walk on the bridge for hundreds of years (hobbits really are amazing creatures), what total distance will they travel?
6. We can view the step sizes as elements of a sequence. Write out a few terms of this sequence and describe its behavior. Can you represent this sequence using an explicit formula?
7. What about the sequence for the distance remaining after each step? Can you represent this sequence using an explicit formula?
8. How small will the distance remaining get?
9. Clearly there is something very strange happening at the end of the bridge, let's look a little more closely at what happens there. Based on your analyses, will Murry and Poppin ever be less than 0.01 meters from the end of the bridge? If yes, how many steps will it take them?
10. Will they ever be within 0.001 meters from the end of the bridge? If yes, how many steps will it take? Try to use your sequence notation from earlier in your argument here.
11. Try to show that for any given distance, k meters, that there is some number of steps, n, such that after Murry and Poppin take those n steps on this bridge, they will be less than k meters from the end of the bridge.

B

Group Roles and Responsibilities

To begin your small group discussion, introduce yourselves to one another. Use the virtual die, or a real one if one is nearby, to assign roles to each group member. https://www.google.com/search?q=dice+roller

Order group members by die role from least to greatest and assign roles in order as follows:

Organizer – Initiates conversation on task to accomplish and tries to ensure no one is left out of the conversation.

Recorder – Documents the group members' ideas and work in the shared space.

Reporter – Reports out to the whole class, if asked to share.

Timekeeper – Keeps an eye on the time with full perspective of the task

10

Linear Regression

Helen Burn, *Highline College*

Topic of lesson	This LAM includes a sequence of five activities focused on linear regression in a technology-mediated introductory statistics course.
Course context	Introductory statistics taught in a community college to a student audience that is diverse and majority health and social science majors. The course is offered as a corequisite format, so no prior exposure to linear equations beyond basic linear equation solving is presumed.
Instructional Challenge	Linear regression is a multi-step, complex process. The challenge is to effectively meet the learning outcomes around linear regression while embedding remediation in a corequisite context, and maintaining cognitive demand by focusing on higher-order thinking (e.g., interpreting and evaluation) in linear regression.
Brief overview of instructional approach	The lesson comprises a sequence of five activities accomplished through a mix of whole-group discussion, student pairs, and student group work. The activities are technology-mediated using a statistical software package included with the textbook I use. The activities begin by activating prior knowledge and exposing students holistically to all concepts associated with linear regression. Following this, I work a complete example that models learning goals around regression, and then provide students with the opportunity to practice putting the pieces together. The final activity aims to bring in complexity and higher-order thinking to linear regression.
Keywords	Linear regression, technology, introductory statistics

10.1 Background Information

10.1.1 Instructional Challenge

Introductory statistics is currently one of the introductory mathematics courses with the highest enrollment at two-year colleges. Linear regression is a traditional topic in the course. The instructional goals for linear regression tend to be modest and leverage students' prior learning of linear equations and functions. My experiences with teaching linear regression suggest to me that students are generally successful in computing and creating linear regression models. However, they are less successful in ultimately making sense of and interpreting the results of linear regression analysis in practical terms, including interpreting the slope of the regression model and its units of measure. Enhancing students' capacity for the higher-order cognitive skills of interpreting and evaluating is essential to helping them grow intellectually. By not achieving the higher-order goals, linear regression becomes merely a rehash of linear equations that does not advance the student. The purpose of this LAM is to explain my approach to teaching linear regression with the aim of increasing students' capacity for interpreting and evaluating the model.

10.1.2 Learning Goals

Linear regression appears in the curriculum after roughly one month of instruction. It is a stand-alone topic about bivariate data but flows nicely from the work we have done thus far with descriptive statistics: sample, quantitative variables, asking questions of data, organizing data into graphs, analyzing through numerical measures.

The main student learning objective (SLO) around linear regression in the course is: Create and evaluate the suitability of a linear model for a data set, and interpret its meaning in everyday language. It is noteworthy that Introductory Statistics within the Washington community college system has a common course number (Math& 146) to ease transfer, but there is no common set of course objectives. Instead, there is a set of content recommendations around learning outcomes for Introductory Statistics developed through a collaboration of statistics instructors from two- and four-year colleges that includes "Descriptive statistics for bivariate data, including linear regression modeling."

The list below was developed within the mathematics department at my college to explicate expectations for assessing the learning objective around linear regression.

- Create scatterplots for bivariate data using graphing technology where appropriate.
- Identify predictor and response (independent/dependent) variables.
- Use technology to find linear regression models and correlation coefficients for a data set.
- Describe whether a linear regression model is suitable for a data set (looking at scatter plot and correlation coefficient).
- Interpret the correlation coefficient in terms of positive/negative/null and strength of correlation.

10.1.3 Background and Context

Highline College, located in a suburb of Seattle, Washington, is a public four-year, primarily associate-degree granting college that operates on the quarter system with a typical class comprising five credits. Formerly a community college, Highline College now offers five Bachelor of Applied Science (BAS) degrees. Corequisite Introductory Statistics is a 10-credit course where students learn traditional introductory statistics topics with embedded support including intermediate algebra remediation. The course prerequisite is prealgebra (signed numbers, basic linear equations, graph reading, unit conversion). In actuality, most students place directly into the course without having to take the prerequisite. Upon completing Corequisite Introductory Statistics, students are transcripted for five credits of Math 87 (Intermediate Algebra for Statistics) and five credits of Math&146 (Introduction to Statistics). The course is taught either in a face-to-face classroom without computers (except for an instructor workstation) or through remote teaching. In the former case many but not all students have laptops or mobile devices they use during class. In the latter case, all students are accessing a computer or mobile device during class. The textbook I use has statistical software included that we use heavily.

The majority of the student audience for the course is health sciences or social science majors. However, each term I have one or two students from the college's BAS in Cybersecurity. The students come from a variety of backgrounds and cultures. The demographic changes in King County brought about by the increased cost of living in the Seattle metro area have led to a remarkable increase in the diversity of South King County residents where Highline College is located. The overall student body at Highline College comprises 77% first-generation college students and 66% students of color [1]. In my typical class, most students are at least bilingual, and many are recent immigrants or refugees from countries such as Somalia, Ethiopia, and Afghanistan. In designing the course, my colleagues and I collaborated with faculty and staff in the college's cultural learning communities, including our college's Umoja Black Scholars Program, to identify and incorporate instructional practices that can enhance the success of racially minoritized students. The practices include attention to community building and support, contextualization of course content to bolster perceptions of relevancy and meaningfulness of learning, and metacognitive activities to build students' capacity as learners.

10.1.4 Rationale for Instructional Decisions

I view instruction as relational involving interactions between students, teachers, and content [2]. Content is typically encountered through academic tasks (e.g., working mathematics problems or taking exams) that mediate learning by determining how students direct their attention, process information, and what students learn [3]. The tasks and activities I design to engage with linear regression are based on the assumption that I am reteaching a concept about which students have prior knowledge but one that they have misconceptions about or that they need scaffolding to activate their prior knowledge. I also make instructional choices based on psychosocial factors, including motivation to engage [4], mattering, and sense of belonging [5]. I use group work systematically and consistently in nearly each class session using techniques acquired through the NSF supported Project CLUME [6].

My perspectives on the learning goals and approaches to linear regression have evolved as I have engaged in several national efforts to modernize introductory statistics. The common call of grants and curriculum guidelines is to make the course datacentric, to focus on asking questions of data, and to engage students in meaningful statistical investigations that leverage open educational resources (OER). There is a sense of urgency because of the high number of students who enroll in introductory statistics at two-year colleges.

Overall, I believe our learning goals for linear regression are quite basic and could be improved to the benefit of the student. For example, it is noteworthy that my department does not require that we assess whether students can interpret the slope of the regression equation. However, my professional collaborations have convinced me of the importance of using the slope as a proxy for "effect size" that can be used along with the correlation coefficient and coefficient of determination to enhance interpretations of the strength and nature of a correlation. I was also made aware that instructors tend to focus first on the independent/predictor variable and to ask whether it correlates with a dependent/response variable. In contrast, a subtle but powerful shift is to start with the response variable (e.g., diastolic blood pressure) and to ask questions of the data such as: What predicts diastolic blood pressure? I also now recognize the limitations of presenting students with a scatterplot based on a single sample. With available OER products, we now have the opportunity to engage students in drawing multiple samples that bring into relief issues around sample variability.

10.2 Lesson Implementation

The lesson comprises a sequence of five activities accomplished through a mix of whole-group discussion, student pairs, and student group work. The activities are technology-mediated using a statistical software package included with the textbook I use. The activities begin by activating prior knowledge and exposing students holistically to all concepts associated with linear regression. Following this, I work a complete example that models learning goals around regression, and then provide students with the opportunity to practice putting the pieces together. The final activity aims to bring in complexity and higher-order thinking to linear regression.

10.2.1 Activity 1: Activate prior knowledge around linear equations and functions.

The first activity around linear regression is designed to activate students' prior knowledge of the topic. Activating prior knowledge is a practice grounded in cognitive science [7]. In the context of a corequisite course, unearthing or triggering students' prior knowledge is an essential practice. Activating prior knowledge is also an equity practice because it gives students a voice in the class, validates them as possessing valuable knowledge, and decenters the instructor as the sole authority in the class.

I draw quadrant I of the $x - y$ plane, along with a scatterplot and a line through the data. I label a single point, such as (3, 5) on the line. I also write $y = mx + b$. Then I put students into small groups that have been established during the first three weeks of the term. I pose the question: What do you already know about anything that I just drew on the board (or tablet if I am doing remote teaching)? After discussing in groups, they return and share with the class, either in a whole-group discussion or through a chat window. Typically, students share things like: slope, rise/run, intercept, coordinates. While I assume no familiarity with the topic in a corequisite course, I generally find that the majority of the students have some familiarity with linear equations and the Cartesian coordinate system.

The activity serves as a soft opening to the topic and can be done in a variety of ways depending on the timing and circumstances. I have presented the activity at the end of a class session or during the day we transition to linear regression. My preference is for the former because then students become aware that we are transitioning to a new topic and can let ideas percolate until the next class session, or students can explore on their own prior to the next class. I share key pages to read in their textbook.

10.2.2 Activity 2: Expose students to *all* concepts around linear regression.

The goal of the next activity is to expose students to all aspects of linear regression. I recommend some sort of mental break between Activity 1 and Activity 2. Adults benefit when they understand why they are learning something, and many benefit from seeing the big picture on a topic before breaking things down into component parts [7]. My students at this point have become accustomed to my instructional approach to address the topic holistically, and they trust that I will break it down for them.

Consistent with past topics, we begin by examining graphs, sharing key vocabulary, and giving a hint at the analysis portion of linear regression. Specifically, I copy two scatterplots directly from the textbook and present them to students. The examples are intuitive in being correlations, such as the age and mileage of a car, or the years of employment at a company and salary. In showing these examples, I share all the relevant vocabulary: bivariate numerical data, scatterplot, independent (predictor) and dependent (response) variable, linear/nonlinear, positive and negative association/relationship/correlation (see Figure 10.1).

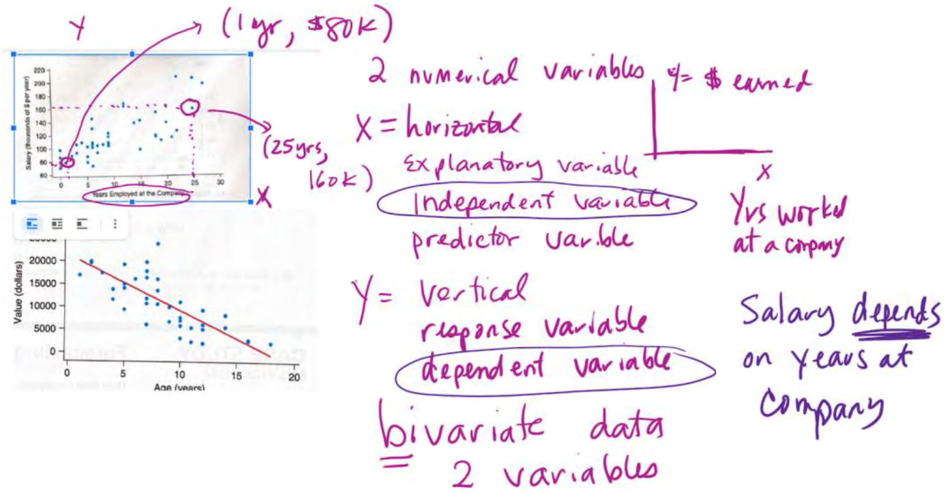


Figure 10.1: Annotated screenshot of initial linear regression examples

Next, I demonstrate for students how to conduct a linear regression with the statistical software used in the course. I point out the correlation coefficient and the standard approach to assessing strength as weak, moderate, or strong. It is common for the notion of lurking variables to emerge here as adults understand the myriad variables that, for example, contribute to the mileage of the car in addition to its age. I compare the correlation coefficient and coefficient of determination, and I show where the regression equation/model appears in the computer printout. However, at this point I do not go into details except to say that we use the regression equation for prediction.

As much as possible, I leverage their prior knowledge around statistics and honor the prior knowledge they shared in Activity 1. For example, I emphasize that a scatterplot is a new type of graph in addition to bar graphs, dot plots, and histograms. I highlight that, similar to our prior work with univariate data, we organize the data into a graphical representation to "see" the trend, we use technology to help with our analysis, and then we interpret the data and draw conclusions based on the analysis. The final part of this activity is to put students back into their groups to check in with each other, to review any questions they have on vocabulary, and to practice using the software.

10.2.3 Activity 3: Setting expectations for learning through working a complete example.

The next activity aims to render transparent the learning goals around linear regression. To be sure, at this point students have an online homework set focused on linear regression, approximately 15 questions. However, being transparent about what students are ultimately accountable for is an equity practice that levels the playing field for students in the class by setting clear expectations [8]. To accomplish this, I show them the SLO and assessment expectations for linear regression created by the department. Then, I create a handout that includes a data set to analyze using linear regression that would be similar to what they would be expected to do in a summative assessment. A powerful example I use is exploring whether there is a correlation between the length of an airplane flight and its cost. I pose questions such as: Do longer flights tend to cost more? I update the data periodically using a travel search engine and identify the cheapest round trip flight between Seattle and several cities. I look for non-holiday dates one month out, departing on a Tuesday and returning a week later. I always include Missoula, Montana, which is close to Seattle but tends to be expensive to fly to. Given the diversity of my students, this data is a wonderful way to engage them in thinking about where they have traveled to. Adult learners have a wealth of experience to share. This example is also excellent for exposing the age-old saying that *correlation does not mean causation* as it is easy for adults to list a plethora of lurking variables that also contribute to the cost of an airplane flight. I work this example start to finish, moving through the handout comprising a standard sequence of questions they will be held accountable for in assessment of linear regression. It is notable that I have not lectured specifically on "slope" in detail. Instead the air flight handout contains a template sentence for interpreting the slope. This scaffolds students' learning of a topic I know they have difficulty with. I indicate which parameter represents the slope and focus on the fact that it represents the growth rate. To demonstrate the effect of the slope, I use EXCEL to create a table of values to demonstrate what happens as I increase the flight miles by 1 mile and that the price goes up by \$0.04 for each additional mile. Also, the topic of outliers or "influential points" is also covered at a very basic level with the advice to run the regression with and without any outliers to determine whether to consider removing an influential point. By the end of this activity, students have all of the basic information on linear regression that meets the current learning objectives.

10.2.4 Activity 4: Students practice linear regression with a follow-up lecture.

Adults need to practice putting the component pieces together in order to make sense of their learning [7]. The next activity is group work activity that is turned in individually. The example I picked explores the question: Does advertising contribute to increased sales? It is noteworthy that this group work has the same format as the handout in Activity 3. It is further worth mentioning that I have not given all the details of linear regression and am assuming that students will help each other fill in the details. Following group work, I do a 20-minute lecture on "prerequisite material" around the Cartesian coordinate system (focusing mainly on quadrant 1), attempting to show how to substitute into the equation $(y = b + mx)$ and the interpretation of the slope using a new example involving a pay scenario I felt the students would understand: the worker gets an initial \$30 "appearance" fee and \$17 per hour. To demonstrate the effect of the slope, I again use EXCEL to create a table of values to demonstrate what happens to net pay with each additional hour. Figure 10.2 shows a screen capture of my class notes. The writing at the bottom of Figure 10.2

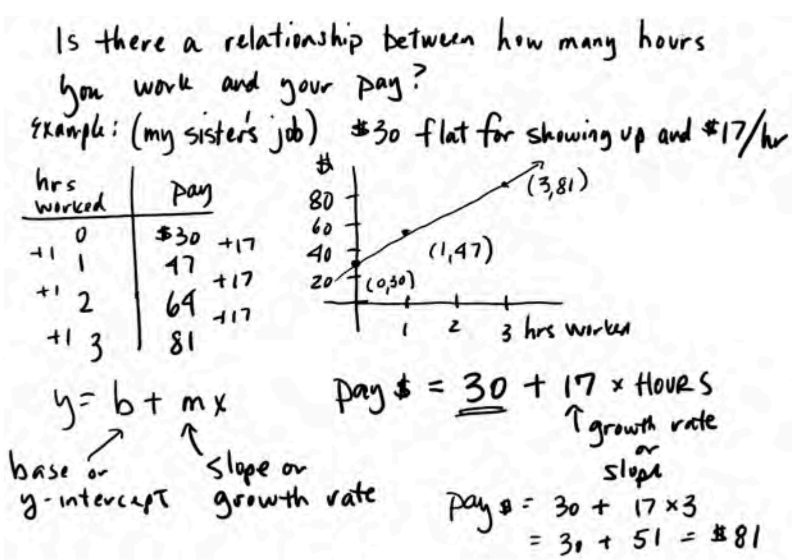


Figure 10.2: Screen shot of board (tablet) work to summarize after group work completed.

shows that I brought in the example from the group work and provided additional instruction without giving the full interpretation of the slope. Also, because my instructional goal is to have students identify and interpret the slope as it appears in the regression model, at this point in the lesson I avoid introducing how to compute the slope using rise/run and related formulas. Students then had additional time to finalize their group work prior to submission.

Several issues emerged around student thinking. First, many had questions about my use of "model" in the group work handout. Thus, I had to explain about how and why we use the word "model." This also is an opportunity to address interpolation versus extrapolation in predicting using the model. Additional questions I have fielded from students include: Do the variables have to be numerical? I answered that in an introductory level course, they tend to be but in fact you can do linear regression analysis with categorical variables if they are "ordered" like a Likert Scale. Students have also asked: How do you find R if you are only given R-squared (a homework problem)?

When I graded the group work that explored the relationship between advertising and sales, the template for interpreting slope seemed to be a helpful scaffold, but there was still evidence that many students were hesitant about interpreting the slope in context. For example, one student said that "for each additional \$1 in advertising, sales increased by 13.9." It is unclear whether they understood this is \$13.90 or 13.9 units sold. A second noteworthy issue is that the computer printout does not use the $y = mx + b$ format but instead formats the model as: Dependent variable = y-intercept + slope Independent Variable (see Figure 10.3). Both the dependent and independent variables appear as words based on how the variable is written in the stacked data, the y-intercept and slope typically have five or more decimal places, and there is an implied multiplication between the slope and independent variable. Many students did not transfer all parts of the equation and were hesitant to round. At the same time, most all of them used the equation/model successfully in predicting sales dollars for a specified amount of advertising. Further, overall the students successfully moved across the multiple representation (tabular, graphical, symbolic) and demonstrated an overall grasp of the concepts of linear regression.

Simple linear regression results:

Dependent Variable: Sales \$
Independent Variable: Advertising \$
Sales \$ = 28645.753 + 13.899614 Advertising \$
Sample size: 12
R (correlation coefficient) = 0.89094958
R-sq = 0.79379116
Estimate of error standard deviation: 2943.794

Figure 10.3: Example computer printout for regression example using adverting and sales dollars.

10.2.5 Activity 5: Focus on complexity and higher-order thinking.

Making linear regression authentic and meaningful for students is key to keeping adult learners engaged and motivated. First, I show students how the correlation coefficient is computed using the standardized scores for the independent and dependent variable. This is out of respect for the students and, in addition, it emphasizes an application of Z scores. Second, the class examines bubble charts using Gapminder World [9] to visually explore correlations for different variables. Gapminder is very appealing to students, and each bubble represents a country, color-coded by continent, with the size of the bubble representing the size of the country's population. I demonstrate positive correlations such as the positive relationship between mean years of schooling and the average life expectancy in a county, and negative correlations such as the relationship between mean years of schooling and average number of babies per women in a country. This is another opportunity to discuss lurking variables and the danger of extrapolating beyond the data. Figure 10.4 shows a bubble chart where the predictor variable is "mean years of schooling for women 25 years and older" and response variable "life expectancy from birth." Clearly, in countries where women receive more education, people in the country tend to live longer. Follow-up questions for students to discuss in pairs or group include: Is education causing people to live longer? How do you explain this? In the second example on babies per women, it was helpful to discuss: In countries where women receive more education, they tend to have fewer babies. How do you

Figure 10.4: Gapminder World demonstration of correlation between education (mean years of schooling for women 25 years and older) and life expectancy for countries in the world.

explain this? What's really going on here, in your opinion? Adult learners tend to have little problem coming up with a causal chain.

As a final activity, I discuss per capita Gross Domestic Product (GDP) as a response variable and pose the question: What is a better predictor of per capita GDP: the level of education of men or women in the country? In advance of class, I download the country data on these three variables: per capita GDP, mean years of schooling for men, mean years of schooling for women. After explaining the variables, students work in groups to explore. My goal is to have them use a combination of the correlation coefficient, coefficient of determination, and the slope of the regression equations to draw a conclusion. After their exploration, students share their conclusions. The format of the sharing depends on whether the class meets face-to-face or remotely. According to the data, the correlation is basically the same regardless of gender, although the slope is slightly higher for men than women. Students tend to focus on that small difference as evidence that educating men may have a better payoff. The final discussion around the topic is an opportunity to discuss the precision of the data, challenges with having only a single sample, and whether the small difference has practical significance. There is also discussion of lurking variables that contribute to a country's GDP.

10.3 Post-Lesson Considerations

In a college-level introductory statistics course, it is essential to elevate the learning of linear regression such that students move beyond their prior knowledge of linear equations and functions. While the reader may not be teaching in a corequisite context where prior knowledge cannot be assumed, my experience and that of my colleagues suggests that most students need assistance transferring their prior learning of linear equations to linear regression. This LAM demonstrated my approach to teaching the topic in a technology-mediated class. The activities could also be done with graphing calculator technology but would be more limited in providing students quick access to data afforded through the statistical software I employed. Without technology, it is doubtful that students could engage with the level of complexity presented here. For a reader in this situation, available OER products such as Gapminder World provide some opportunity to demonstrate concepts to students. An extension of the topic that I have yet to actualize in a class is to have students explore a host of independent variables to determine which might be the best predictor of a response variable (e.g., what predicts diastolic blood pressure: weight, age, BMI, age of first child, marital status). Related to this, I would also like to build examples where the independent and/or dependent variable is an ordinal variable, which is a frequent situation in the social sciences.

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11

Proving, Analyzing, and Deepening Understanding of a Structural Property in Abstract Algebra

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Topic of lesson	This lesson focuses on a structural property theorem (isomorphism preserves the abelian property) with attention to analyzing and refining proofs and statements.
Course content	This lesson takes place in an introductory, undergraduate abstract algebra course taken by mathematics majors and future high school teachers. The course was taught at a large, Hispanic-serving, state university.
Instructional challenge	In undergraduate mathematics classes, theorems are often treated as finalized. The overarching goal for this lesson was to develop a task sequence where students could analyze and refine an existing theorem and proof approaches via engaging meaningfully with each other and the mathematics.
Brief overview of instructional approach	This lesson forefronts a structural property in abstract algebra. Students bring initial proof approaches to class that tend to vary in terms overarching structure and what assumptions in the theorem are explicitly used. By having students present and compare their proof approaches, students are positioned to notice important differences which can then lead to a rich discussion of what assumptions in the statement are actually needed to arrive at the conclusion. Further, the approaches vary in the degree they use the conclusion of the statement to structure the proof. Through this process, students engage with each other's thinking and use numerous tools (such as analyzing proofs for use of assumptions, creating diagrams to connect between the formal proof statement and informal ideas, and generating counterexamples). The lesson concludes with reflection on these activities and the impact of proof structuring decisions.
Keywords	Abstract algebra, proof, proof analysis, weakening or strengthening theorems, proof frameworks

11.1 Background information

A common task in an introductory abstract algebra course is proving statements of structural properties:

Suppose G and H are isomorphic. If G has property X, then H has property X.

My colleagues and I have observed that students commonly prove such statements by making arguments about the image of elements in G rather than arguments directly about elements in H. Such an approach unintentionally weakens the theorem. This difference is often subtle for students and reflects a reliance on going directly from assumptions to conclusions rather than using the conclusion to structure a proof. In this LAM, we situate this proof approach not as an "error," but rather an opportunity for students to explicitly address assumptions and conclusions of statements.

This lesson focuses on the particular theorem:

If G and H are isomorphic, and G is abelian, then H is abelian.

We have found that majority of students actually produce the proof of the weaker statement:

If G and H are isomorphic, and G is abelian, then $\phi(G)$ is abelian.

Juxtaposing the standard approach (that proves the stronger statement) and non-standard, but common student approach (that often proves the weaker statement) can provide grounds for fertile discussion on these topics via contrasting the arguments, considering alignment of proofs with the theorem statement, and making modifications to both proofs and statements. We value students engaging in activity that more closely mimics the work of mathematicians, and this particular theorem provides a meaningful and accessible setting for students to not just produce proofs, but also to analyze and alter proofs and statements.

11.1.1 Background and Context

In order to situate the instructional challenge and ensuing lesson, I[1] begin by addressing my positionality as an instructor. I teach abstract algebra using an active approach with mix of existing inquiry-oriented curriculum [2] and tasks my project team and I have developed to engage students in various types of proof activity. I value creating a classroom where students are positioned as contributors to mathematics – that means I ask my students to not just create proofs, but also reinvent key ideas, analyze proofs, create conjectures, generate examples, and connect between intuitive ideas and more formal versions. During class, I often have students work in small groups to explore and generate new ideas prior to formalizing them in a whole class setting. Further, I believe that such a classroom is only productive if students feel comfortable working in mathematical messiness where they both voice and value their own and others' mathematical ideas at all stages (not just final, "correct" versions.) As an instructor, I strive to incorporate opportunities for all students to engage with, and contribute to, the mathematics of a lesson.

This LAM includes a series of activities that have been developed, tested, and refined at a large, Hispanic-serving, state university. This lesson is intended to be taught within a single 80-minute class session of an abstract algebra course. At our university, abstract algebra is a one semester course that acts as an introduction to the basics of group theory with the option for a brief introduction to the theory of rings. The course is typically taken by students pursuing a bachelor's in pure mathematics or teacher certification in mathematics and occasionally by graduate students who require leveling before advancing further in abstract algebra. The students are usually in the final year of their degree plan and have taken prerequisite courses including introduction to proof and linear algebra. The class size for this implementation was 18 students.

11.1.2 Instructional Challenge

I have found it difficult to find opportunities for students to engage with theorems as open, rather than closed pieces of mathematics that cannot be altered. Undergraduate students are typically exposed to the strongest versions of theorems (at least in relation to the content which they have been exposed.) However, weakening and strengthening assumptions is an important type of mathematical activity that is often tied up in the process of conjecturing and proving. As such, this LAM is addressing the overarching challenge of engaging students in analyzing and refining a theorem driven

[1]Because this task was developed as part of a multi-person project, *Orchestrating Discussion Around Proof (NSF DUE- 1836559)*, this manuscript will use "we" to represent the full author team, and "I" when the first author/implementing instructor is speaking.

by student thinking and varied proof approaches. Our overarching goal is for student ideas to drive the process of analyzing and refining theorems and proofs.

11.1.3 Learning Goals and Important Mathematical Ideas

We will outline a series of activities where students are positioned to prove, analyze, and modify this structural property. Our abstract algebra content goal is to enrich understanding of structural properties by:

- recognize the roles (or lack of role) of onto, 1-1, and homomorphism in making an argument about a structural property which may include de-entangling ideas of well-defined and 1-1
- deepen their understanding of the components of isomorphism and what it means to preserve a structural property. Our proof content goals and activities include:
- recognizing the impact of using a conclusion versus hypotheses to structure a proof,
- developing skills in analyzing and modifying proofs, and
- developing skills in analyzing and modifying statements.

The Theorem and Standard Proof Approaches. *Theorem: If G and H are isomorphic, and G is Abelian, then H is Abelian*. A variant of this theorem is ubiquitous to introduction to group theory. The crux of this theorem is that isomorphism preserves commutativity of elements. In our work as instructors and researchers ([5]) we have found that students provide one of two standard proof approaches:

Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian H is abelian.*

Proof. Let $c, d \in H$. Since ϕ is onto, there exists $a, b \in G$ such that $\phi(a) = c$ and $\phi(b) = d$. Then

$$c \cdot d = \phi(a) \cdot \phi(b) = \phi(a * b) = \phi(b * a) = \phi(b) \cdot \phi(a) = d \cdot c.$$

Thus for all $c, d \in H$, $cd = dc$. So H is abelian. □

Theorem. *Suppose G and H are isomorphic groups. Then if G is abelian, H is abelian.*

Proof. Let $a, b \in G$. Then

$$\phi(a) \cdot \phi(b) = \phi(a * b) = \phi(b * a) = \phi(b) \cdot \phi(a).$$

Thus, for all $\phi(a), \phi(b) \in H$, $\phi(a) \cdot \phi(b) = \phi(b) \cdot \phi(a)$. So H is abelian. □

The first approach is a more "textbook" approach where the student begins by selecting arbitrary elements from H, leverages that the isomorphism is onto, and concludes that these elements must commute. In the second, more common student approach, the student begins with elements from G and argues that their images commute. This version never explicitly draws on the fact that an isomorphism is "onto." You may be tempted to think that students might just have left this notion implicit, but the interviews we have conducted with students reflected that usually they were not thinking about arbitrary elements from H and did not see the role that *onto* played [5].

Beyond this major difference, we can see the mechanics of both proofs are pretty similar. Commutativity of elements in G is leveraged to argue about commutativity of elements in H. This argument relies on the homomorphism property. What you might also notice is that these arguments do not rely on 1-1 at all. In fact, the statement itself could be made stronger:

Suppose there exists an onto homomorphism between G and H[2]. If G is Abelian, then H is Abelian

The Underlying Mathematical Content and Student Thinking. In this section, we briefly outline some of the ideas related to isomorphism that come into play when engaging with this theorem. Isomorphism is a particularly complex subject for students (see [3]). To understand that two groups are isomorphic requires an appreciation for abstract structure: the two groups are "essentially the same" or the same group with different labels. We use an isomorphism to codify this relabeling scheme where the bijective property guarantees all elements are relabeled, and the homomorphism property guarantees the re-labeled elements interact the same way. The link between sameness and the

[2]Alternatively, this could be phrased "Suppose H is a homomorphic image of G." However, this language was not introduced in this course's curriculum

isomorphism map is often tenuous for students (see [9]). Spending time on structural properties can aid in supporting a sense of structural sameness for students. If two groups are isomorphic, they share all of their structural properties.

When digging into a structural property theorem, students are also positioned to gain a deeper appreciation for the various properties of the isomorphism map including: 1-1, onto, homomorphism, and well-definedness. We have found that students working with isomorphisms may not even be aware that this map meets the function requirements or even what it means to be a function in this formal setting [7]. While this is certainly not the first time students encounter functions, this is a situation where there is a meaningful way to compare and contrast the properties of a function being everywhere-defined (any element in the domain has an image) and *onto* (any element in the co-domain has a pre-image), as well as well-defined (any element taht has an image, has unique image in the co-domain) and 1-1 (each element in the image has a unique pre-image in the domain.) Many students conflate these pairs of properties, respectively. By analyzing the proof of this theorem, students can spend time identifying whether and where such properties are being used. This can provide grounds for building deeper understanding of functions and their properties.

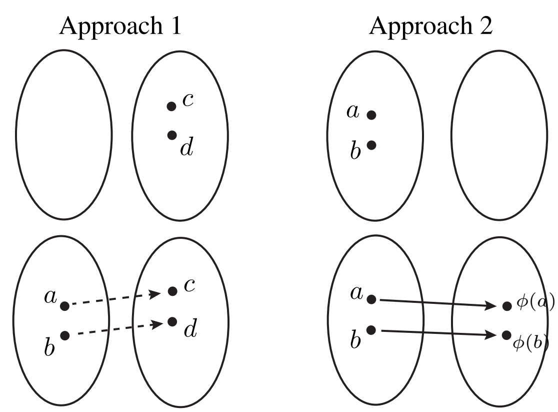
Additionally, this theorem highlights the essential role of homomorphisms in preserving structure. Within the mathematics education literature and our experience, it is not uncommon to find students focusing on the existence of a bijection without necessarily attending to the homomorphism property. This is unsurprising as students bring with them knowledge of bijective functions (such as for cardinality arguments), but homomorphisms are likely a new concept.

Rather than necessarily going through a suite of these theorems, we have found it productive to dive deeply into this one standard example theorem to aid students in increasing their understanding of isomorphism overall, as well as functions and various function properties. Students can then spend time working with other such theorems in their homework after working deeply with this example theorem in class.

11.1.4 The Underlying Aspects of Proof and Student Activity

We developed this lesson not just because of its links to important content, but also because this particular theorem and proofs can provide opportunities for students to engage in activities beyond producing a formal proof. Much of the undergraduate approach to proof is in a definition-theorem-proof (DTP) format where we present definition(s), a relevant theorem, then its proof. In many ways such an ordering makes sense with the mostly simple and fundamental proofs in the early courses. However, activity around proof goes beyond this formalism and includes activities like comprehending proofs, refining proofs, and refining statements. The theorem we choose is ripe for this type of activity because there is space to refine both standard proof approaches and the theorem itself. It also affords these activities in a setting that we find is comparatively accessible to students.

The Role of Diagrams in Comprehending Proofs. We launch the lesson by highlighting the two proof approaches above, having students engage in making sense of them, and comparing them. This type of activity reflects proof comprehension goals. We leverage function diagrams as a means to differentiate the two approaches:



Diagrams in particular have been documented to serve an important role for mathematicians and students in making sense of formal arguments.

The Role of Proof Frameworks. Further, comparing the proofs positions students to notice elements of the *proof frameworks* (see [8]). In the math education literature, it is common to refer to the general structure of a particular kind of proof as a proof framework. A given statement will usually be aligned with a proof framework, such as proving a claim about all elements of a set by first selecting an arbitrary element. A first level proof framework for the statement in our lesson is:

Suppose G is an Abelian group and G and H are isomorphic.
...
...
Therefore H is an Abelian group.

A standard second level proof framework would be:
Suppose G is an Abelian group and G and H are isomorphic.
Let $c, d \in H$
...
...
Then $cd = dc$.
Therefore H is an Abelian group.
The more common student approach is as follows:
Suppose G is an Abelian group and G and H are isomorphic.
Let $a, b \in G$
...
...
Then $\phi(a)\phi(b) = \phi(b)\phi(a)$.
Therefore H is an Abelian group.

Students often take this approach because it begins by moving from the hypothesis (in G) to the conclusion (in H). Discussing proof frameworks can be powerful for thinking about how a proof is aligned with a statement. In this context, students often differ in their second-level proof frameworks which can draw attention to the particular role that the conclusion plays in structuring the proof. It often takes time for students to develop the lens needed to explore the alignment of a statement and proof framework.

The Role of Analyzing and Refining Proofs and Statements. In addition to these aspects of proof comprehension, this theorem and proof provide grounds for *proof analyzing*. Students often notice that in one approach onto is used and in another approach, onto does not appear to be used. This noticing can lead to discussion of revising the statement to just what is needed. In our experience, students almost univrsally dismiss the need for onto and one-to-one arriving at a statement like:

Suppose there exists a group homomorphism from G to H. Then if G is abelian, H is abelian.

At this point, this statement can be analyzed for potential counterexamples. The production of a counterexample is likely a trivial task for a mathematician (pun on the trivial homomorphism intended), but challenging for students. Depending on the curriculum being used, an accessible counterexample might be using the trivial map to go between a familiar abelian group (such as Z_4) to a familiar non-abelian group (such as D_3 or a group of matrices). Developing examples and counterexamples is essential to the work of mathematicians, but not always asked of students. In the next sections, we discuss some of the scaffolding that might be needed. In particular, students will not quickly or easily create or recognize a counterexample without some explicit consideration to properties a counterexample would have (i.e., the assumptions are true, and the conclusion is false). Students often do not have an approach to generating examples and counterexamples. Trying to haphazardly produce abelian groups, non-abelian groups, and particularly a useful homomorphism can be unproductive. While experts quickly look for examples such as simple cases (the trivial maps), boundary cases, or generic cases, we have found most students have not developed this strategic knowledge.

Testing with counterexamples is important for analyzing and refining proofs and statements. A second important activity is the analysis of the proof itself. In this case, counterexamples can be found for the statement because the onto property of the homomorphism is necessary. The proof can however be a powerful tool for students to identify where onto is needed. This is especially true if they are analyzing a proof where onto did not explicitly play a role in justifying a particular statement. For example, a justification can be inserted into the standard student approach to patch the proof. $\phi(a)\phi(b) = \phi(b)\phi(a)$ can imply H is abelian if paired with the warrant[3]: "because ϕ is onto, all elements of H are images of elements from G." More briefly, because ϕ is onto, $\phi(G) = H$.

In contrast to onto, one-to-one does not in fact play a role in this proof. A new, stronger, valid statement can be created:

Suppose there exists an onto group homomorphism from G to H. If G is abelian, then H is abelian.

Students often desire for 1-1 to be necessary for the proof. We attribute this struggle to three causes:

1) students are not used to having theorems where not all the assumptions are needed.
2) students conflate well-defined and 1-1 and often think 1-1 is needed in the proofs for claims such as "$ab = ba$" implies "$\phi(ab) = \phi(ba)$."
3) students want to guarantee that c and d are distinct elements.

Each of these occasions is a worthwhile discussion about the nature of proofs and definition of 1-1 versus well-defined.

11.2 Lesson Implementation

This lesson implementation occurred over the course of one 80 minute lesson. I reflect on implementing a lesson that my colleagues and I originally developed in a laboratory setting with small groups of students ([6]). This lesson involves alternating between partner discussions, student presentations, and whole group discussions. An outline of the lesson is as follows:

Promoting Students Opportunity to Access the Theorem (10 minutes)	• Reflecting on what is needed to begin a proof • Identifying key terms and relevant definitions
Sharing and Sense-Making of Each Other's Proof Approaches (25 minutes)	• Exchanging proof attempts with partners • Students presenting the two proof approaches
Comparing Across Proof Approaches (10 minutes)	• Identifying key similarities and differences in the proofs
Analyzing and Refining Proofs and Statements (20 minutes)	• Conjecturing which of the assumptions from the statement are needed • Modifying the statement accordingly • Testing the new statement(s) with examples and by analyzing the proofs
Conclusion and Wrap-Up (5 minutes)	• Summarizing and reflecting on big ideas

In each case, we share not just the type of task, but the nature of the activity (whole group, pair discussion, individual think time) and provide a rationale for these decisions.

11.2.1 Promoting Students Opportunity to Access the Theorem

Prior to the class, I had students think about the theorem at home and prompted them to make an initial proof approach. I made this decision for a few reasons. First, my main goal in this lesson was for students to engage with proof approaches, and so I did not want to lose substantial class time for proof construction. Second, by allowing students to work on this at home, I am not advantaging students who are quicker to put proof ideas on paper.

I then launched the lesson with a whole class discussion around the components of the proof. In prior iterations, our project team has found that recalling definitions and decomposing the statement can create barriers for students to

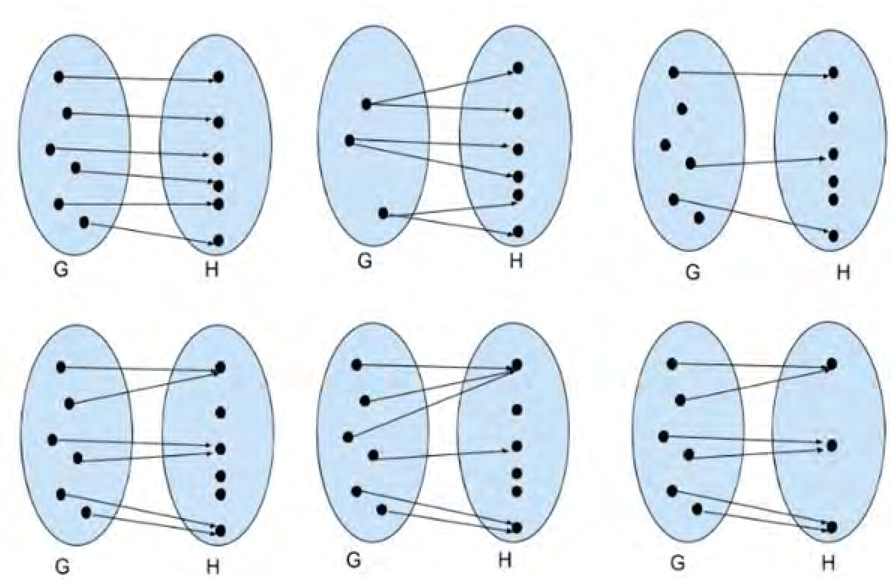
[3]By warrant, we mean the reason or justification that a given line of a proof is valid. They may be explicitly stated or left implicit. See Alcock & Weber (2005) [1].

engage richly. Thus, I began the lesson by providing the theorem and prompting students, "[B]efore we even go about starting to prove something, what are some things that we might think about?" Several students contributed ideas like looking at assumptions and conclusions ("what we want to prove"). I used these suggestions to write a public record on the board that including a list of relevant assumptions and conclusions. We also spent time defining each of the relevant terms. This was particularly important for isomorphic groups where students contribute properties 1-1 and onto. I have found that many students think of the groups of having these properties, rather than a function. This provided a chance for me to ask, "What has the property 1-1?" and focus on the fact that a function needs to be introduced (an essential step in the proof approaches.)

I made the choice to include the list of assumptions, conclusions, and definitions on the board, so we could refer to these throughout the class as students worked to improve their proofs and refine the statement. This also was an access decision where this information could be available for students to use at any time.

Additional Relational Sort Activity

Since 1-1 and well-defined, and onto and everywhere-defined function properties have many similarities, I have found it worth spending extra time disentangling these properties. In this implementation, I had students work with partners for 20 minutes to decide which properties the set of diagrams below have. This could be done as homework or on another day when the properties are first introduced. In future implementations, I will likely move this activity to a different day to allow more time for the focal activity.



11.2.2 Positioning Students to Share and Make Sense of Each Other's Proof Approaches

After students spent time in the additional activity and we debriefed as a class, I returned students to their partners to exchange their proof approaches. I had the partners decide who would be "partner A" and who would be "partner B." As an instructor, I have often found it difficult to support all voices being heard in groupwork. By structuring partner work, I hope to engage students in meaningfully contributing and valuing each other's ideas. In this case, I prompted the partners to address:

What is one thing about this proof approach that makes sense to you?
What is one thing that you have a question about?

After each student had time to think about their partner's approach, I had partner A share their sensemaking and question, then after a few minutes, partner B. By having students share something that makes sense, I was hoping to reinforce the norm that everyone contributes something valuable and to avoid conversations driven by just corrections and criticisms.

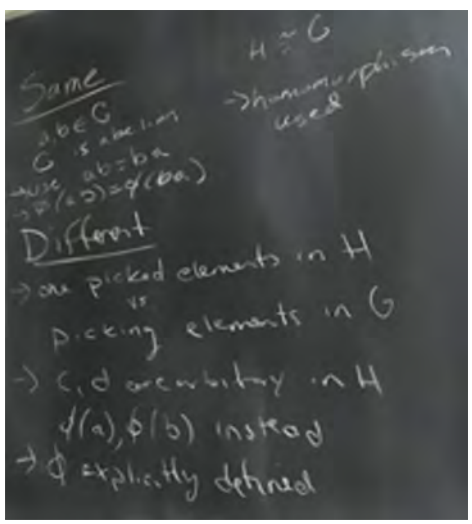
> **Additional Information**
> I provided an empty diagram on the board so the students had a better sense of what I was asking them to do.

While students engaged with their partners, I walked around to find a "G-first" proof approach and an "H-first" proof approach. After finding these students, I asked if they would be willing to share their approach in a bit. I provided this information in advance so the students would have time to prepare (and hopefully) feel more comfortable sharing at the whole class level. Because I was focused on finding these approaches, I found that I had less time to focus on interacting with the students during partner work. This was a tension for me, and I hope to improve at attending to both aspects as I walk from group to group.

After the partner talk was completed, I had the two selected students share their approaches on the overhead. I prompted the students to listen and create a function diagram that represents what is happening in the proof. I paused the presentations after key steps, such as when the student explained picking the elements from G or H first in order to allow for students to encode that choice into their diagram. I wanted students to notice the important structural difference between where initial elements were selected. After each student shared their approach, I had students return to brief partner talk, again for the purpose of including everyone in the discourse. They were prompted to again think about one thing that made sense and one thing they had a question about. I had a few students share out ideas to emphasize the norm of engaging in proof approaches this way. I cycled through this process for both the student that provided a "G-first" approach and the student who provided an "H-first" approach.

11.2.3 Comparing Across Proof Approaches

Following the presentations and discussions around the two proof approaches, I projected the approaches next to each other and asked students to think about, "What do you see that is the same, what do you see that makes them different?" At this point, I gave the students some private think time. I know that different students process differently, and I did not want the conversation to be dominated by those who process out loud. After a few minutes, I had partner A share a similarity and partner B share a difference. I again relied on these partner structure to try and create more equitable participation.



I then switched to whole class discussion and created a record on the board of similarities and differences. I found students usually notice the difference in picking elements in G and H, the difference of c and d used instead of $\phi(a)$ and $\phi(b)$, and the similarity of using the warrants: G is abelian and the homomorphism property. At this point, students often notice that the H-first argument used "onto." In this class, the students did not suggest this difference, and so it became a question I asked later. The public record helps students recall what similarities and differences were mentioned throughout the rest of the lesson and can provide grounds for exploring and altering statements in the next section of the lesson. The goal of this comparison is for structural elements of the proof to be noticed that may otherwise stay hidden. In my experience, students see both proof approaches as valid, and often do not consider them different arguments until explicit analysis.

11.2.4 Analyzing and Refining Proofs and Statements

After the comparison is complete, I asked the students if we needed all the assumptions (that are listed on the board from the first part of the discussion.) Because of the comparison, students have noticed that not all assumptions were used in both approaches. I had students again spend time talking with their partners asking, "Do we need all of the assumptions in the statement?" and "How might we change the theorem to leave out the extra assumptions that are not needed?" I anticipated that students may modify the statement where ϕ is a homomorphism instead of an isomorphism due to the fact that the G-first approach did not leverage 1-1 or onto. The small group time gave students time to engage with this idea and allowed me to hear the various conjectures students made.

I then returned to whole class discussion to ask for what assumptions were needed. A student contributed that the G being abelian was essential. Another student shared the isomorphism was needed. While a third student countered, "We don't have to specifically use all the properties of isomorphism in the proof" with another student expanding, "I think you need homomorphism." At this point I wanted to encourage students to grapple with whether 1-1 and onto were needed and did not endorse a particular statement. I formalized the alternative conjecture on the board: "Let ϕ be a homomorphism from G to H. If G is abelian. Then H is abelian."

I returned students to partner discussion to "come up with examples that may help show this conjecture is true or produce a counterexample to disprove our conjecture." After partner talk, I brought students back together for a more heavily guided exploration of creating a counterexample. I wanted to emphasize how I think about creating counterexamples because I know how complex it is to think about the logical quantification in a statement and develop a useful counterexample (see Lynch and Lockwood (2019) for a discussion of student and mathematician example-related activity [4]). I spent time asking students to address each part of the statement, "If you wanted to construct a counterexample to this, what would that look like? What would need to be true about G? What about H? What about ϕ?" Students voiced that G had to be abelian, H had to be non-abelian and ϕ had to be a homomorphism. I continued the interactive lecture by asking students to provide examples of abelian and non-abelian groups they were familiar with. After a little struggle to produce a non-abelian group example, a student shared the dihedral group. I once again leveraged a function diagram because I thought this would help illustrate the structure and could side-step the difficulty of using a symbolic representation of the function, which is often what students prefer to construct first. Further, the property of "onto" is easier to see in the diagram. I also asked for students to think about the "easiest" homomorphism as a way to emphasize some of the strategic choices made when trying to construct examples. We arrived at the counterexample $\boldsymbol{Z}$ to D_3 where all elements map to the identity.

Because I did a lot of the heavy lifting in terms in the counterexample creation, I had students think about, "I claim that this is a counterexample to the statement that only having a homomorphism is enough to guarantee that if G is abelian, H is abelian. What is our problem and how does it relate to onto?" and share out in whole discussion to reinforce why this example is a counterexample to the statement. Further, I wanted to make sure all students saw that this function was clearly not onto. This led to some discussion and a decision that onto was a needed assumption where I endorsed that if we do not have onto, "we can essentially have elements hanging out that don't commute."

At this point, students are naturally curious as to whether 1-1 is also necessary. I gave students time again to talk with their partners. I instructed them to go back to the proof approach and see if 1-1 was needed anywhere in the arguments. Due to time constrains, I quickly took up some ideas that it may be needed because c and d are treated as unique students. We discussed that they do not have to be unique just because they have unique names. Ultimately, many of the students voiced that they did not see anywhere 1-1 was used. I endorsed this idea arriving at the version of the statement:

"Let ϕ be an onto homomorphism from G to H. If G is abelian. Then H is abelian." Finally, I had students go back to the two proof approaches for one final round of analysis. The counterexample gave us insight that onto was needed for the statement to be true, but I wanted the students to see if they could identify if and where onto was needed in the two student proof approaches. They were able to easily identify the role of onto in the H-first proof where the arbitrary elements come from the co-domain. However, the role of onto was obscured in the G-first approach, and the proof itself would need modification to become valid. Due to the class period running out, I had a brief discussion of the fact that using $\phi(a)$ and $\phi(b)$ does not guarantee arbitrary elements from H and the need to include additional argumentation for why those images could represent arbitrary elements.

11.2.5 Class Wrap-Up and At-Home Activities

I concluded the lesson by summarizing some of the big ideas by tying back to the differences noticed when comparing the arguments. Sometimes I have led a more complete reflection on the type of activity involved. For this implementation, I explained the final reflection I wanted them to do for homework with a focus on the role of the conclusion in structuring the proofs:

> What I'm going to have you do then, on your homework reflections is, think about these two approaches and why we might start one way and start the other way and why it seems like when we put the arbitrary elements inside of H, we didn't end up accidentally making some assumptions or hiding things about that were really important about making an argument that these are truly arbitrary.

In their journals, I had the students reflect on:

> One of the major differences between these approaches was selecting arbitrary elements from G and H to begin. That is, these proof approaches had pretty different structures.
>
> 1. How might the conclusion of a statement help you to figure out how to start the proof of that statement?
> 2. Some of our proofs didn't make use of all the needed assumptions initially. How were we able to correct these proofs by identifying those unused assumptions?
> 3. Did any of these activities make you think about how you engage with proofs (constructing, reading, etc.)? If so, which activities and how?

My focus in these questions was to help students reflect on the important role of looking at the conclusions. However, the fifth question was meant to have students reflect back on modifying the statement and the larger process of how analyzing proofs and statements can work together.

11.3 Post-Lesson Considerations

11.3.1 Reflecting on Equity and Modifications for Future Iterations

After implementing this lesson, I think many of my goals were successful. First, there was a high degree of engagement from many students. The work with partners seemed particularly productive. Further, we arrived at the lesson agenda goals: conjecturing and modifying a statement in conjunction with proof analysis. However, I found myself spending more time on the first parts of the lesson and less time on the concluding aspects. In the next iteration of the lesson, I would have students engage with the relation sort activity in a prior lesson rather than spend twenty minutes of class time on it. In this way, I can allow more time for students to work with partners analyzing the proofs for various properties and engaging richly with the modified statement. This could include using a function diagram to illustrate that each step of the proof works without the one-to-one requirement.

Additionally, I reflected back on whose voices were heard and for what purpose. While many of the women in my class engaged in smaller parts of the conversation, the men in my class often contributed more substantively. This was particularly noticeable because the two proof strategies came from two of the most vocal men in the class. As an instructor, I often feel torn between the wanting to move the mathematics agenda forward in particular ways and making sure that I lift up a diverse set of voices. I struggled in this class because many of the students had very partial proof approaches, and so I selected the two students who had mostly complete proofs. Since the focus was on analyzing (and not constructing) proofs, I made this tradeoff. However, in the future I have several modifications I am considering. First, I asked students to "make an attempt" at home. I will state the directions to create a proof so that more students might bring in mostly completed attempts. Second, if I find that most students do not have complete proofs, I will have two sample versions that I can hand out and then charge students with making sense of them. That way more students can be in a position to present them. In fact, I had this in my back pocket if the two approaches did not come out. I think equity reasons are just as valid as content reasons to introduce sample student work.

11.3.2 Considerations for Adapting this Lesson to a Different Context

When implementing this activity, instructors might wish to adapt the project to fit the structure of their course. As noted in the implementation section, certain background activities can be separated out and used very early in the semester, either as a short in-class activity or as a homework assignment. In general, it benefits students to develop a deep understanding of the algebraic content goals early in the semester. An instructor that primarily lectures may want to consider adapting just a portion of this task. For example, samples of the two proof approaches could be projected and the class could engage in a short conversation about the differences between the approaches. This could lead to an interactive lecture about why *onto* matters and the importance of using the conclusion of a statement to structure a proof. Alternatively, many of the more active portions of this lesson could be turned into homework activities.

For courses structured to explore the concept of isomorphic groups shortly after groups are introduced, the current structure of the activity is likely to align well with course materials. Other courses and accompanying textbooks are structured to focus on homomorphisms as the key functions to study early on and then bring in isomorphisms later when discussing preservation of structural properties. For these courses, minor modifications can be made in order to conduct the activity early in the semester, with the option of revisiting or referencing it when structural preservation is later introduced. An instructor could simply state the theorem using "one-to-one and onto homomorphism" in place of "isomorphism," or could open with an exploration of the question "If there is a group homomorphism from G to H and G is abelian, does H need to be abelian as well?" Students who say "yes" can work in small groups to attempt a proof, while students who say no can search for a counterexample and try to correct the statement. The results can be brought together in a class discussion following the broad outline of the activity above. If this modification is used, the instructor should be prepared to follow the conjecture phrasing found in footnote 2.

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12

Developing Integrals in Multivariable Calculus: The Boysenberry Patch

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Topic of lesson	This lesson is focused on helping students form conjectures and generalizations in the context of multiple variable functions. This leads to their development of a 'chop, multiply, and add' approach to integration.
Course content	The activity is appropriate for a multivariable calculus course with or without vector calculus. The activity is used prior to formal discussion of integration.
Instructional challenge	Students create models for the amount of Boysenberries picked along various paths drawn on contour maps and/or 3-dimensional 'surface' manipulatives representing the berry density. These models lead to opportunities for students to propose (and refute) conjectures using data from their case; These models also let students develop a 'chop, multiply, and add' approach for accumulation (integration) in the multivariable setting. Students are not given formulas for the berry density, encouraging students to discuss concepts and relationships needed for the path integrals.
Brief overview of instructional approach	Students develop reasoning, develop models, and recognize patterns while working in small groups which are then shared publicly during one of two whole-class discussions. Student groups work with one of six different contour maps representing the density of boysenberries. Students rank the amount of berries that could be picked along three long paths and one short path which extend through the berry patch. Groups provide their ranking and explanations during a class discussion and that inevitably leads to conflicting conjectures, each of which can be refuted or supported with student-generated arguments. Back in their groups, students develop estimates for the amount of berries that could be picked on a path. These models obviously account for path length and a representative berry density, but unit analysis and the act of picking berries in a wild bramble requires student incorporate a width, or arm reach, into their model. Group discussion directed toward improving these models leads to a 'chop, multiply, and add' approach toward accumulation (integrations) which is used throughout multivariable and vector calculus.
Keywords	Conjecture, Multivariable Calculus, Integration, Whole Class Discussion

12.1 Background Information

The activity discussed in this LAM was developed for the *Raising Calculus to the Surface* project (NSF DUE-1246094). We discuss here the intentions of the activity and its use in the classroom.

12.1.1 Instructional Challenge and Learning Goals

Are undergraduate mathematics students like history students, who study history, or like cooks, who study cooking techniques but also create new dishes?

I want to engage students in doing mathematics in the classroom in an authentic way: Rather than *telling* students the important ideas and *showing* them how to use those ideas, I want to hear *students propose ideas* and hear *students discuss them*. I want to build the course content on their notions of important concepts. In this classroom setting, students have an obligation to share their hunches and conjectures, to critically analyze and question the ideas of their peers, and to draw out patterns or abstractions from the arguments shared with classmates. In this classroom setting, it is important for student voices to be invited, involved, and valued in this discussion.

There are several behaviors and expectations that students and instructors must address in order to create such a classroom:

1. This classroom breaks the stereotypical role of instructor-as-teacher and student-as-learner, and can be uncomfortable for both student and teacher.
2. Students may be hesitant to share ideas, especially when they contradict existing knowledge.
3. Instructors must trust student contributions will occur and invite those contributions (whether right, wrong, or ill-defined!).
4. Everyone must trust that student contributions can lead to valuable discussion that build course content.
5. Everyone must trust that definitions and theorems that capture important concepts will arise from student observations within the course.

All of these challenges pose a threat to building the course upon student ideas. In order to overcome the first two challenges, students need to be able to share and discuss their ideas – and they need to feel comfortable posing questions or discussing ideas arising from their own observations or confusions. The last three challenges often conflict with mathematics culture, where ideas are typically presented only after significant revision: Someone else (typically a professional mathematician) has carefully crafted definitions to capture nuance and circumstances; Someone else (typically a professional mathematician) has provided necessary conditions and precise conclusions for theorems; Someone else (typically a professional mathematician) has summarized significant ideas with specific notations. Rather than limiting these activities to professional mathematicians, I want students to engage in all of these activities. In this light, it is necessary that students (mis)use concepts and definitions before they are well defined, use (and confuse) notations as they discuss relationships, and (mis-)apply theorems to situations to explore the necessity of conditions. These actions are not deficits on the part of students, which might usually limit student engagement, but are seen as necessary ways to help students discuss and explore concepts and relationships.

This LAM describes a *discovery activity* called *The Boysenberry Patch* that lets students create mathematics conjectures and propose content as they explore solutions to a contextualized problem in small groups. Further information about *discovery activities* are included later in Section 12.1.4, but student groups explore their own specific situation and the instructor facilitates a whole class discussion in which students share their results and develop (and refute) conjectures. This small group and whole class discussion is aided by the design of the activity: It utilizes context to provides meaning and an additional access point to the underlying mathematics, minimizes notation so as to encourage exploration of relationships between math objects, and it utilizes tangible *surfaces*, contour plots, and measurement tools (See Figure 12.1) which represent mathematical objects while also serving as common work areas for students in the small groups. The activity for this LAM can work without the Surface materials, as students work primarily with the contour maps.

12.1.2 Specific Learning Objectives

In addition to engaging students in mathematical practices, this LAM also helps students development the *Chop, Multiply, and Add* conceptualization of integrals in multivariable and vector calculus. Students coordinate quantities

Figure 12.1: Students make measurements and discuss ideas surrounding multivariable functions using a *Surface*, a tangible representation of a function of two variables with a dry erase finish.

along paths traveling through a boysenberry patch, developing the *precursor* to the line integral which provides a total amount of berries picked along the path. We use the term *precursor* since students account for quantities and relationships that are represented in the line integral form, but do not necessarily write down a formula for it using correct notation. One advantage of the development of the precursor form is that it is usually developed geometrically and thus generalizes for all coordinate systems.

There are several specific learning objects related to Integrals in Multivariable Calculus:

1. Students will develop a model for the total amount of a quantity which utilizes context, units, and approximations
2. Students will investigate the association between path length, scalar (density) quantities, and accumulated quantities.
3. Students will develop a 'chop, multiply, and add' approach to find a total amount of a quantity.
4. Students will utilize length, density, and context to describe a quantity on a small segment of a path.

In order to help promote a 'chop, multiply, and add' approach to integration, the activity has students develop and then refine a model to estimate the total amount of berries that can be picked along a path winding through a boysenberry patch. Due to the path's sharp corners, the path is initially divided into four parts. Students are then free to subdivide each of those paths into smaller segments using techniques which they deem appropriate. Students who pay attention to units and context will recognize a need to incorporate *arm reach*, or a *width* of picking berries off the path, in order to have their model match the physical action of picking berries.

The activity includes several learning objectives related to the creation and ownership of mathematics:

1. Students find answers to an important question (*What is the total amount?*) by developing a model.
2. Students communicate a summary of the results and justification found for their specific case to their classmates.
3. Students propose patterns and conjectures based upon reports from small groups.
4. Students challenge and/or support conjectures using evidence from their specific cases.
5. Students discuss connections between various mathematical objects.
6. Students recognize the value of contradictions or objections to forming mathematical statements.

Items 3–6 above occur within whole-class discussion. Although not every student will engage in proposing conjectures or make connections between various mathematical objects, the emphasis is that this action is done by students in the course rather than the instructor.

12.1.3 Course, Institutional Context, and Student Audience

This activity has been used by several instructors at a range of institutions across the United States. At the author's home institution, a regional university in the upper midwest United States, the activity is used in a third semester calculus course which combines multivariable and vector calculus content. This course meets four times per week for fifty minutes. Of the 30 students in the course, roughly half are physics or engineering majors and a third are math education or statistics majors. White males typically account for greater than three-fourths of the students. As described later in Section 12.2.3, students often associate integrals with computations to find area under a curve or as rules for finding antiderivatives, rather than as accumulation.

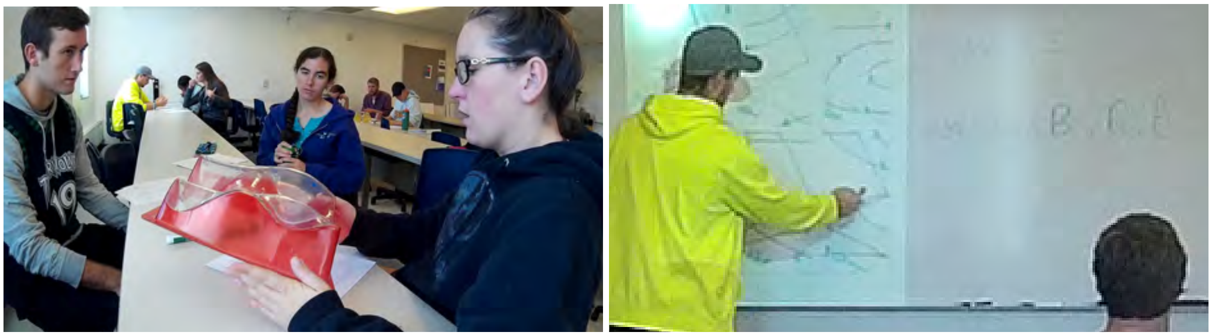


Figure 12.2: Students use a contour map or a tangible Surface, or graph, of the berry density to discuss the amount of berries that could be picked on a path. During whole class discussion, students share which path would produce the most berries and why they chose that path.

The Boysenberry Patch activity is part of the *Raising Calculus to the Surface* project [1, 2, 3], which consists of a set of instructional activities for multivariable calculus courses where students use plastic surface models ("surfaces") and accompanying contour maps, all dry erasable, to explore conceptual and geometric relationships. During *Raising Calculus to the Surface* activities, students work in groups to explore context-rich problems involving functions of two variables. The functions are presented to students either as a 3D plastic graph model or a contour map (not as equations). During the context-rich tasks, students explore the conceptual and geometric foundations of multivariable calculus concepts, such as level curves, partial derivatives, the gradient, line integrals, etc, before formal instruction is given on these topics. Students use tools to make measurements on the surface models and contour maps (See Figure 12.1, Figure 12.2), and different groups consider different functions so that conversations focus on concepts and processes rather than numerical answers. During whole class discussions, students ideas are valued by the instructor and students are positioned as *doing* mathematics (rather than *learning about* mathematics).

The instructor introduces a new topic in each week of the multivarible calculus course. This LAM is used on the first day (Monday) of the new content involving line integrals. The LAM provides students the chance to find relationships between different mathematical quantities, recognize the quantities that must be accounted for in line integrals, and begin to develop the *chop, multiply, and add* formulation of line integrals. The instructor formally covers content on Tuesday, beginning the class by asking students to explain the important concepts they found during the activity. Because they have previously grappled with the content, students often grasp the new definitions, concepts, and processes fairly quickly - leaving plenty of time for the instructor to work through example problems. In Thursday's class, students work in small groups on practice problems or activities from *The Vector Calculus Bridge Project* ([5]). The instructor uses Friday's class period to wrap up the week's content and set up the next week's content. For example, on the Friday before using this LAM, the instructor and students developed expressions for $d\vec{r}$ in rectangular, cylindrical, and spherical coordinate systems so that students had some experience describing small pieces of a path with $|d\vec{r}|$.

12.1.4 Teaching Practices, Content, and Instructional Decisions

The activity shifts between work in small groups and whole class discussion both to help pace the activity and to help promote student ideas throughout the classroom.

- In small groups, students explore the activity, discuss their interpretation of the context and situation, and develop their solution process.
- During whole class discussion, students explain the reasoning for their particular case and engage other students in mathematical debate.

While students work in groups, I actively listen to their ideas to decide how to order the student presenters. I listen for incorrect ideas and decide whether to address them in the moment, during whole class discussion, or in a subsequent lecture. I also listen for ideas that will drive the discussion toward the week's new content. Letting students explain their reasoning helps keep student voices central to the activity.

I usually put students into new groups for each class period, trying to ensure students are comfortable with each other in the group while also mixing together the various majors in the course. Using new groups for the activity helps ensure that a variety of prior student experiences are represented in each group. If students work in the same groups from one activity to the next, I've observed that groups often rely upon one primary approach to solve the problems and are more likely to ignore the approaches proposed by their peers. By mixing together students of different majors, it is more likely that students will discuss the differences in their understandings of key terms, like *small*, *accuracy*, and *error tolerance*. Ensuring groups have pairs of women or pairs of minorities can help students feel more comfortable sharing their ideas, but it is important to discuss this action with the students and let them tell you their preference for working with others.

Perspective on the Mathematical Content in the Lesson

Integrals describing physical situations in Multivariable and Vector Calculus (see Table 1) can be constructed geometrically using a *Chop, Multiply,* and *Add* technique using vector differentials

Rectangular Coordinates	$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$
Cylindrical Coordinates	$d\vec{r} = dr\,\hat{r} + rd\phi\,\hat{\phi} + dz\,\hat{z}$
Spherical Coordinates[1]	$d\vec{r} = dr\,\hat{r} + rd\phi\,\hat{\phi} + r\sin(\phi)\,\hat{\theta}$

To construct an integral, students use the problem context to choose the appropriate integral form and coordinate system. After chopping up the region of integration, students use one, two, or three vector differentials $d\vec{r}$ to describe the region. After taking the appropriate product to form $d\vec{r}$, $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$, or $dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3$, the resulting quantity (or its magnitude) is multiplied by the scalar or vector field to find the desired quantity on that small region. The limits of integration are chosen based upon the orientation of the vector differentials $d\vec{r}$ used to chop up the region. This technique provides a unified approach for scalar and vector integrals in rectangular, polar, cylindrical, and spherical coordinate systems ([4, 5]).

With this approach, it is important that students understand the geometric relationships between regions of integration, vector or scalar fields, and any curves, surfaces or volume regions used in the problem. Students need to associate dot products to projections between two vector quantities, and to recognize the need to use a cross-product to form a directed area between two vectors. Thus, it can be helpful for students to practice visualizing vector and scalar fields (e.g., surfaces), how surfaces or curves cut through vector or scalar fields, and to move between various representations of these objects.

Rationales for Instructional Decisions at the Whole-lesson Level

In this subsection, we discuss the purpose of incorporating this discovery activity in the course and using it to introduce integrals in Multivariable Calculus.

Discovery: There are several reasons this activity was designed as a *discovery* activity that should be completed before the instructor has formally introduced line integrals:

1. The students participate in authentic mathematical practices as they discuss new ideas and explore relationships.
2. The ideas developed and utilized in the activity can be owned by students.
3. The process of *telling algorithms to students* is avoided, or delayed, until *students connect and discuss mathematical ideas*.
4. The resulting mathematical concepts and processes have meaning for students when created in response to meaningful questions.
5. Pushing students to defend / critique their models or conjectures engages students in exploring the underlying connections and their significance on the results.

The *discovery* nature of the activity lets students wrestle with concepts and various representations of functions, make connections between them, and develop deeper understandings of concepts before the instructor provides formal definitions. In the instructor's experience, less class time is needed to cover new concepts or explain definitions because students have already worked with, and have an understanding of, the underlying concepts.

Several specific features of the activity are designed to support student discussion and development of the path integral concept:

Integral	Scalar	Vector
Line Integrals	Scalar Line Integral $\int_a^b \rho \lvert d\vec{r} \rvert$	Vector Line Integrals $\int_a^b \vec{F} \cdot d\vec{r}$
Surface Integrals	Surface Mass $\iint\limits_A \rho \, dA$ where $dA = \lvert d\vec{r}_1 \times d\vec{r}_2 \rvert$	Flux through a Surface $\iint\limits_A \vec{F} \cdot d\vec{A}$ where $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$
Volume Integrals	Mass Integrals $\iiint\limits_R \rho \, dV$ or Divergence $\iiint\limits_R \vec{\nabla} \cdot \vec{F} \, dV$ where $dV = (d\vec{r}_1 \times d\vec{r}_2) \cdot d\vec{r}_3$	

Table 1: Forms of integrals in Multivariable and Vector Calculus

1. The *Boysenberry* context: Students are likely familiar with picking berries, whether along rows (like strawberries or blueberries) or among brambles (like raspberries). The context spurs discussion about how berries are picked, a detail which must be captured in their resulting model.
2. The activity starts with a ranking task to encourage student discussion about the quantities that impact the amount of berries that will be picked on each path, without getting caught up in the specific computational details.
3. The paths are drawn on the contour map, but students are free to draw the paths on the surface manipulative. The computations are easier to complete using the details from the contour map. Although not necessary, the ranking task is generally easier to complete when using the surface manipulative. The activity does not direct students to use the surface, but its value often emerges during whole class discussion.
4. The units for the density of berries are given as liters per square meter, a reasonable unit for berry yield. However, this unit introduces a problem if students try to simply multiply the density by a length along the path. One feature of this mismatch is that students have to match their computational estimate with the physical action of picking berries, which requires extending an arm length or two off the path into the berry patch.
5. The first whole-class discussion serves as an early checkpoint, allowing students to correct misconceptions on the ranking task before they proceed to computing the amount of berries on specific paths. The discussion also serves to make sure students are attending to the path length, not just the berry density, in their ranking.

The whole class discussion provides a means to help student groups focus on important concepts within their small groups. First, the discussion helps the instructor broadcast the distinction between the value and the density of the contour lines, only one of which is relevant for the ranking task and subsequent computations. Second, by asking questions or highlighting student arguments, the instructor emphasizes features like path length, units, and berry density value that students will incorporate into their subsequent computations. Third, when students think critically about their reasoning and the reasoning of their peers, they evaluate which quantities matter for their problem. Finally, the discussion engages students recognizing patterns, forming conjectures, and critically supporting or refuting claims, all of which are important mathematical practices.

Integration: Whereas first and second semester calculus courses often focus on integration techniques, the emphasis of integrals in multivariable and vector calculus is on constructing a sum to find the total amount of a quantity. The technique used throughout the course is the *Chop, Multiply, and Add* technique discussed earlier. As an instructor, I've seen the process of setting up integrals baffle students. Some are willing to put numbers in places without a rational reason. They sometimes put the lowest number in the lower limit for an integral and the highest number in the upper limit for an integral. They may also re-order iterated integrals and try to simply swap the limits on integrals. Students do a variety of incorrect things because they are confused.

As an instructor, I also want student to analyze the situation to choose the appropriate integral form, rather than choosing the form that was most recently covered in class or contains symbols matching the problem. I like *The*

Boysenberry Patch activity because it lets students develop the habit of constructing integrals that account for the quantities in the process of picking berries.

By providing students with geometric tangible representations of the function and path – but not the equations for the path or function itself – the students must discuss the meaning of the quantities within their small groups and connect those meanings to mathematical concepts we've previously discussed. This approach is again utilized when students encounter problems to find the mass of chocolate on the surface of an ice cream cone, the number of calories in a piece of cake, or the amount of water flowing through a fishing net (e.g., flux integrals): When students chop up the region into small segments, construct a quantity on each small segment, and use an integral to find a sum across all segments, students construct an integral which makes sense and reflects their understanding of the problem.

Equitable Teaching Practices

Providing a place in the classroom to practice and develop these mathematical practices increases the chance that all students can participate in the classroom and in the discipline. Pushed a bit further, when the mathematics arises from student ideas, then students have an obligation to share their ideas whether they support or contradict those of their peers or instructor. Students might be hesitant to raise objections or voice concerns if they are not confident their observations are correct. In this instructor's view, those concerns should be raised precisely because they help **all students** develop better understandings of how concepts are (and are not) related to each other.

The activity design itself helps foster student discussion of their ideas. Since the activity is done prior to formal lecture, and because it asks for solutions to open-ended questions without providing algebraic formulas, students are likely to begin their discussion by sharing their interpretations of the task and to focus on conceptual meanings.[2] Students are put into small groups to work so that they can discuss concepts or ideas before sharing them with the entire class. However, the whole class discussion is also used to reinforce student ideas and concerns which may have been dismissed within the small group. In order to raise those issues, it is very important that the instructor listen and observe how students solve the problem within their small group - and to note the important points that are made, or ignored, in the small groups.

12.2 Lesson Implementation

In this section, we discuss the specific implementation of *The Boysenberry Patch* activity. The activity and detailed contour maps are included in the appendix to this LAM.

12.2.1 Lesson Components and Flow

The Boysenberry Patch includes three distinct sections where students work in small groups. It also includes two important whole class discussions which do not appear on the activity sheet.

To setup that activity, the instructor divides students into groups of three and assigns each group a specific case by color of the surface manipulative. Students pick up the activity sheet and appropriate surface, measurement tools, and dry erase markers. While students are working on the first part of the activity, the instructor hands out the laminated contour maps and listens to the discussion in the small groups.

For the activity's first task, labeled **On your Mark**, students work in small groups to rank the four paths based upon the amount of berries that would be picked on each path. This task does not involve computation, but comparison of berry density values and path lengths. A common point of confusion for students is whether the value of the contour lines, or the density of the contour lines, is relevant for this ranking task.

After about eight minutes of work, the instructor calls students to share their results. Groups take turns reporting the path which had the most amount of berries and the least amount of berries. They also provide justification for their choice. As discussed in the next section, the instructor organizes this presentation so that both incorrect and correct ideas are presented, providing opportunities for students to recognize patterns and propose conjectures relating the

[2] Some students defer to other students who are proficient at algebraic computations; Removal of the algebraic computations encourages students to begin their discussions by sharing their understandings and interpretations of the task. The *Raising Calculus to the Surface* activities are specifically written with some ambiguous language, some missing information, or some ill-defined questions to encourage group discussion and prevent one person from completing the activity by working alone.

amount of berries to the value or spacing of the berry density contour lines. These conjectures lead to a whole class discussion in which students support or contradict conjectures using evidence from their particular case.

After this first whole class discussion, students work on **Get Set** in small groups to estimate the amount of berries on specific paths. There are several ways to answer this question. Students who attend to units will notice that the product of the berry density $\left(\frac{L}{m^2}\right)$ and path length (km) produces a density, not volume, of berries. Once they recognize this conflict, some students re-evaluate how they pick berries in a patch of wild berries. If several groups simultaneously raise this issue, the instructor will have a tall and short student stand up while asking the class whether they will pick the same amount of berries while walking down a path. This brief question generally encourages students to incorporate arm reach into their model.

The final whole class discussion occurs after students have solved the second question for **Get Set**[3] and leads to the development of the *Chop, Multiply,* and *Add* formulation of line integrals. The instructor will ask students to share how they found the amount of berries that will be picked along Path C. Students usually address the following issues, but the instructor will prompt them to explain how they estimated the length of the path, whether they subdivided the path into smaller segments, and how they chose a berry density value for each small segment. This discussion also incorporates the *width* or *arm-reach* factor needed to more accurately model the action of picking berries on the side of the path. This process is written on the class board once it is apparent that most groups solved the problem in a similar way.

To wrap up the activity, the instructor asks students to identify the group whose technique was most accurate, or to propose a more accurate estimate. When students suggest dividing the path into smaller segments, the instructor encodes their process using the mathematical formula for line integrals:

$$\text{Amount of berries along a path} = \int w\rho|d\vec{r}|$$

At the end of the class period, students complete a short questionairre on the back of the activity sheet asking:

- What was the main point of the activity?
- Describe one thing you understand as a result of this activity.
- Describe one thing that is confusing after completing this activity.

After reading the student responses, the instructor addresses any issues in the next class period.

12.2.2 Rationales for Instructional Decisions at the Specific-moment Level

Two meaningful whole class discussions occur during this activity. In this section, we describe how the instructor prepares and facilitates the discussion. We also provide the rationale for setting up the discussion in this way.

The first is demonstrated in Vignette 12.1, and we encourage reading this short vignette to set the stage for the discussion of the instruction decisions described below. The second whole class discussion serves as a wrap-up for the activity, and occurs at the end of the class.

The first whole class discussion occurs approximately 8 minutes into the activity, after students have worked in small groups to rank the four paths based upon the amount of berries that would be picked along each path. The ranking should account for path length. While the amount of berries should be proportional to the value of the berry density, a common mistake is for groups to connect it to the density of the contour lines rather than the value of the contour lines. Having overheard student discussions, the instructor first selects a group which is confident, articulate, and has the incorrect justification for their ranking[4]. The instructor usually has a second group present which again uses the incorrect reasoning to help establish a plausible argument within the classroom. Thus, when the third group presents using the correct argument, the stage has been set for a mathematical debate. This is especially effective if two groups arrive at conflicting answers for the same case (e.g., the red surface).

When students present, the instructor asks them to identify the paths on which they would pick the largest quantity of berries and to explain how they arrived at their solution. The instructor repeats their reasoning, emphasizing the conflict between representing berry density by contour lines or the density of contour lines. Identifying these competing

[3]The **Go** question is included to ensure students stay for the final whole class discussion and do not finish the activity early.

[4]The value of the contour lines on the red and blue surface models are largest when they are also closely packed together. The instructor usually picks a group working with one of these surfaces to start the whole class discussion.

For ten minutes, students have been working in groups with their particular contour map and surface. The instructor had carefully listened to group discussions while tracing the six contour maps and paths onto the front whiteboard. Now, the instructor interrupted the group work and called successive groups to the board:

- The first group, working with the contour map for the red surface, said they would pick the most berries along Path B since the contour lines were closest together.
- The second group, working with the yellow surface, arrived at the same conclusion: Path B had more berries because it crossed a lot of contour lines.
- A third group disagreed. They also worked with the yellow surface. For them, Path C went through a higher density of berries than Path B.

The debate was on! The instructor repeated the conjectures made by the previous groups:

- *Claim 1:* A path has more berries where the contour lines are dense.
- *Claim 2:* A path has more berries where the value of the contour line is higher.

What did you think? the instructor asked.
The first group conceded Path B matched both conjectures for their contour map, so their situation couldn't rule out one claim. Then, someone working with the orange surface held up their surface. They'd drawn the four paths on their surface with a marker. *"The contours are more dense along Path B"*, the student said, *"but that just means the berry density is changing a lot."*. They pointed to a high point on their surface. *"Path C runs along this ridge, where there are a lot of berries."*
The second group pulled their surface over and started tracing their trails onto it. Seeing students point and talk in their groups, the instructor wasn't quite sure if everyone was convinced the first conjecture was wrong. "The Blue group will present next - but before they do," the instructor told the class, *"I want your group to predict which path will have the most berries."*

Vignette 12.1: Classroom Discussion in which students share and challenge conjectures developed in the small groups.

justifications as conjectures, the instructor asks students to support or challenge the claims using evidence from within their group work. Most groups reason using the contour maps, but a few groups might draw the paths on the surface manipulative. These groups usually recognize that the berry density corresponds to the 'height' of the surface and not the spacing of the density contours. When a student from one of these groups incorporates their surface as part of their argument, the disagreement within the class usually resolves itself. As a side benefit, the clarity of this final argument helps reinforce the benefits of translating between different representations of functions.

After this discussion, the instructor asks one final group to present their ranking. However, before they present, the instructor has all the students make a prediction about their results. This whole-class discussion serves to engage students in mathematical debate evaluating student-provided conjectures, brings misconceptions to the forefront of the class discussion, and helps ensure students proceed through the rest of the activity with a more sound understanding of the mathematical objects being coordinated as students develop integral quantities.

The second meaningful whole class discussion occurs at the end of the class period and helps formalize the *Chop, Multiply,* and *Add* formulation for integrals. It starts with students sharing their model for estimating the number of berries that were picked along *Path C*. The instructor usually selects a group with the following model:

The amount of berries = (berry density value) (path length)

where a specific density is chosen to represent the berry density for the whole path. This model is easy for students to explain and understand, and the instructor highlights or praises how the model accounts for path length and a choice of berry density value. Nevertheless, students are quick to point out how this model has the wrong dimensional units. The instructor calls upon one of the groups to share how they solved the problem, and their model typically resembles:

(arm reach length) (berry density value) (path length)

At this stage, the instructor can then ask a sequence of students how they improved upon the model. Using information gathered while listening to students work in their small groups, the instructor can organize the student presentations so that earlier presentations discuss various choices for the berry density[5] and later presentations incorporate two or more subdivisions of *Path C* to obtain a more accurate estimate of the amount of berries.[6]

The instructor organizes the student ideas in this way to demonstrate better ways of obtaining estimates for the problem, and to align student ideas with the goal of introducing line integral content. This discussion naturally leads students to the revised model:

The amount of berries = A Sum of [(arm reach length) (berry density) (length of sub-path)]

where the entire path has been divided into two or more smaller path segments. The instructor can emphasize this model requires chopping the path into small segments and describing the amount of berries along each piece as a product of the arm reach factor w, the berry density ρ, and the length of the path segment (found using the distance formula). Thus, when the instructor converts the above written formula into the notation

$$\text{The total amount of berries} = \sum_{\text{path}_i} (\text{ width })\, \rho^{\star}\, (\text{ length of path}_i\text{ })$$

or

$$\text{The total amount of berries} = \int_{\text{path}} (\text{ width })\ \ \rho\, |d\vec{r}|$$

using $d\vec{r}$ notation, the resulting model captures the *Chop, Multiply*, and *Add* approach to integration that was first proposed within the student presentations.

12.2.3 Student Mathematical Thinking

In this section, we discuss four instances of student thinking which occur during the activity. In the first case, we share how student's prior notions of integration can strangely impact their approach to the estimation problems in this activity. Next, we share several methods that students develop for finding the amount of berries along a particular path during the **Get Set** part of the activity. The third instance of student thinking illustrates how quickly students can turn seemingly incorrect ideas into productive ways of dealing with the activity's questions. The last case is used to illustrate how a meaningful question or observation can lead students to generate useful ideas and lead to meaningful whole-class discussions about the appropriateness of making assumptions when solving problems.

Student Ideas on Integrals. Based upon entrance surveys, students entering multivariable calculus generally think of integrals as:

1. Algebraic Rules
2. Process for *undoing* derivatives
3. Ways to find the area under a curve
4. Limits of sums
5. Accumulation, or ways to find a total amount

The activity does not mention the word *Integration* or *Integral*, yet students will sometimes try to incorporate integration into the ranking task in **On your Mark**. This can lead to seemingly surprising behavior which is explained by the student's prior notions of integrals. For example, a student who thinks of integrals as finding area under the curve may argue that *Path A* will have the most berries since it is the highest up on the contour map and therefore has the most *area beneath it*. Students who think of integrals as rules or as ways of undoing derivatives have also tried to develop a formula for the berry density, which they would then integrate.

When I've faced these situations, I've often asked students to explain what they are doing and why they are doing it. This provides time for me to listen to their reasoning and find a way to re-route their approach in a productive way. When the student gets to a strange jump in their process or logic, I'll ask them *why* they did that next step. Some students approach math by repeating prior processes and are uncomfortable with *developing* new ideas, and often their

[5]Popular choices for the density value include using an average value, a value in the middle of the segment, or a maximum or minimum density value for the entire path.

[6]If students have not broken the path into smaller sections, the instructor will ask students to develop a way to improve the accuracy of a given model.

reasoning is based upon having remembered some step like it in the past. However, they often recognize it is strange. At this stage, I can then talk with the students about the meanings involved in the question and ask the student leading questions to help them move into a productive direction.

Estimates for Berry Quantities. As he walked around the room and listened to groups working on **Get Set**, the instructor heard many different ways of estimating the amount of berries on *Path C*. We briefly identify these techniques:

- Groups often use a minimum or maximum berry density value for the entire length of the path.
- One group multiplied an average berry density by a width for the entire path
- One group broke the path into two equal length segments, choosing a sample density for each segment
- One group broke the path into several segments at each major contour line. They used a known sample berry density to find a berry total for each segment, then added these totals together for the entire path.

These techniques exchange one unknown for another: Either the density is measured, and the length estimated, or the length is known and the density is estimated. This can lead to a useful discussion, especially since the estimation techniques preferred by mathematicians differ from those of physicists or engineers.

Many students recognize a unit issue when they multiply the berry density and the path length. Some students realize that they need to multiply the berry density by an area, or a quantity measured in terms of *meters squared.* The problem does not tell students how berries are picked. The instructor may be tempted to *tell* students how to resolve this particular issue by telling them to incorporate arm reach into their model. However, it does not take long for students to incorporate an arm reach into their model, especially if the instructor asks the class if two students of different heights will pick the same number of berries on the path. Once students incorporate this feature into their model, the instructor can have students justify why arm reach is a reasonable parameter for this problem. In this instructor's view, discussing models and validating assumptions is meaningful discussion to have with students learning multivariable calculus.

12.3 Post-Lesson Considerations

The danger with using the *Raising Calculus to the Surface* materials is that they become disconnected from the regular multivariable calculus curriculum. To help the content build on student ideas explored during this activity, the instructor begins the subsequent lecture by asking students about the main ideas and features that were involved in *The Boysenberry Patch*. The instructor writes down student ideas on the front board, and then connects those ideas to the formal ideas being covered in class that day. In this way, the subsequent content seems to follow from ideas that students developed while completing the activity.

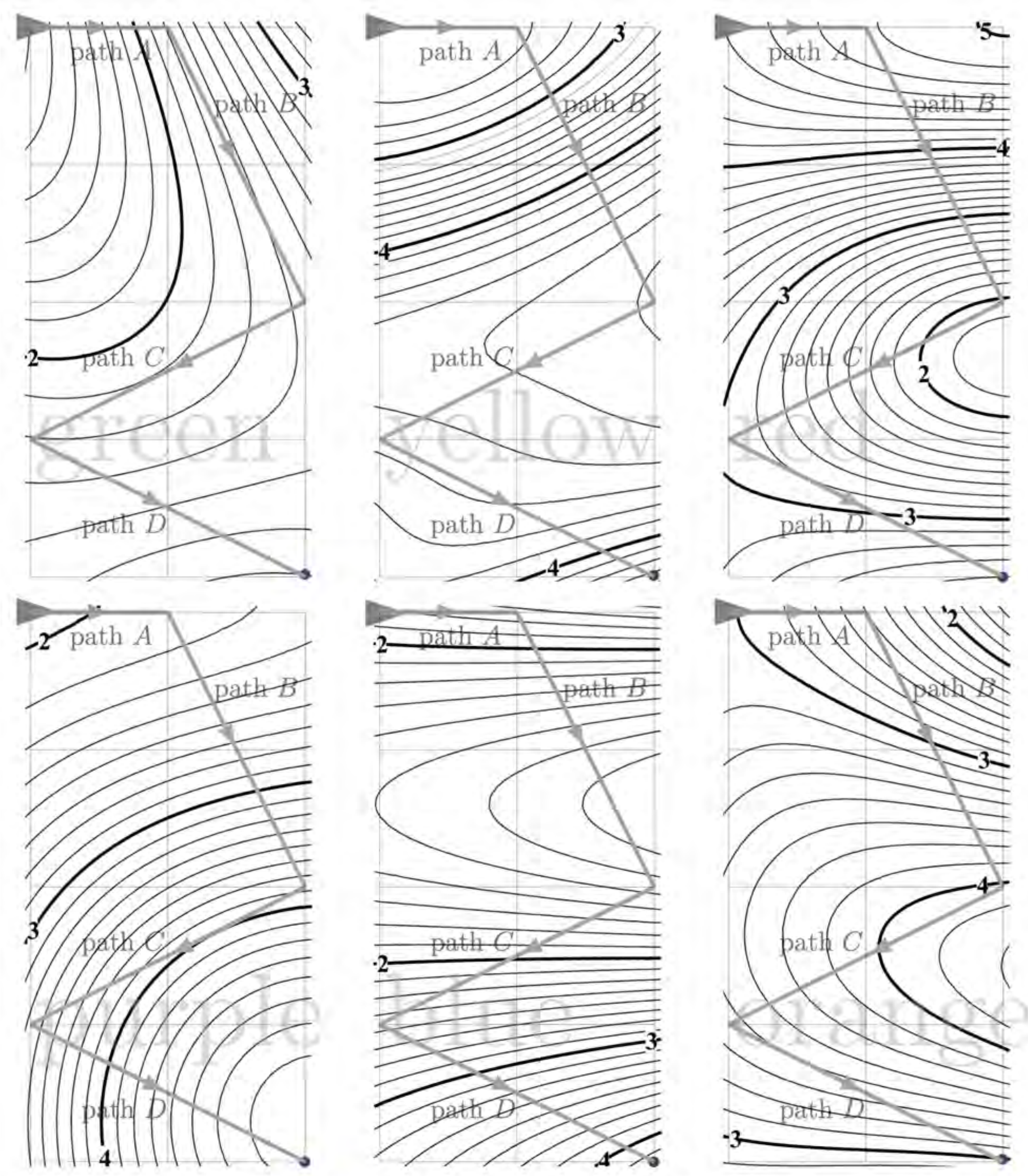
Having read the short evaluation summaries that students completed as part of the activity, the instructor is also able to address concerns that students might have raised during the activity. For example, students often note that they aren't sure how the exact path equation is incorporated into the resulting integral, or how to find the exact length of the path for use in the integral. The instructor will acknowledge these issues, and use them to lead into the new content (which has to address these issues!)

Students work with the contour map and surface manipulative during *The Boysenberry Patch*, and the instructor likes to have students solve a similar problem utilizing a table of data in their homework.

12.3.1 What Might Be Done Differently?

The instructor implemented this LAM within a course where students discovered new ideas on the first day of each week. This is **not** a requirement for using the activity. In fact, the *Raising Calculus to the Surface* materials are designed so that instructors can incorporate as many activities as they feel comfortable using into their course. While this instructor uses the activity to introduce content, other instructors have used the material to reinforce or review content which they've previously covered. This also has benefits, although it does appear to be a less efficient use of time.[7]

[7] When the activity is used after lecture, the time penalty occurs because the lecture has to cover many of the significant ideas that students discuss and discover before they do the activity. More time is spent on the lecture this way than when the activity is done prior to lecture.


path A
3
path B
2
path C
green
path D
path A
3
path B
4
path C
yellow
path D
4
5
path A
path B
4
3
path C
2
red
3
path D
2
path A
path B
3
path C
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4
2
path A
path B
path C
2
blue
3
path D
4
2
path A
path B
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4
path C
orange
path D
3

Name: ____ **Surface Color:** ______________

Task Master: ________________________ **Cynic:** ______________ **Reporter:** _________________

The Boysenberry Patch

Working in groups of three, solve as many of the problems below as possible. Try to resolve questions within the group before asking for help. Each group member should write up solutions in their own words on this paper. Show your work when appropriate, and explain your solutions.

On your Mark: You plan to go on a hike and pick Boysenberries along the way. The contour map shows the density $\left(\text{in } \frac{L}{100m^2}\right)$ of Boysenberries along the trail. The path is divided into four segments, and you plan to pick berries along each. Rank the four segments according to the total amount of berries you'd pick.

Amount of Picked Berries: Least ______ ______ ______ ______Most

Why did you rank it this way?

Get Set: Path A is $1km$ long. Estimate the total amount of berries on paths A and C.

1. Path A:

2. Path C:

Go: Exactly how many 3 liter buckets are necessary to pick berries along Path C? Explain how you arrived at your answer.

Activity Evaluation

What was the main point of this activity?

Describe one thing you understand as a result of this activity.

Describe one thing that is confusing after completing this activity.

Instructor's Notes: The Boysenberry Patch (Accumulating Mass along a Path)

Essential Ideas

1. Students develop concept of linear Riemann sum with multivariable functions.
2. Students rank mass quantities based on segments from contour map.
3. Students estimate total mass using contour values and length.
4. Students match seemingly mismatched units to physical situation.

Activity leads into: Lecture summarizing process of *chopping up* a curve for integrating along a curve. Applications of arclength, mass, and center of mass integrals along a curved wire or other 1-dimensional path in 3 (or more) dimensions.

Activity Information

Type: In-class Group Activity (30-40 minutes, depending on discussion)
Materials for Activity: 1 Surface, 1 contour map handout, 1 inclinometer, 3 markers per group

Classroom Timeline

In-class Preparation: Students should be familiar with integration from single variable calculus. Students do not need to have seen arclength or $d\vec{r}$. The formula for $ds = |d\vec{r}|$ or $ds = \sqrt{dx^2 + dy^2}$ follows from the activity.

Part	Comments / Discussion Points	
Prep	None necessary.	
On your Mark (7 min)	Beware:	Rankings should account for path length and density values.
	Beware:	The variation in the density is not a factor for the rankings.
	Comment:	Ask groups for explanations. Remember groups with partially correct explanations for the next discussion.
	Comment:	Notice which groups gravitate to the contour plot or the surface.
Discussion (7 min)	Comment:	Project the contour plot handout. Let groups present using it.
	Question:	*Tell us which path had the least, and the most berries. Explain.*
	Comment:	Call groups with incorrect or partially incorrect explanations first.
	Comment:	Have groups which used just the contour plot, or just the surface, also present.
	Comment:	Encourage discussion of the group's explanations. Groups may not initially realize the explanations are different, or that they are using different representations. Have groups compare their rule to previously presented rules.
	Question:	*Does this group's rule give the correct result for your contour map?*
	Note:	It can be helpful to have other groups predict the results before a group presents.
	Note:	Path A is always the least, but for the Red surface, it is only slightly less than path C.
	Challenge Q:	*Is path A always the worst path for picking berries?*
Get Set (10 min)	Comment:	Path A is $1km$ long and path C is $\sqrt{(1km)^2 + (2km)^2} = \sqrt{5}km$.
	Comment:	Students conscience of units realize an issue: Density (L/m^2) × Length $(m) \neq$ Volume (L)
	Comment:	Once students find the issue, discuss dimensions for the problem. Have students raise the issue.
	Question:	Open-ended: *Why is there a problem?* or *What could cancel the other meter?*
	Question:	Directed: *How do you pick berries along a path?* or *How wide is the path?*

Continued on the next page...

Part	Comments / Discussion Points	
	Comment:	Berries might be picked within $0.5m - 2m$ of the path. Let groups set their width.
	Comment:	The sum is roughly $\sum(density)(width)(ds)$, or $\int f(2m)ds$.
Go	Comment:	Groups transition seamlessly from **Get Set** to **Go**.
(5 min)	Comment:	It is OK to start the wrap-up before every group has finished **Go**.
Wrap-up	Wrap-up:	Project the contour plots on the wall, and let students draw on it.
(10 min)	Wrap-up:	Have a group or two present their answer for **Get Set** Path A.
	Wrap-up:	Have a group present their solution for Path C or for **Go**.
	Comment:	To deal with the varying berry density, students can use the average density along the path, or break up the path into pieces.
	Question:	*What do you do about the changing berry density?*
	Comment:	Students may suggest breaking up the path into smaller sub-paths.
	Question:	*How would you find the length of these smaller paths?*
	Comment:	This is set up for the Pythagorean Theorem, with the path giving a relationship between $\triangle x$ and $\triangle y$ (or dx and dy). Let this lead into a calculation for ds
	Comment:	Form a small triangle on the map with sides (dx, dy) and hypotenuse (ds).
	Question:	*What if the path were $y = x^2$? How would you find the length?*
	Comment:	The answer for **Go** should be $\frac{10\sqrt{5}}{3}(avg.levelcurve)(width)$. It is about $\frac{10\sqrt{5}}{3}(2)(width) \approx 15(width)$ for the blue surface.
	Comment:	These are some open questions which can help generate discussion:
	Challenge:	*Is this a good model of picking berries?* (Do we pick every berry?)
	Challenge:	*Is Path A a good path to pick?* (It's shorter, but fewer berries.)

Suggestions and Pitfalls

The dimensions of volume density (L/m^2) and length (km or m) don't multiply to a volume. The path is infinitely thin, but people pick berries a half meter or so from each side of the path. This *width* of $0.5m - 2m$ makes the dimensions work.

This calculation is an estimate. It can be done with a single line integral since the path's length is significantly longer than it's width. Hence, the berry density will vary along the path's length. The density won't vary much along the path's width, providing a symmetry enabling the answer to be found using a single path integral.

Wording: Possibly change the wording from *The contour map shows the density $\left(in\ \frac{L}{m^2}\right)$ of Boysenberries along the trail.* in **On your Mark:** to *The contour map shown the volume of berries (in L) you pick per meter along the trail.*

Activity Leads Into…

Lecture summarizing process of *chopping up* a curve for integrating along a curve. Applications of arclength, mass, and center of mass integrals along a wire or other 1-dimensional path.

12.4 Supplementary Material

Homework and Supplementary Materials

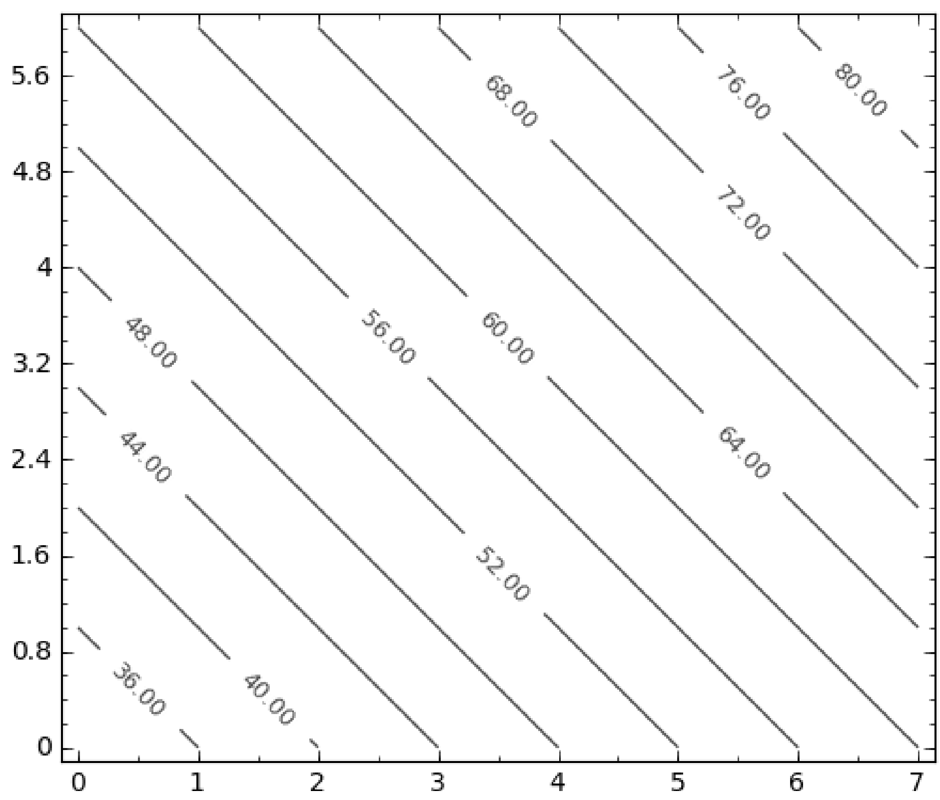
1. The table below gives the height (in inches) of a surface, with x and y measured in inches. Estimate the length of the path extending from $(0, 2)$ to $(3, 8)$ using the following curves:

   (a) $y = 2(x-1)^2$

   (b) $y = 2x + 2$

$x \setminus y$	-1	2	5	8	11
-3	3.91	3.04	4.61	7.16	9.96
0	2.50	0.50	3.50	6.50	9.50
3	3.91	3.04	4.61	7.16	9.96
6	6.50	6.02	6.95	8.85	11.24
9	9.34	9.01	9.66	11.10	13.09

2. The following contour plot shows the height (in m) of a hill over a 7 mile by 6 mile sand dune. Estimate the length of the path along the sand dune from $(2, 1)$ to $(5, 4)$ and then to $(2, 5)$ using straight-line paths. Include units.



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13

Developing Students' Covariational Reasoning of Functions Through Active Learning

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> If, however, some quantities depend on others in such a way that if the latter are changed the former undergo changes themselves then the former quantities are called functions of the latter quantities. This is a very comprehensive notion and comprises in itself all the modes through which one quantity can be determined by others. If, therefore, x denotes a variable quantity then all the quantities which depend on x is any manner whatever or are determined by it are called its functions.
>
> —attributed to Euler, 1755, by Rüthing, 1984, pp. 72–73.

Topic of lesson	This lesson is focused on building students' covariational reasoning abilities through active learning and engagement. Students will work in pairs to make sense of covarying quantities through the use of dynamic situations.
Course Context	This lesson is focused for a Calculus 1 course at the post-secondary level and could be taught on Day 1–2 of the course. Ideas promoted in this activity can set the stage for future course ideas involving instantaneous rate of change, concavity, inflection points, etc.
Instructional Challenge	Students are typically not provided opportunities to think about varying or covarying quantities prior to their calculus courses.The notion of covariational reasoning, however, has been shown in research as critical for students' ability to develop robust understanding of rate of change and the concept of function, which are central to the mathematical ideas in calculus. Students need the space and time to hone their reasoning abilities to retrain their minds away from graphs as merely depictions of shapes and move towards graphs as a dynamic result of covarying quantities in tandem.
Brief overview of instructional approach	This lesson is designed to be a whole-class discussion, with students paired at the whiteboards, that provides opportunities for students to grapple with changing quantities as presented in dynamic situations (through videos and mental imagery).
Keywords	Covariation, active learning, changing quantities, calculus triangles

13.1 Background Information

13.1.1 A Cultural Challenge

All too often students' experiences with mathematics are ones that exacerbate their working knowledge that mathematics is boring, non-joyful, lacking creative thought, and something they were not "born being able to do." My experience teaching at the community college level has demonstrated that students bring this gloomy mindset to the classroom that often serves as a barrier to their success. I believe that my role as their instructor is to not only guide their learning of mathematics, but also to reshape their mindset of mathematics and reduce (or eliminate) their propensity to view mathematics negatively. This is a big challenge and one that many instructors might say is impossible to achieve in one semester, but I contend that we need to interrupt the narrative that it is acceptable to "not be good at math" and that "math is boring." As a result, one of my primary instructional goals is to bring a sense of joy and excitement to the learning of mathematics which, I believe, can best be accomplished through fostering student engagement and implementing active learning strategies in the classroom. I will explain in more detail later about the components of active learning that I strive to promote, though next I will turn to the mathematical content for which I will center my pedagogical discussion.

13.1.2 An Instructional Challenge

Covariation is a tool embedded in functional thinking and understanding that provides a lens with which to view and make sense of functions. Research in mathematics education has revealed that students' inability to reason about changing quantities impedes their ability to think about changing rates of change, for example [5]. Researchers have proclaimed that covariational reasoning is fundamental to understanding functions in preparation for calculus topics such as limit ([1], derivative [7], and the Fundamental Theorem of Calculus [5, 6]. The notion of covariation – that is, the idea that quantities vary and that we need to track how changes in one quantity relates changes in the other quantity – is one that is often ignored or (at best) tangentially taught in precalculus or below courses. Carlson et al. argue that instructional activities should emphasize ways to progress students toward a more coordinated image of instantaneous rate of change with continuous changes as an advanced notion of the coordinated image of two variables changing in tandem.

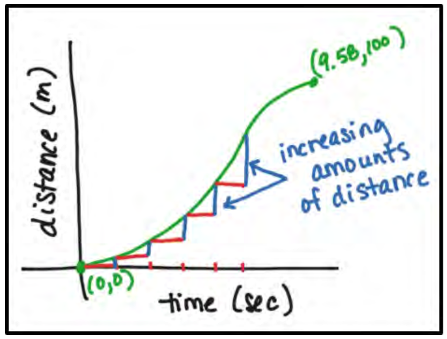
Carlson et al.'s [1] study of covariational reasoning in the context of modeling dynamic events proposes a covariation framework encompassing detailed mental actions that students exhibit as they engage in mathematical activities. The following outlines the five mental actions from Carlson et al.'s framework:

- Mental Action 1 (MA1): Coordinating the value of one variable with changes in the other.
- Mental Action 2 (MA2): Coordinating the direction of change of one variable with changes in the other variable.
- Mental Action 3 (MA3): Coordinating the amount of change of one variable with changes in the other variable.
- Mental Action 4 (MA4): Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.
- Mental Action 5 (MA5): Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (p. 357).

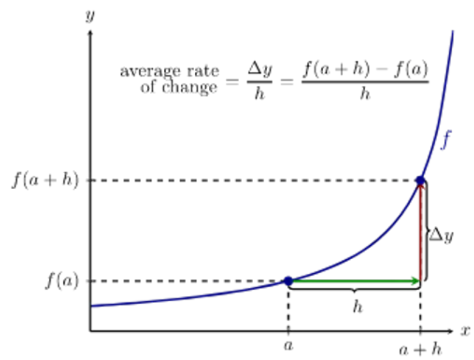
This framework is hierarchical and inclusive given that each mental action is associated to a specific level of covariational understanding and inclusive in the sense that higher levels of mental actions are obtained once students exhibit the current level as well as lower levels of mental actions. For example, reasoning covariationally at Level 4 is obtained only after exhibiting the mental actions associated with Mental Action 1 (MA1) through Menta Action 4 (MA4).

13.1.3 Calculus Triangles: A Tool to Help Students Analyze Covariation of Quantities

In this lesson I use a technique which I refer to as calculus triangles. This technique is used on the coordinate plane to help students reason through how much one quantity changes as another changes. The strategy, which is used in some textbooks, draws horizontal lines to represent a change in x (usually a small change) and vertical lines to represent a change in y (usually the corresponding change in y based on the small change in x). The figure below shows examples of these "triangles" drawn next to a time-distance graph.



I use calculus triangles differently than they are used in calculus textbooks. In textbooks they are usually used only as a step in reasoning about ideas related to instantaneous rate-of-change, limits of secant lines, and the definition of a derivative. A typical example is illustrated below.



However, I use this technique so that students have tools and a language to reason about change. For example, drawing multiple "triangles" allows the class to analyze how the rate-of-change changes across the values of the input variable. Drawing some triangles large and some small allows the class to explore that it is the ratio of the changes, not the size of the changes, that really captures the essence of measuring the rate of covariation. Allowing students to get very comfortable analyzing change in curves by drawing the string of calculus triangles helps to solidify their understanding and transition well to thinking about change without drawing the triangles, and even helps to think about the rate of change of quantities in contexts where there is no graph.

13.1.4 Mathematical and Instructional Goals of the Activity

To address the challenge of developing students' covariational reasoning abilities, the primary mathematical goals of the activity are to:

- Allow students the opportunity to grapple with dynamic events of changing quantities through a qualitative lens (minimizing the formalization of mathematical notation),
- Explore the meaning of linearity and non-linearity as they arise within the activity, and
- Develop the notion of "calculus triangles" as a tool for making sense of constant and changing rates of change, where students compare vertical (output) changes relative to horizontal (input) changes to determine rate of change. This idea will be further explored in a later section.

Within the context of these mathematical goals, this activity addresses the cultural conundrum related to the teaching of mathematics – that is, to provide positive experiences on Day 1 of Calculus 1 for students as they learn mathematics. Regardless of the mathematical content that students need to learn, I approach the teaching of mathematics with the following instructional goals:

- Teach energetically and demonstrate for students the excitement in the learning process,
- Build a sense of community in the classroom where mistakes are expected, respected, inspected, and then corrected (by anyone in the classroom),
- Allow students the mental and physical space to grapple with mathematical ideas and contexts, and

- Create an environment where students can be authentic and where they find value and enjoyment in attending class.

13.1.5 Instructional Setting: The Classroom

The community college offers a unique setting to explore different instructional strategies given that the class size of mathematics courses tends to be small and the student population is typically highly diverse. Mathematics classes at my community college are around 28–30 students and comprise students who are first generation to college, working professionals looking to retool for other work, looking to transfer to a university, or who just want to learn for the sake of learning. Recently, I had a student enroll in my Calculus 1 course because he had broken his leg and was unable to start classes at a university in another part of the state. He decided to stay close to home so that he could have the necessary surgeries he needed and yet he was able to take classes at my community college in the interim. His father, a senior level engineer, reached out to me to inquire about enrolling in the course alongside his son. As he stated, he wanted to have a meaningful learning opportunity with his son while learning calculus again. For this reason, and many others, the community colleges serve a unique and diverse population of students wishing to increase their knowledge. The experience for both the son and father – as well as for the entire class – was such a positive one that I missed greatly having this duo in the follow-up Calculus 2 course the next semester.

Given that my class sizes are small and that my classes are held in classrooms that are conducive to active learning, I strive to create learning opportunities for my students that involve high levels of student-to-student engagement. The furniture in the classrooms, which include rectangular tables for groups of four students, allows for informal dialogue about mathematics and participation in group activities which promote deep thinking. In addition, most of the classrooms have sufficient vertical whiteboard space for students to work on problems together. Utilizing the whiteboards for paired board work is an instructional strategy that I leverage in every class session. It is the activity of sharing student thinking via visible and public display of mathematics that allows for me, as the instructor, to cultivate discussions and meaningful reflection of the mathematics being learned. Precision (or imprecision) of students' notation and language can be highlighted in real-time, questions can be posed and challenged, and class sessions become more of a "thinking factory" of ideas that can be used as springboards for further development and understanding of mathematical concepts. Liljedahl [3] refers to this type of instructional activity as *building thinking classrooms* where the goal of the instructor is to design experiences for students that are situated around valuing, leveraging, and promoting students' thinking as a lever for instruction.

13.2 Lesson Implementation

13.2.1 Task 1: The Sprinter Problem

The following activity described below should be implemented over at least two class sessions of Calculus 1. On Day 1 of the activity (which is typically the first day of class), I implement the "Covariation Tasks" in a specific sequence to begin building the mathematical goals described above. There are two main tasks completed in class (Sprinter Problem and Skateboarder Problem) with three additional tasks given for homework. Each of the tasks are described below, including the "big ideas" that can be generated through whole class discussion.

Task #1: The Sprinter Problem (Usain Bolt) – 45 minutes
https://www.youtube.com/watch?v=3nbjhpcZ9_g
Sketch a graph that could represent Bolt's distance from the start as a function of time. Be sure to label the axes!

Before any instructions about the task are given, I ask students to pair up with their shoulder partner and move to their desired location at the whiteboard. Once students are positioned at the whiteboard, I set up *Task #1: The Sprinter Problem* by asking students to first watch the video of Usain Bolt's sprint. Then, I let students know that they will need to sketch a possible graph of Bolt's run based on two quantities that we will track: (1) distance from the start of the race and (2) time since the race began. Since time can be graphed on the vertical as well as the horizontal axis, I allow students to determine how they want to set up the axes. However, as an instructor, I am prepared to explore both options of graphs based on students' graphs. I replay the video 2–3 more times and I note that the distance from the start to the finish line is 100 meters and takes Bolt 9.58 seconds. Students are given about 5–7 min to grapple with a possible graph and then, as a whole class, we explore students' thinking that generated their graph. It is important to consider the sequencing of ideas at this stage of the activity, so I typically sequence the discussion starting with a "linear" graph, where I probe students thinking about the meaning of the line connecting the endpoints. Next, we examine a graph that students describe as the "accelerating" graph – first where the horizontal axis is time and next where the vertical axis is time (if a group provides this type of example). During the whole class discussion, I not only probe students' thinking about their graph but also critique their explanations by pointing out vague and imprecise language along the way. Below are examples of student generated graphs, along with a discussion of ideas that can be developed from these graphs.

The "Linear" Graph. A dominant method for some students to imagine the graph of this context is to first consider the endpoints of (0,0) and (9.58,100) and then simply connect these points (see Figure 13.1 below). In many cases, students do not consider what happens (or could happen) between these endpoints, thus connecting them with a straight line seems reasonable to them.

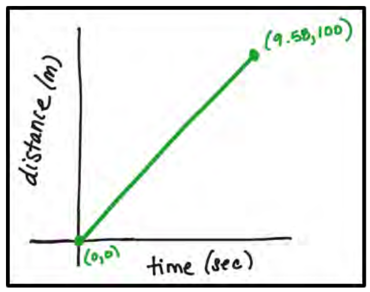


Figure 13.1: Linear graph of The Sprinter Problem.

When students provide their thinking of the graph above, they often say things like "Bolt ran at a constant speed until he crossed the finish line." In order to truly know the speed that Bolt ran during the race, we would need to determine his speed at every instant from 0 sec to 9.58 sec. I explain to students that this is not our goal at the moment, but rather we now want to understand what it means when we say that "Bolt ran at a constant speed." For the moment, I tell students that we are going to remove the ability to say "constant" or "speed" to describe what we think is going on here. Rather, we need to use the quantities given — distance from the start of the race and time since the race began. Describing Bolt's moments using just these two quantities is more challenging for students than we think, and they often need some encouragement to think about the amount of distance covered in some amount of time. In fact, it is at this stage of the discussion that I ask students to imagine the "time axis" being partitioned into equal chunks (e.g., 1-second intervals, 0.1-second intervals, or any other chunk of their choosing). Now, I ask them to describe how the distance from the start of the race is changing in each chunk of time. In some cases, students will say "it's increasing!" at which time I remind them of two critical things: (1) I do not know what "it" is referring to, so we need to avoid the use of pronouns and (2) while the distance is "increasing", this term is too vague for us to know how the distance is increasing. The goal here is to continuously refine students' language for describing the changing quantities until they can confidently describe the graph as showing that "for every equal chunk of time that elapses since the race began, Bolt's distance increased by equal amounts." This is shown in the graph below (see Figure 13.2).

Figure 13.2 highlights for students the idea that the distance since the race began is increasing by the same amount (blue vertical line) for each chunk of time that elapses (red horizontal line). We call these "calculus triangles" as they will be used as a tool for describing changing rates of change in later graphs.

The "Accelerating" Graph with Time as Independent Quantity. Once the idea of the calculus triangle is developed, at least for linear graphs, I move to a graph similar to the one shown in Figure 13.3 below.

Similar to the prompt I previously provided students when discussing the linear graph, I ask a group who drew the above sketch to describe their thinking using only the two quantities of distance from the start of the race and time since the race began. However, inevitably, students will describe their graph as showing that "Bolt's distance is increasing

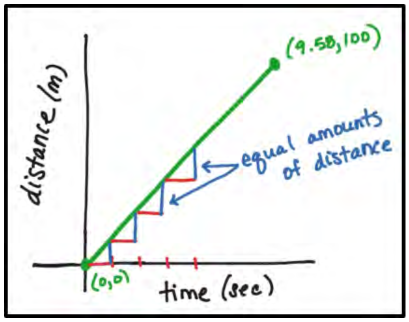


Figure 13.2: Linear graph of The Sprinter Problem with "calculus triangles."

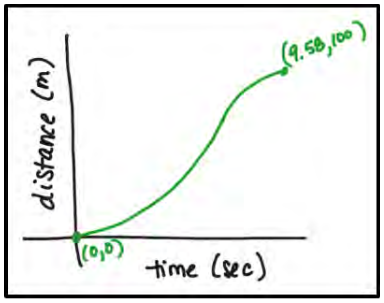


Figure 13.3: Changing rate of change graph for The Sprinter Problem

as time goes on" or "Bolt is speeding up (or accelerating) until he gets closer to the finish line, then he slows down (or decelerates)." These descriptions, especially the first one, illuminates that students are not attending to how the two quantities are changing in tandem. Certainly, the graph shows that the distance is indeed increasing. But so does the linear graph depicted in Figure 13.1. This needs to be pointed out to students so they recognize the need to describe the graph with more precise language. Furthermore, since we are focusing on developing language to describe how the distance from the start of the race is increasing, such phrases as "speeding up/down" or "accelerating/decelerating" should be suspended to perturb students' thinking towards meaningful language. The push towards describing the graph using the quantities also avoids students' "shape thinking" that is often embedded in their minds when looking at graphs (this is where students view the graph as a literal shape of something, rather than the result of covarying two quantities).

Finally, to help students connect the changing quantities with appropriate language to describe the changes, the calculus triangles could be deployed. In Figure 13.4 below, the idea of chunking the time axis into equal intervals (as evidenced by the red horizontal lines) and then analyzing the increase in the associated distance quantities (in blue) reveals that the distance is increasing by increasing amounts. The goal here is to continuously refine students' language for describing the changing quantities until they can confidently describe this graph as illustrating that "for every equal chunk of time that elapses since the race began, Bolt's distance increased by increasing amounts." It is

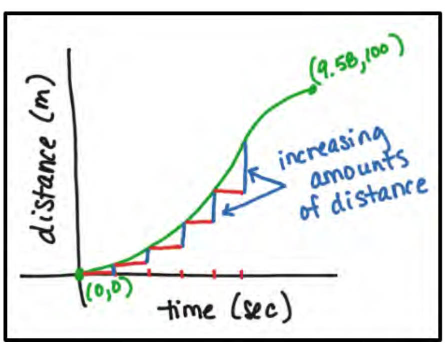


Figure 13.4: Changing rate of change graph for The Sprinter Problem with "calculus triangles."

important that students understand that this description is the underlying reason why we can also say (at least for the beginning portion of the graph) that Bolt was speeding up (or accelerating). Developing this language to describe the graph is vital for students to conceptualize the notion of changing rate of change in future work in Calculus 1.

The "Accelerating" Graph with Time as Dependent Quantity: In some instances, students will conflate their thinking that Bolt is speeding up with the lack of consideration for the orientation of quantities that reveals this idea. As a result, a possible graph that students might sketch is shown in Figure 13.5 below where distance from the start of the race is on the horizontal axis and time is on the vertical axis. While, in the context of the Bolt sprint, it is possible to have time on the vertical axis, students also need to attend to how time increases if we chunk the distances into equal amounts. This is a big challenge for students! It is difficult for them to organize their thinking in terms of first chunking the horizontal quantity into equal chunks, then analyze how the vertical quantity is changing. Instead, students often are compelled to chunk the vertical quantity into equal intervals, then describe what is happening with the horizontal quantity. Or, worse yet, not chunk either of the quantities at all. As a result, students' descriptions of the graph lack the precision needed to make sense of the scenario. The tool of calculus triangles can be very useful to illuminate for students that the notion of Bolt speeding up cannot be correct given the graph below since for every equal chunk of distance, the time increases by increasing amounts. In essence, Bolt is slowing down.

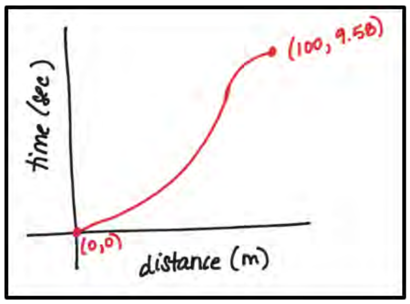


Figure 13.5: Changing rate of change graph for The Sprinter Problem time as dependent quantity.

It is often difficult for students to wrap their minds around how the graph in Figure 13.5 is actually illustrating that Bolt is decelerating. This is often due to students' impoverished ways of thinking about how the graph is the result of covarying the changes in the time quantity relative to the changes in the distance quantity. To help shed light on this complex issue, I will often create this graph for class discussion when students do not offer this graph as a possibility of Bolt's run. If time is limited during class, this task can be assigned for students to think through for homework.

13.2.2 Task 2:The Skateboarder Problem

In the next task, students will once again watch a video of a dynamic event (a skateboarder skating a half-pipe). However, in this situation, students will have to attend to the quantity of horizontal distance despite what they see is the skateboarder's horizontal and vertical movement. This task can be tricky for many students as they must consciously attend to the horizontal distance as a quantity while visually seeing the skateboarder's vertical distance also changing.

> **Task #2: The Skateboarder Problem** –25 minutes
> https://www.youtube.com/watch?v=pxKgNmKGUxc&feature=youtu.be
> Sketch a graph of the skateboarder's horizontal distance from start (the left edge of the half-pipe) as a function of time since the video began. Be sure to label the axes!

Similar to the introduction of the Sprinter Problem, this video is initially shown to students in a whole class setting (although students are still in the same pairs at the whiteboard) without describing the quantities that they should attend to. Once students have seen the video, I ask them to re-watch the video while tracking the skateboarder's horizontal distance from the start and time since the video began. I reiterate the need to pay close attention to the horizontal distance, but I do allow students to make mistakes here as it is part of the process of mentaly focusing on the appropriate quantity (horizontal distance) even though visually we see the skateboarder also moving vertically. Students are given about 5–7 minutes to sketch a possible graph and then, just like with the Sprinter Problem, we

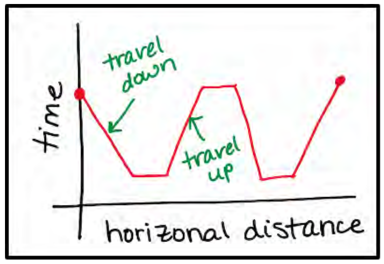


Figure 13.6: Possible student graph of The Skateboarder Problem.

discuss various graphs as a whole group. Often, I have found that at least one group will provide a graph such as the one in Figure 13.6. This is an example of what I use to start the whole class discussion since, based on the graph, students were "shape thinking" by graphing the literal movement of the skateboarder with attention to the vertical distance rather than the horizontal distance. Mentally, it is more challenging to do the "mental gymnastics" of tracking horizontal distance despite visually witnessing the skateboarder also move vertically.

During the whole class discussion, and consistent with the Sprinter Problem discussion, I not only probe students' thinking about their graph but also critique their explanations by pointing out vague and imprecise language along the way. Figure 13.7 includes examples of student generated graphs.

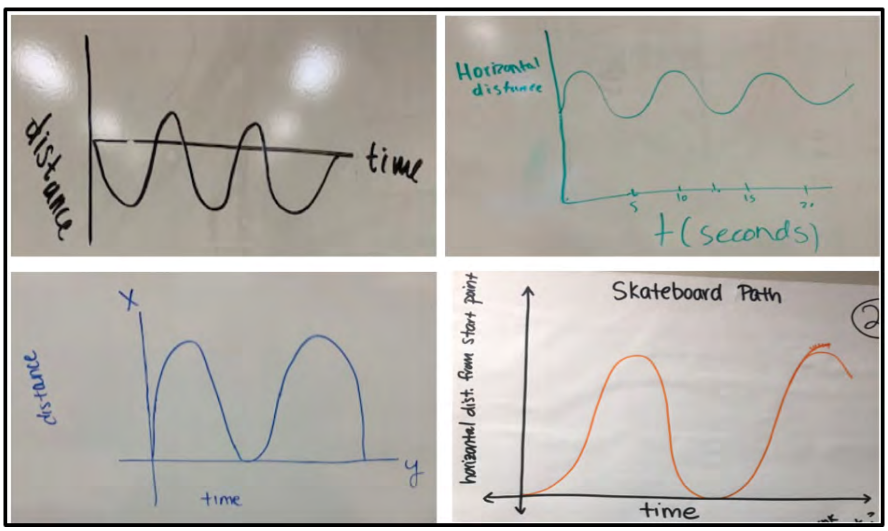


Figure 13.7: Examples of students' graphs for The Skateboarder Problem.

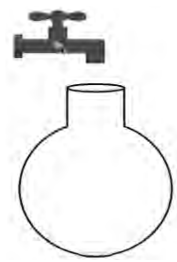
In these student graphs, we can see a variety of thinking that provides a glimpse in how students are conceptualizing the Skateboarder Problem. The upper left graph in Figure 13.7 is another depiction of students' shape thinking where they have drawn a graph of the skateboarder with the horizontal axis representing the height of the platform from which the skateboarder launched. Here again, the students are not thinking about the horizontal distance, but instead have focused on vertical distance. The graph in the lower left appears to indicate that the students are thinking about the horizontal distance of the skateboarder relative to time, but they have not yet correctly attended to how the horizontal distance is changing given the concavity of the first portion of their graph. Concavity is often a challenge for students to think about, especially in the skateboarder context. To help facilitate students' development of concavity, the use of calculus triangles in this graph will highlight the issue that as time progresses in uniform amounts, the increased distance in each time interval is less than the previous. This would indicate that the skateboarder is slowing down as he drops off the platform, which is not possible. In contrast, the graphs in the upper and lower right indicate that the students are thinking about the horizontal distance (which is also indicated on the vertical axis) while attending to the notion that as time increases in uniform amounts, the increases in horizontal distance are increasing in the initial drop of the skateboarder. It is important to highlight these differences in graphs to help students build up to the notion that concavity has meaning and is important when sketching graphs, especially when thinking about rate of change.

Additionally, when highlighting these complexities with students, it is important to scaffold the unfolding of the ideas in such a way that the story of the mathematics unfolds and builds along the way. As a result, given the students' graphs above, highlighting in the order of upper left, lower left, then the right graphs help to build the storyline for students.

13.2.3 Tasks 3 and 4: Filling a Water Bottle

For the next two tasks, students will be asked to imagine a bottle of water filling up. Both of these tasks can be assigned as homework extensions to the in-class portion of the activity from Tasks #1–2. While the aim is to have students think carefully about the changing quantities through mentally imaging the bottle filling, they can also benefit from watching a short video of the bottle filling to help their thinking along. The video provided here connects to Task #3: https://www.youtube.com/watch?v=XzHcBFiya4E.

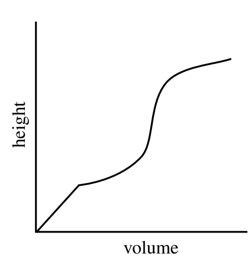
Task #3: The Bottle Problem (Bottle to Graph) – 30 minutes
Imagine the bottle below filling up with water. Sketch a graph of the height of water in the bottle as a function of the volume of water in the bottle. Be sure to label the axes!



In Task #3, the challenge is for students to move completely away from the quantity of time since the focus is on the quantities of height and volume. This is a real challenge for students because many initially think about this task as involving time, where they believe that the speed of the water filling the bottle needs to be taken into account. This, of course, is not the case and should be discussed during the next class session (in anticipation of students thinking in this way). The Bottle Problem has been researched in mathematics education and has been instrumental in determining the level of students' and teachers' covariational reasoning ability. In Carlson et al.'s (2002) study [1], they used their covariational reasoning framework to characterize common reasoning behaviors elicited by students while working through the bottle problem. Of the 20 students in the study, they found that only two students were able to correctly sketch all aspects of the graph. Notably, 11 students constructed an increasing concave-up graph while three students constructed an increasing concave-down graph. This illustrates the challenge for students to consider changing rate of change – and thus changing concavity with a graph – as dictated by the contextual situation of the bottle problem.

Lastly, Task #4 challenges students to think about the shape of a bottle that would produce the given graph as indicated below.

Task #4: The Bottle Problem (Graph to Bottle) –15 minutes
Sketch a bottle for the height of water in the bottle as a function of the volume of water in the bottle.



One of the (mathematically unproductive) heuristics that students often leverage is that they think the shape of the graph is merely the shape of one side of the bottle. As a result, they "fill in" the bottle shape by pretending that the bottle is simply leaning over, thus they produce a drawing such as the one indicated in Figure 13.8 below.

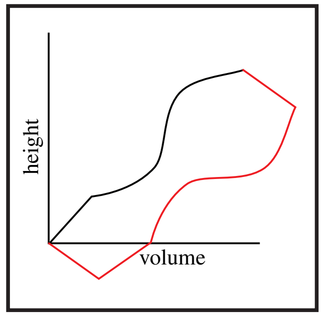


Figure 13.8: Examples of unproductive student work for The Bottle Problem (Graph to Bottle).

In this situation, the student has completely ignored the representation of the graph as the resulting covariation of the quantities of height and time. Instead, they have resorted to the notion of shape thinking as though the graph is the shape of the bottle. It is important to remind students, once again, about the power of using calculus triangles to make sense of whether the increases in height are constant per volume added (as in the first section of the graph), whether the increases in height are increasing per volume added (as in the middle section of the graph), or whether the increases in height are decreasing per volume added (as in the final section of the graph). Once students are reminded about the power of the calculus triangles, they are often equipped with productive ways of thinking about the graph to produce a bottle that is rather different than their initial thinking.

13.3 Post-Lesson Considerations

13.3.1 Summary

The notion of covariational reasoning has been shown in research as critical for students' ability to develop robust understanding of rate-of-change and the concept of function ([1], [2], [5], [6], [7]). The set of covariation tasks presented in this paper highlight an activity that can be conducted during class and facilitated through high levels of engagement and discussion. With the follow-up extension tasks for the Bottle Problem, students will have an opportunity of continuing to grapple with covarying quantities that can then be leveraged in the next class session. It is important to note that students' conceptions of covariation and their ability to reason about quantities needs to be cultivated beyond this one activity. They need the space and time to hone their reasoning abilities to retrain their minds away from graphs as merely depictions of shapes and move towards graphs as a dynamic result of covarying quantities in tandem. Thus, as students encounter graphs later in the course and in future mathematics, they should be encouraged and supported to leverage their thinking of graphs as illustrations of covarying quantities as often as possible. This is especially critical as they develop notions of changing rates of change and derivatives, where they can use calculus triangles, for example, as a tool for building understanding of instantaneous rates of change and concavity. Ideally, this way of thinking will become a natural way of thinking for students, but experience has shown that this transition takes time before students freely view graphs as a dynamic display of covarying quantities.

In the event that a classroom is not set up or ready for a highly engaged active learning lesson as described in this chapter, there are still components that can be incorporated into instruction. For example, in large lecture courses where it may not be possible to get students up to whiteboards to sketch their graphs, students could be paired up with another student sitting near to them and asked to sketch a graph on their paper once the instructor showed the video to all students. The instructor could then roam around the room to monitor students' thinking and, if a document camera was provided in the classroom, the instructor could select 2-3 sample graphs to highlight student thinking. Given that mathematics classrooms have a variety of limitations for implementing active learning, there are components of the lesson that can be leveraged and modified so that the ideas of reasoning about quantities and building dynamic conceptions of graphs can still be explored.

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14

An Instructional Treatment of Proof by Mathematical Induction

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Topic of lesson	Proof by Mathematical Induction (MI)
Course context	University-level transition to proof course, mainly lecture-based, about 100 students, mathematics and other STEM majors.
Instructional challenge	To help students understand the logical basis of proof by mathematical induction, particularly that it is not circular reasoning. It proves a sequence of statements labeled by natural numbers by using earlier statements to help prove later ones.
Brief overview of instructional approach	MI is developed from an intuitive type of recursive reasoning called process pattern generalization. The lesson begins with this problem: define a sequence by the rules $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ thereafter. Numerical values suggest that this is an increasing sequence and each term is less than 2. As students try to justify these conjectures, they focus on the key issue for induction: how can earlier examples of a pattern be used to justify later examples? The Principle of MI emerges as a summary of their reasoning. The lesson continues with a purposeful sequence of more challenging problems introducing variations on the technique, such as strong induction, as well as broader applications and more demanding reasoning and notational challenges.
Keywords	proof, mathematical induction, inductive step, recursion, natural numbers, process pattern generalization

14.1 Background information

14.1.1 The Instructional challenge

Proof by mathematical induction (MI) is a standard topic in a transition to proof course. Although it is based on fairly intuitive reasoning, common teaching approaches often lead to students applying this technique mechanically, and being unsure of its logical basis, even viewing it as a form of circular reasoning. They may also associate it narrowly with proving a few stereotypical summation formulas. The challenge is to help students understand the validity of inductive reasoning and apply it flexibly to a variety of problems. The problem is complicated by the numerous variations of the basic proof method (strong induction, well-ordering, etc.), the subtle conditional reasoning required in the inductive step, the wide variety of statements that can be proved in this way, and a range of algebraic and notational challenges that must be met to write a correct proof even when the key ideas are clear.

14.1.2 Specific Learning Goals

Students should understand the logical basis of proof by MI, in particular that it is not an instance of circular reasoning. They should understand the roles of the base and inductive steps, and apply them reflectively, not mechanically.

Students should recognize that MI is a method of proving a sequence of statements $P(n)$ labelled by a natural number n. (My convention is that the set of natural numbers $\mathbb{N}$ starts with 1, not 0, so "natural number" means "positive integer". Students with backgrounds in computer science or discrete mathematics may be more accustomed to including 0 as the first natural number.) It depends on recognizing the implication relationship between consecutive statements $P(n)$ and $P(n+1)$, or more generally between earlier statements and later ones. It is not usually useful for proving statements about rational or real numbers, because such a number does not have a unique successor.

Students should recognize basic types of theorems to which MI is likely to apply and be able to write such basic proofs clearly and correctly in terms of notation, language, and logical flow. These types include not only the standard examples of summation formulas (such as sums of powers of the first n natural numbers), but also inequalities, divisibility results, and explicit formulas for recursively defined sequences.

Students should flexibly use the appropriate variations of MI: strong induction, inductions not starting with $n = 1$, well-ordering, and proof by minimal counterexample.

In addition to these lesson-specific goals, the lesson advances more general instructional goals for the course, especially the meaningful use of mathematical language and notation, and seeing algebra as a tool for *reasoning*, not just calculation.

14.1.3 Mathematical Content

The Principle of Mathematical Induction is fundamental for proving universal statements about the set of natural numbers; indeed, it is one of the Peano Axioms for that set. In a rigorous treatment, essentially *every* claim about natural numbers must be proved by this technique. The textbook I normally use [2] states it this way:

Principle of Mathematical Induction: For each positive integer n, let $P(n)$ be a statement. If (1) $P(1)$ is true, and (2) the implication "If $P(k)$, then $P(k+1)$" is true for every positive integer k, then $P(n)$ is true for every positive integer n.

Thus, every proof by MI has two parts: the *base step* (1) and the *inductive step* (2). The assumption in step (2) that $P(k)$ is true is called the *induction hypothesis*. The Principle is often stated instead in terms of *inductive sets*, nonempty subsets S of $\mathbb{N}$ having the property that whenever $k \in S$, then $k+1 \in S$ also. In this form it says that an inductive set containing 1 must be all of $\mathbb{N}$. This is equivalent to the textbook statement if we choose S to be the set of all natural numbers n such that $P(n)$ is true.

One can also deduce the Principle from the fact that the set $\mathbb{N}$ is well-ordered, meaning that any nonempty subset T of $\mathbb{N}$ has a least element. Alternatively, well-ordering can be used to rewrite a proof by MI as a proof by contradiction, often called a proof by minimal counterexample. To do so, we assume for the sake of obtaining a contradiction that $P(n)$ is *not* true for some natural numbers n, meaning that counterexamples exist. Then the set T of counterexamples is nonempty and must have a least element k. The base step shows that $k \neq 1$, and therefore $k \geq 2$. Since $k-1$ is then a natural number not in T, we know $P(k-1)$ is true, and then the inductive step leads to the contradiction that $P(k)$ is true as well.

In general, I find that such derivations of the Principle confuse students without adding to their understanding or confidence in this proof method. In a course at this level, where the set $\mathbb{N}$ of natural numbers is not formally constructed or defined, it is admittedly hard to give a rigorous answer to the question of why the Principle is true. I simply present it as an axiom about $\mathbb{N}$, a basic property characterizing it, that captures the intuitive picture of a discrete set ordered in a sequence beginning with 1. This is the familiar image of an infinite line of dominos that all fall down when the first one is pushed, because each knocks over its successor.

There are variations on the technique that are easier to apply to certain types of proof problems. One can prove that some statement $P(n)$ is true for every natural number greater than or equal to 5, for example, by simply modifying the base step to check that $P(5)$ is true. A bigger modification leads to the "Strong" Principle of Mathematical Induction, in which the induction hypothesis is expanded to the assumption that $P(1), P(2), \ldots, P(k)$ are all true. All such variations are actually equivalent to the original Principle.

My course does not specifically cover the principle of recursive definition, although we use it implicitly. This says, for example, that one can uniquely define a sequence a_n for every natural number n by giving a starting value a_1 and a rule of the form $a_{k+1} = F(a_k)$ for computing successive values thereafter. This is intuitively acceptable to students although its formal justification by MI is rather subtle [10].

The literature in mathematics education, and in philosophy of mathematics, distinguishes "proofs that convince" from "proofs that explain" ([11], [12]). Proofs by MI are often claimed to convince without explaining, since conjectured formulas are *proved* but rarely *discovered* using this technique, and the inductive step often seems to succeed "by magic". One should be prepared for students to have such reactions to some proofs by MI. On the other hand, some MI proofs have the form of recursive constructions of a desired object, and these seem to explain its properties perfectly well.

14.1.4 Instructional Context

At one time it was assumed that students entering college with a mathematics major would have learned to construct proofs in their prior geometry classes. Even if this assumption were correct, geometry proofs form a specialized subgenre, and college-level proofs are not written in the traditional two-column format. Accordingly, most college mathematics departments have created their own "Transition to Proof" courses to prepare students for advanced proof-based classes such as analysis and abstract algebra. At my institution, a large research university, this course is required for all mathematics majors and is taken by many other STEM majors as well, particularly computer science, economics, and engineering majors. It is a prerequisite for most upper-division mathematics courses, which are of course proof-based. Multiple sections of the course are taught each quarter, with 70–120 students enrolled in each. Most students take it in their sophomore year, following the calculus sequence, although some postpone it until their junior or even senior years. This is a one-quarter course, and because of the volume of material to be covered it is primarily lecture-based without much active learning. The instructor needs to compensate for this by anticipating and addressing what is challenging for students and by asking questions and encouraging discussion during lecture. There are weekly discussion sections of at most 30 students, run by a teaching assistant, where more interactive learning is possible. My institution has a diverse student population, including many non-native speakers of English. Because proof depends on precise use of language, and I make connections in class between colloquial and formal mathematical language, extra attention to these students' understanding is needed.

The course is a general introduction to mathematical proof techniques, and the associated language of mathematical reasoning. It begins with elementary logic and set theory, then covers the basic proof techniques: direct proof, proof by contrapositive, proof by cases, proof by contradiction, and then proof by MI about midway through the course. Illustrations of the proof techniques come from multiple content areas, but with an emphasis on basic number theory: divisibility, and the line of reasoning that begins with the Division Algorithm and concludes with the Fundamental Theorem of Arithmetic. MI is crucial for both of these results. I use the textbook *Mathematical Proofs: A Transition to Advanced Mathematics* by G. Chartrand, A.D. Polimeni, and P. Zhang [2], which is the most popular textbook for this course in the United States according to David & Zazkis [4]. Many of my colleagues, however, use Eccles [5] instead. The lesson described here extends over three class sessions.

14.1.5 Overall Rationale for Instructional Decisions

Setting the stage. A couple of weeks before this lesson, I assign the following homework problem, based on a claim made without formal proof early in the textbook [2, p. 86]: Prove as carefully as you can that every natural number is either even or odd but not both. One can prove by contradiction that no natural number can be both even and odd: the existence of a number that can be written in both forms $2m$ and $2n + 1$ leads easily to the contradiction that 1 is even. It is trickier to show that each natural number must be in one or the other category. Many students approach this by showing that if a number is even then the next consecutive number is odd, and vice-versa. They are not sure, though, if this suffices for a proof. In fact, this would be the inductive step in a proof of the claim by MI. My goal is simply for them to realize that some principle of logic is required to justify this sort of reasoning, even if they do not yet know what it is. This follows Harel's Necessity Principle: in order for students to learn a mathematical concept, they need to experience a problem situation for which that concept is required, forming the key to its solution [8].

Introducing the technique. My overall approach to introducing the technique of MI is based on research by my colleague Guershon Harel [7], [9], who suggests the following introductory problem consistent with the Necessity Principle. Let's define a sequence of numbers x_n by the conditions that $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ thereafter. Thus, the sequence begins $\sqrt{2}$, $\sqrt{2+\sqrt{2}}$. $\sqrt{2+\sqrt{2+\sqrt{2}}}$, and so forth. One can conjecture from the numerical values of these initial terms that the sequence is increasing, and that each term is less than 2. (One might also conjecture that the limit of the sequence is 2, but that is a more complicated concept.) Students are asked, how can these conjectures be justified?

I situate this question in the more general context of, how can a plausible mathematical pattern be proved or justified? Harel and Brown [9] distinguish two kinds of pattern-based reasoning. Result Pattern Generalization (RPG) focuses on a regularity in the *result* of a process or computation, in this case the concrete numerical values of the terms. If a sample of these values fit some pattern, then the pattern is accepted as valid. Scientific induction, generalizing from instances of a pattern to a general law, is essentially a more sophisticated version of this. It is of course not a valid principle of logical deduction. Process Pattern Generalization (PPG) focuses on the regularity in the *process* that generates the terms, that ultimately accounts for the pattern. This is the intuitive version of the Principle of MI, which *is* logically valid. Here, the process that generates the terms x_n ensures that if one term is less than 2, so is the next one. The key is to uncover the mechanism that ensures this. It is not simply a coincidence, as it might be if merely based on a sample of numerical values. To give a proof is to expose and explain this mechanism. I do not use the terms RPG or PPG in class, but I emphasize looking for connections between successive numbers in a sequence, and logical relationships between successive instances of a sequence of claims.

14.2 Lesson Implementation

14.2.1 Harel's Sequence Problem

When asked why each term $x_n < 2$, students often volunteer the idea that a particular term will be less than 2 if the quantity under its outermost square root is less than 4. Since that quantity is 2 plus something, the something has to be less than 2 itself. Since that something is in fact the previous term, the property of being less than 2 is indeed "self-perpetuating" from one term to the next. It is important that students express this idea intuitively first, without trying to express it formally in algebraic notation at once. The formal algebraic derivation can look "mechanical", just an instance of symbol-pushing, whereas the intuitive formulation captures the key idea of PPG. As students become more familiar with the template for MI it is all the more important that they keep this idea in focus and not let the proof process become mindlessly automated. The introduction of algebraic notation also carries its own challenges as students need to distinguish the kth and $(k+1)$st terms, be clear which one the inductive hypothesis applies to, and so forth. Following a clear intuitive argument, we write the formal proofs of both conjectures, showing how the intuition translates into a normative mathematical exposition. The proof that the sequence is increasing is trickier than the claim that each term is less than 2 since it involves pairs of terms rather than individual ones. The inductive step for that proof assumes that $x_{k-1} < x_k$ for some k. Then $2 + x_{k-1} < 2 + x_k$, so $\sqrt{2 + x_{k-1}} < \sqrt{2 + x_k}$, which is the desired conclusion $x_k < x_{k+1}$. We also revisit the proof that each natural number is either even or odd, showing that MI is the needed proof technique that was missing earlier. I do not formally state the Principle of MI until we have constructed and understood both of these proofs. The Principle should be a summary of students' reasoning, letting them take ownership of it, rather than an authoritative textbook pronouncement.

14.2.2 Further Examples

The sequencing in this lesson begins with "standard" examples in which the recursive structure of the sequence of claims is more obvious and the needed calculations more mechanical. This helps students see how the Principle of MI formalizes their intuitive PPG reasoning. Later problems have less obvious connections between successive statements, more challenging requirements for the careful use of symbols, and more creative reasoning in the inductive steps.

Due to the constraints of my lecture format, I present most of the examples myself. Instructors with more time and/or an inquiry-based classroom should pose as many as possible as tasks for the students, perhaps in small groups, followed by whole-class discussion. The first goal in these tasks is for students to articulate the RPG reasoning in their own words and defend it against possible objections and loopholes. Then the instructor can help them formalize the reasoning and translate it into a normative proof. This involves directing their attention to the sequencing and justification of the claims, the use and meaning of symbols and notation, and the stylistic conventions of proof.

Although summation formulas should not be overemphasized, they are essential first examples. I prove that the sum of the first n natural numbers is $n(n+1)/2$ to show how this works, and I refer the students to additional similar examples in the textbook. These are among the most mechanical examples of MI: you "turn the crank", and it works. Students should learn to turn the crank but should not come to expect every proof by MI to fit this pattern. At one time, students could be expected to have encountered these summation formulas for powers of integers in their calculus classes, in evaluating Riemann sums, but instructors can no longer count on this.

Following the summation example, I return to Harel's $\sqrt{2}$ sequence and prove the explicit formula $x_n = 2\cos(\pi/2^{n+1})$. This clarifies the distinction between an explicit formula and a recursive one. The inductive step is almost automatic by "turning the crank", but it depends on the trigonometric half-angle formula $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$. Students don't have to know this by heart (I don't), but this example shows the value of being aware that such formulas exist and where to find them when needed. The explicit formula then "explains" why the sequence x_n is increasing and bounded above by 2, and indeed shows that its limit is 2 as conjectured earlier.

14.2.3 Divisibility

I pose the (textbook) problem of showing that for every natural number n, 6 divides $n^3 - n$. The recursive structure of this problem is not as obvious as in the preceding examples: the key question for MI is always, how is any specific case $n = k$ helpful in proving the next case $n = k+1$? Some exploratory algebra shows that $(k+1)^3 - (k+1) = (k^3 - k) + 3(k^2 + k)$. Students need to realize that the goal of the algebra is not to *simplify* the quantity on the left, but to *connect* it to the corresponding quantity $k^3 - k$ in the induction hypothesis. At this point in the course we have seen that $k^2 + k$ is always even, so this shows that the two quantities *differ* by a multiple of 6. Therefore, if one of them is divisible by 6, so is the other. This problem promotes the general course goal that algebra is a tool for *reasoning*, not just calculation. Algebra can suggest the next step in a proof, but it must serve the overall goals of the reasoning, not be applied mechanically.

There are also approaches that do not use MI, at least not in an obvious way. Students may suggest that since $n^3 - n = (n-1)n(n+1)$, this is the product of three consecutive integers, of which one must be a multiple of 3 and one must be even, so that 6 necessarily divides the product. This depends on the Division Algorithm, though: the fact that each integer is congruent to 0, 1, or 2 modulo 3, in a cyclically repeating pattern: the mod 3 version of the even/odd warm-up problem discussed in "Setting the stage," above. My textbook [2] assumes this provisionally in early chapters before later giving the proof, which of course uses MI. In general, MI problems assigned to students should genuinely require MI, but discussion of alternate solutions in which the use of MI is "hidden" can also solidify their understanding.

14.2.4 An Inequality

I introduce inductions that don't start with the case $n = 1$ through the example of proving that (sometimes) $2^n > n^2$. A numerical table suggests that this holds for $n > 4$, which is what we prove. Following the base step $n = 5$ the inductive step requires us to show that $2^{k+1} > (k+1)^2$ when $k \geq 5$, an unfamiliar task for students. Using the induction hypothesis we obtain $2^{k+1} = 2 \cdot 2^k > 2k^2$. Thus, it will suffice if we can show that $2k^2 > (k+1)^2$, or even $\geq$ here. Students are not always comfortable with such uses of the transitive property. Setting up this kind of *subgoal* in a proof may not be familiar to them either. It represents progress because we are now comparing two quadratic functions rather than an exponential and a quadratic. One way to achieve the subgoal is to simplify it to the equivalent inequality $(k-1)^2 > 2$, which is clearly true for $k \geq 3$. The written proof needs to make it clear when a subgoal is introduced, with a warning such as, "We will now show that… ." Otherwise, the stylistic convention is that proofs are read linearly, so that any claim made is a logical consequence of the previous ones.

Students are often puzzled that the inductive step works for $k \geq 3$ whereas the theorem itself holds only for $n \geq 5$. This is an opportunity for them to revisit the need for both the base step and the inductive step. *If* we had a valid base step for $n = 3$, we could begin the induction there, but that case does not hold.

14.2.5 Invisible Inductions

I point out to students that they have seen examples in prior courses of results that have "invisible" statements and proofs by MI. For example, calculus textbooks show that differentiation is linear, in the sense that $(f+g)' = f'+g'$ for any two differentiable functions $f(x)$ and $g(x)$. This is then taken as a license to differentiate polynomials, for example, term by term, even when there are more than two terms. Such generalizations without proof from two terms to n terms are common in mathematical exposition. They have simple inductive proofs, once the reader is aware that this is what's expected. I present such a proof in class for this case of linearity of differentiation. This example helps to "enculturate" students into the mathematical community by "letting them in on a secret" as to its norms and expectations. Students can be asked for other examples they may have seen, and how those proofs would be constructed. These might include formulas for simplifying the logarithm of a product of several numbers, or the determinant of a product of several matrices.

14.2.6 Another Calculus Application

MI can be used to prove the differentiation rule $\frac{d}{dx}\, x^n = nx^{n-1}$ for natural numbers n. For the base step we need to know that the slope of the graph of the function x (a straight line) is 1. For the induction step we need the product rule: $\frac{d}{dx}\, x^{n+1} = \frac{d}{dx}\, xx^n = xnx^{n-1} + 1x^n = (n+1)x^n$. It's useful for students to see that MI applies in calculus contexts, and it may help them to understand the meaning of "we induct on n," even though there is also a variable x in the problem. However, students know that the power rule actually holds for all real n and may be confused that MI only proves it for the subset of natural numbers. It gives the correct formula, but not for the full range of values of the parameter. Students are likely to ask whether induction can be used to prove such claims for all integers n, by treating the positive and negative integers separately. This is a good example for exploring this suggestion.

14.2.7 Cautionary Tales

I discuss at least two types of errors in applying MI, to warn students against doing so mechanically and to solidify their understanding of the logic of this technique. First, there are false "theorems" for which the inductive step nevertheless works fine, to emphasize the importance of the base step. A simple example is the false claim that $n > n + 1$, for which there is no true base instance. Second, logical or verbal swindles in the inductive step. My favorite example here is the "proof" that all people have the same sex (meaning gender; this gets the students' attention, although more mundane predicates like having the same name or hair color would make the same point). More formally, I present the following argument that for each natural number n, given any set of n people, they all have the same sex. Base step: in any set consisting of one person, that person clearly has the same sex as him/herself. Induction step: suppose the claim is true for every set of k people, and let S be a set of $k + 1$ people. List those people in some arbitrary order. Let S' be the set obtained by removing the first person from S, and let S'' be the set obtained by instead removing the last person. (I draw a line of several stick-figure people on the board and cross out either the first or the last.) Then by the induction hypothesis everyone in S' has the same sex, and likewise everyone in S'' does too. But, from the diagram on the board, the sets S' and S'' overlap: there are people belonging to both. So, the sex of those people is the sex of everyone in S, which proves the inductive step.

Having presented this "proof" I ask students what they think is wrong with it. Knowing that the conclusion is false is not the same thing as detecting the flaw in the reasoning. I recommend that instructors collect several student suggestions without endorsing their correctness. Ask clarifying questions ("What did you mean by that?") but let students debate the reasoning themselves. My students often object to applying the induction hypothesis to the sets S' and S''. The resulting discussion leads them to appreciate the meaning and the strength of the induction hypothesis that *every* set of k people have the same sex, regardless of how such a set was constructed. Sometimes they ask about the meaning of "overlap". Eventually they focus on this claim that the sets S' and S'' have nonempty intersection. This looks obvious in my diagram because each set has several elements, and one has to examine the case $n = 2$ specifically to see that it is false in this case: if S has two elements, each set S' and S'' has one element, and these are distinct. The conclusion is that the inductive step doesn't bridge the gap between the cases $n = 1$ and $n = 2$ although it works in every subsequent case! Indeed, if it were true that every set of two people have the same sex, then the general claim would hold. The moral is that an inductive argument that sounds plausible for generic values of n should also be checked for specific values, often small ones.

At this point I present an example of what Richard Guy [6] terms *the Strong Law of Small Numbers*: "there aren't enough small numbers to meet the many demands made of them," meaning that an apparent pattern cannot be trusted on the strength of a few examples involving small numbers. This is intended to make students skeptical about RPG and more reliant on PPG. My favorite example is the chords-of-a-circle problem: Choose n points on a circle in "random" positions and connect them by drawing all possible chords. How many regions does this divide the circle's interior into? The pattern begins 1, 2, 4, 8, 16 consistent with the formula $2^n - 1$ but incredibly the $n = 6$ term is 31, not 32. Students really need to draw the diagrams and count the regions themselves to believe this. (If the points are not chosen "randomly", then there may be three or more chords that coincidentally meet at a common point inside the circle. This is a missed opportunity to create more interior regions.) My only regret about this example is that it is too much of a digression to derive the correct formula, which is $1 + \frac{n(n-1)(n^2-5n+18)}{24}$.

14.2.8 Strong Induction

Thinking about PPG, there may be examples where a single instance of a pattern does not suffice to make the next instance true. Each instance might be logically connected to prior instances, but not the *immediately* preceding instance. This is what strong induction addresses. I often explain strong induction in the following way. Imagine starting with the base step $P(1)$ and then proving each successive instance of $P(n)$ from the previous one via the inductive step. By the time you reach the proposition $P(k)$, you actually know that $P(1), P(2), \ldots, P(k)$ are all true. So, you may as well make use of all that information in proving $P(k + 1)$. That becomes the new "strong induction" hypothesis.

The first application of strong induction that I present is to a sequence having a two-term recursive definition. I begin with my textbook's [2] example of the sequence defined by $a_1 = 1$, $a_2 = 4$, and $a_n = 2a_{n-1} - a_{n-2} + 2$ thereafter. Numerical examples suggest that $a_n = n^2$, although it looks like a coincidence in each case. We prove this by strong induction. Apart from the need for two base steps, it is "turning the crank" again. For the induction hypothesis we assume that for some k, $a_{k-1} = (k - 1)^2$ and $a_k = k^2$. The key calculation for the induction step is $a_{k+1} = 2k^2 - (k - 1)^2 + 2 = (k + 1)^2$. I think this *does* play an explanatory role because it shows exactly how the recursion relation has been crafted to ensure that it always produces perfect squares.

An important instructional choice in presenting strong induction is deciding how many base steps such a proof requires. For a two-term recursion one clearly needs to know the first two terms of the sequence in order to compute more, so I feel that the most natural answer is two, namely $n = 1, 2$. In other problems, like the postage stamp problem presented next, the number is less clear. My textbook [2], in order to prescribe a uniform method for all MI problems, insists that there is only one base step $n = 1$ in all cases: additional small values of n requiring special treatment are handled at the start of the inductive step rather than included in the base step. I prefer including them in the base step even though it may not be clear how many are needed until the inductive step has also been completed.

The postage stamp (a.k.a. Frobenius, or chicken nuggets) problem is to show that for each integer $n \geq 8$, there are nonnegative integers a and b such that $n = 3a + 5b$. Of course, there are innumerable variations in which 3 and 5 are replaced with other (relatively prime) numerical values. The interpretation is that combinations of 3 and 5 cent postage stamps can pay any total postage of at least 8 cents. Students should be encouraged to work out several examples for various n to build intuition for how this works. They are often reluctant to do so, having been told by their teachers that examples are not proofs. That's true, but examples *are* opportunities to notice the patterns that will make the proofs work. Here students should notice that the combination of stamps giving a specific total is not unique, but once one has obtained three consecutive totals then one way to obtain any higher total is to just add more 3 cent stamps. This suggests the following inductive proof with three base steps.

Base steps: $8 = 3(1) + 5(1)$, $9 = 3(3) + 5(0)$, $10 = 3(0) + 5(2)$. Inductive step: Assume that $k \geq 10$ and that the claim is true for every $n = 8, 9, \ldots, k$. We will prove the claim for $n = k+1$, which is at least 11. Then $n-3 = k-2$

is at least 8, so by the inductive hypothesis there are nonnegative integers a and b such that $k-2 = 3a+5b$, which implies $k+1 = 3(a+1)+5b$. Then $a+1$ and b are the desired nonnegative integers in the case $n = k+1$.

This is a situation where students face challenges with expressing the underlying reasoning correctly in mathematical notation. Some of my students tried to use the same letters a and b to denote the integers that work in the separate cases $n = k$ and $n = k+1$. Some tried to distinguish them with subscripts or in other ways. Some tried to give explicit formulas for a and b in terms of n, which is neither necessary nor productive. Some were unsure whether the induction hypothesis applied to case k, case $k-2$ or case $k-3$. Some did not appreciate the importance of verifying that $k-2 \geq 8$, which is what guarantees that we have checked enough base steps. It would be a great class activity for students to construct proofs together in small groups and then share with other groups the intuitions they drew upon and how they solved the notational problems of presenting the proof.

If one drops the requirement that the integers a and b are nonnegative, then *every* integer n can be represented in the specified form. This appears later in the course when we discuss relatively prime integers and greatest common divisors in detail. MI still plays a role in the proof.

If there is time for more examples (or homework), then the Fibonacci sequence exhibits innumerable patterns that can be proved by strong induction.

14.2.9 Half of the Fundamental Theorem of Arithmetic

The proof of the Fundamental Theorem of Arithmetic, that each natural number greater than 1 has a unique prime factorization, is a major content goal of this course. I prove the easier existence part now as an application of strong induction, postponing the uniqueness claim until more machinery (Euclid's Lemma) has been developed. This is another new wrinkle on the technique for students, since it needs the full strong induction hypothesis: the specific part of the hypothesis that is used depends on which case is being considered. The base step $n = 2$ holds because 2 is prime. In the inductive step we may assume that each $n \in \{2, 3, \ldots, k\}$ has a prime factorization and then prove that $k+1$ has one. If $k+1$ happens to be prime, then it is its own prime "factorization". If on the other hand $k+1$ is composite, then it has some factorization $k+1 = ab$, where a and b belong to the set $\{2, 3, \ldots, k\}$. Even though we don't know which elements of the set they may be, each has a prime factorization by the induction hypothesis, and multiplying these gives a prime factorization of $k+1$. Students often ask whether the single base step we checked was sufficient; in other strong induction problems the number of base steps was greater than one and could be specified in advance. It can be useful for them to write out the justifications of several early cases $n = 3, 4, 5, \ldots$ in sequence to see that no additional base step is needed. It's also helpful for them to compare the induction step here with the recursive process one actually carries out to completely factor some given integer, such as 60.

14.2.10 Proof by Minimal Counterexample

At one time I thought that deriving the Principle of MI from well-ordering could effectively counter students' beliefs that MI is circular. I've changed my mind, since the derivation is logically and notationally tricky, and students tend to be puzzled rather than convinced by a "proof" of one obvious fact from another. However, reasoning about the minimal element of some set is an important technique that is needed later in my course. Therefore, I choose one of the easier MI proofs and reformulate it as a proof by minimal counterexample for students. This is to make them focus on the logic of the reformulation rather than the mathematical content of the proof itself. I can't say that this works very well. I think what is needed is a necessitating situation, like Harel's $\sqrt{2}$ problem, that will lead students to the minimal counterexample form of reasoning themselves. I leave this to the reader as an open problem of instructional design!

14.2.11 Discussion of Student Thinking

Inductive thinking about patterns (PPG) is natural for students to some extent, but they encounter many obstacles in trying to formalize it as a reliable proof technique. Instructors should expect to see and address the following kinds of student thinking, some of which have already been mentioned. The lesson design tries to anticipate and address many of them, but I don't have effective solutions to them all.

The original instructional challenge was that many students think the logic of MI is circular. That is, one is trying to prove $P(n)$, but the induction hypothesis is to *assume* $P(n)$! How can that be valid? This error is the result of ignoring quantifiers. We are trying to prove that $P(n)$ is true *for all* n. The induction hypothesis is that $P(n)$ is true *for some specific* n, for the purpose of proving the next instance $P(n+1)$. And in fact, having completed the base step, we *know* $P(n)$ is true at least for the specific case $n = 1$. One example can be worth a ton of theory: the initial proofs involving Harel's $\sqrt{2}$ sequence are clearly valid and noncircular. I also try to avoid this error by changing notation: using $P(n)$ for the general claim but $P(k)$ in the induction hypothesis.

A distinct form of "circularity" is that students often replace the inductive step with its converse, assuming $P(k+1)$ and deducing $P(k)$ from that. This behavior is not specific to MI but originates from students' prior experience of algebra. When solving an equation or proving an identity, students are taught to replace an equation by an equivalent equation at each algebraic step. They often prove an identity by reducing it to a trivial one such as $0 = 0$ by a sequence of such steps. This is fine if each implication step is really an if-and-only-if biconditional. But students generally don't attend to the direction of logical implication in algebraic work and are not aware of how algebra is used to prove one-way implications. They may not realize that the first of a sequence of algebraic steps plays the logical role of a hypothesis whereas the final step is a conclusion. These are not interchangeable unless all the steps are reversible. I put considerable emphasis on this aspect of algebraic reasoning throughout the course.

Students have additional difficulties writing formal proofs by MI that stem from the use of algebra to express the reasoning involved. The inductive step can be notated as $P(k) \to P(k+1)$ or as $P(k-1) \to P(k)$, but these do not seem interchangeable to some students. In the postage stamp problem, some students did not make the meaning of their variables a and b clear, specifically which value of n they were associated with. Using algebra as a language with clarity and precision is an overall goal of a transition to proof course that takes much more than any single lesson to address.

Although students have seen sequences of real numbers in calculus, they may not be comfortable with how the notation is used to distinguish terms such as x_n versus x_{n+1} or x_{k-2}. They may confuse the *label* $n+1$ of the term x_{n+1} with the *value* of that term. It may not be clear when n is used as a generic label versus when it has been assigned a specific numerical value. This use of variables is more subtle than those students are used to: either placeholders for a single unknown value, or continuously covarying quantities as in calculus.

Students are often unclear about the differences between the sets of natural numbers, integers, rational numbers and real numbers. In their experience of algebra, it rarely matters what the domain of a variable is, because the rules for manipulating them are the same in all cases. Thus, students may not realize that it can be crucial to keep track of this. They may not have internalized key differences such as: the natural numbers have a least element; the rational numbers are dense while the real numbers are complete; divisibility makes sense for integers but not for rational or real numbers, and so forth. This can lead them to try induction over a set of real numbers, or to claim that 5/7 divides 5.

Students also learn very little about inequalities as compared to equations in algebra, which is a handicap when proving claims such as $2^n > n^2$ for $n > 4$. They may learn to "solve" restricted classes of inequalities, but very little about reasoning with them, for example establishing an upper or lower bound for some quantity or writing a chain of inequalities connecting two quantities. Sometimes they are explicitly taught to work with inequalities by changing them to equations, manipulating those, and finally restoring the original inequality symbol. Many know and use the fact that functions cannot change sign in the intervals between their zeros but are unaware that this depends on continuity and the Intermediate Value Theorem.

There are at least three aspects of student thinking that are relevant to a unit like this that covers a particular proof technique. One is student understanding of the specific technique itself, its logical basis and the typical examples to which it applies. Another is students' facility with mathematical expression, the ability to use variables, operations, algebra, and explanatory text to present their arguments clearly and correctly to a reader. Finally, there is the background knowledge, sometimes termed *resources*, that students must draw upon when constructing proofs. This may include knowledge about divisibility, trigonometric identities, number systems (e.g., integers versus real numbers) and so forth. When instructors ask what is hard for students about MI, or any other proof technique, they should expect the answers to involve all three aspects.

14.3 Post-Lesson Considerations

14.3.1 Additional Examples

I do not have time in my course for additional in-class examples of MI, but I can recommend some to other instructors. Earlier in my course I proved the arithmetic-geometric mean inequality for a pair of positive real numbers. A beautifully simple inductive proof for arbitrarily many positive numbers is given by Chong [3]. I also presented Euclid's proof that there are infinitely many prime numbers. An alternative "proof from The Book" that uses MI works by showing that any pair of the Fermat numbers $F_n = 2^{2^n} + 1$ are relatively prime [1]. These help to convince students that MI proves interesting and significant mathematical results, not just textbook exercises. A nice application of MI in a visual/geometric context is the following tiling problem. Given a $2^n \times 2^n$ chessboard from which any one square has been removed, and an adequate supply of tiles each in the L-shape of the $n = 1$ case, prove that the board can be completely covered by these tiles. The proof provides a recursive procedure for actually constructing such a tiling.

14.3.2 A Challenging Problem

How do we know that the sum of a list of numbers doesn't depend on the order in which we add them up? Addition is done on two numbers at a time, after all, so the "sum of a list of numbers" is not even well-defined until this question is answered. If there are two numbers, this is the content of the commutative property, and if there are three then it is not hard to prove using the associative property as well. This suggests a proof by MI that the sum of n numbers is independent of their ordering for all $n \geq 2$. I sometimes suggest this problem to ambitious students. It is challenging to make sure that one's proof has accounted for all possible orderings, and to many students it is not even clear exactly what is at issue here. The claim is so "obvious" that it is hard to avoid taking some instances of it for granted during the proof.

14.3.3 Later Use of MI

Proof by MI should not be confined to a single unit within a course, or a single homework assignment. Students should be assigned additional MI problems later in the course, so that they have to choose MI as an appropriate proof technique from their full repertoire of methods. The instructor should apply MI in later proofs to confirm its importance in mathematical reasoning beyond the basic examples. My course finishes with the elements of number theory leading up to the Fundamental Theorem of Arithmetic. Here MI is needed to prove the Division Algorithm, the properties of the greatest common divisor of two integers, and Euclid's Lemma that a prime dividing a product must divide one factor. And of course, what you test for is what you (really) value: exam problems on MI need not be algebraically complex but should be nonmechanical and require genuine reasoning rather than simply "turning the crank".

14.3.4 Conclusion

The elements of this instructional sequence are based on the research of Harel and others, but I don't have research-based evidence that this instructional design is superior to some comparison method of "traditional instruction." My impression from my teaching experience (discussions in class and during office hours, student performance on homework and exams) is that students taught this way are less likely to believe that MI is circular reasoning, and that they perform well on routine problems. They still struggle with notational issues, with the number of base steps in strong induction, and applying MI to problems where it is not appropriate.

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15

Discussion of Lesson Analysis and How to Get Involved

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The project that this book grew out of attempted to tackle an important problem in the field of undergraduate mathematics education: creating a space and a method for reporting on the detailed components of teaching a lesson. Such a method can open a new range of professional dialogue around how we teach and how we deal with the multi-dimensional space of choices available. In this book, we have focused mainly on a written method for sharing such analyses, that we have called *Lesson Analysis Manuscripts (LAMs)*. In future work we (the book editors) hope to explore more deeply the work of *Lesson Analysis (LA)* itself. In this concluding chapter, we examine how LAMs fit within other types of teaching scholarship and offer some reflections on our process. In doing so, we wish to be clear that this section contains our views (the book editors') and might not necessarily reflect the views of all of the conference participants, though we did solicit feedback from them on the content of this chapter. Similarly, while we have based our descriptions of the Scholarship of Teaching and Learning and of Design-based Research according to what we have found in published journals and books, others might have different understandings of these programs.

15.1 Relationship between Lesson Analysis and Other Types of Teaching Scholarship

The work of LA, and its accompanying LAMs, are a cousin to other, more established, forms of teaching scholarship. Two forms we touch on in this section are The Scholarship of Teaching and Learning (SoTL) and Design-Based Research (DBR). We explain what SoTL and DBR offer by way of building knowledge for teaching and then explain the relationship we see LA has with them. We wish to highlight that SoTL, DBR, and LA, all share similar goals and can work together to build knowledge for teaching, but that their products have distinct foci.

15.1.1 The Scholarship of Teaching and Learning

The Scholarship of Teaching and Learning is a type of scholarly work that encourages instructors in tertiary institutions to apply appropriate research methods to answer meaningful questions about their own teaching and their students' learning. A perusing of two SoTL journals (Journal of the Scholarship of Teaching and Learning, International Journal for the Scholarship of Teaching and Learning) reveals that the bulk of the articles fit the form of standard educational research papers. Many use pre- and post-tests or survey results to establish learning gains or changes in affective domains, and the focus is typically on larger curricular units, entire courses, or on tools that transcend individual lessons. In fact, in a brief analysis of 100 SoTL articles (The 50 most recent articles available online from each of the two journals previously mentioned, as of Spring of 2021), only one was specific to an individual lesson and it spent less than a page describing the lesson itself. The goal of this type of research, at least as represented in these two prominent journals, tends to be on generalizable knowledge that is applicable across contexts. As one prominent reviewer put it, "While I enjoy reading about how a mathematician struggles to help her students understand the nature of mathematics proofs, or how a literature professor tries to help his students understand Beowulf, I would also like the piece to include something that I, as a political scientist, can use in my class" (Bernstein, 2011). The LAMs in this book may not help a political science professor, but they are specifically designed to be very helpful to mathematics professors in their classroom teaching. Perhaps LAMs could be thought of as a way to capture content-specific instructional knowledge, which is of course very important knowledge in teaching [7].

Does LA simply fall under the SoTL umbrella? Some might say yes while others no, and the answer probably depends on one's perspective of SoTL. One SoTL scholar, Weimer (2006), takes a broad view of what is included in SoTL work, categorizing SoTL studies as either *Research Scholarship* or *Wisdom of Practice*. Research Scholarship is the result of formal research to answer questions related to a professor's (or multiple professors') instructional activities and student learning. Wisdom of Practice are insights about teaching and learning drawn mainly from experience, perhaps through years of varying instruction and reflecting upon the results, without doing a formal research study. However, Wisdom of Practice pieces tend not to focus on the detailed decisions made during teaching a lesson. In this broad view, we feel that there is room in the SoTL umbrella for work like LA, although it might not fall under either of these categories.

15.1.2 Design Based Research

The second type of scholarly work that has connections to LA is Design-Based Research [2]. Authors whose work is classified as DBR strive to understand issues of teaching and learning by designing or "engineering" instructional interventions and studying student learning and means of supporting that learning. DBR includes cycles of interventions and revisions over an extended period of time and takes place in a variety of authentic or naturalistic settings. DBR draws heavily on learning theories to guide the design and interpretation of data and seeks to build instructional principles and curricular materials that speak to pressing matters of practice. The goal of this type of scholarship is often focused on establishing domain-specific *theory*. In fact, the intertwining of developing learning environments/interventions and developing theories of learning is a hallmark of DBR [3].

Does LA simply fall under the DBR umbrella? Again, the answer might be yes or no, depending on one's perspective. LA seems to fit closely with the purpose of DBR and its use of the design cycle. They both take seriously the need for developing solutions for instructional challenges and develop the solutions in specific contexts. Yet, LAMs may be considered to have a different scope than many DBR products and may use that fine-grained scope to focus on details that are not usually captured in DBR products. LAMs also would likely not emphasize the development of domain-specific theory, even though they could certainly use domain-specific theory to provide rationale for certain decisions or to explain how students think at one point in a lesson. DBR helps to develop instructional knowledge that is definitely related to the knowledge LAMs are meant to capture, but LAMs may offer a genre for sharing lesson-specific parts of that instructional knowledge.

15.1.3 Lesson Analysis

Like SoTL and DBR, the focus of LA is on developing engineered solutions to specific instructional challenges. However, in the instructional materials that are currently accessible, we are missing a kind of knowledge that is central in the practice of teaching. To reiterate this point from Chapter 1, we share again this quote from Hiebert around this idea:

> In addition to unpacking the details of teaching, studying teaching means seeing the cause-effect relationship between teaching and learning that infuse an ordinary lesson. Many teachers do not appreciate that slight changes in lessons - in the ways they interact with students around content - influence directly what students learn. When teachers see the effects of the changes they make on what and how well students learn, they can begin to appreciate the powerful impact of studying the details of teaching [4, p. 53].

A LAM is a deep dive into the myriad instructional decisions and an opportunity to connect student thinking to the instructional decisions at the natural rhythm and level of daily instruction. LAMs focus on a specific instance of teaching and lay out the many small and large decisions, as well as how these decisions address an important instructional challenge. In trying to provide a possible solution to the instructional challenge, instructors account for constraints and manage a variety of dilemmas native to teaching [6]. Thus, LAMs provide a much-needed space for dialogue about detailed instructional practice that is difficult, uncommon, and not well established.

LAMs broaden the opportunity for other instructors to build on the teaching knowledge of others, and to continue to grow important knowledge in the community. Of course, there are many instructors that are already generating instructional knowledge and doing what we would call LA, but we lack a written product to share that knowledge and a well-defined scholarship around that type of product. We believe coordination and building a shared knowledge base in the form of LAMs promotes the sharing of and evolution of the work engaged in by reflective instructors, and it also emphasizes the importance of that work.

15.2 Reflection: The Benefits of Limited Scope and Specific Details

In summary, and in reflecting over our process, we (the book editors) feel that one of the most important contributions of the LAM genre is the way we can capture specific details not commonly found in other types of published records. For example, a lesson plan often might offer a suggestion such as, "Use the result of the students' work to have a five-minute discussion that develops the idea of a basis of a subspace." Unfortunately, that directive does not help the reader/user see *how* to hold such a discussion, what students might say during that discussion, or how the teacher orchestrates that student thinking toward the learning goals. Such gaps are likely to be filled in by the reader's own skills, goals, and philosophies, which may differ from the author's. By including a specific instructional challenge, explaining rationales for teaching decisions, describing student thinking, and including other important components of teaching, a LAM is situated to give deep insight into how a lesson designed to address a specific purpose plays out, in a given setting.

LAMs' ability to focus on these details are derived in great part from the fact that LAMs are limited in scope to a single lesson, or perhaps to a sequence of two or three lessons. We realized that if we tried to describe too many lessons, even if they did build on each other, much of that detail described here would be lost. Those descriptions tended to revert back to the "Hold a five-minute discussion" style of description, without being able to adequately give the details in the LAM on how to hold that discussion. We hope that the LAM genre might provide a space for preferentially capturing the fine-grained details of a lesson, as opposed to covering a wide stretch of course content. We believe that such fine-grained accounts, even though they do not cover a large swath of curriculum, are a necessary part of building up a knowledge base for teaching undergraduate mathematics.

15.3 Reflection: Some Gaps in our Current Work and How We Plan to Address Them

In reflecting back across our process of trying to build up this specific genre of scholarly writing, we realize that in many ways this was a "build the bridge while crossing it" project. By this we mean that we (the entire conference group) were pioneering a new type of written product and tried the best we could to learn from each other's ideas, each other's attempts at writing up LAMs, and the feedback we gave each other during that process. Thus, we in no way claim that the outcome of our collaborative work represented in this book is a finished product. There is much work to do in understanding how to carry out Lesson Analysis, and what content in a LAM is the most impactful. For example, we are still learning the best way to incorporate student thinking or small classroom excerpts into a LAM, and the LAM authors in this book used different methods to try to include student thinking. It might be helpful to know that this project was done during the summer when few of us were teaching, so the lessons we were writing about had already been completed. Thus, while the outcome of our work together was the proposal of a useful genre of capturing and sharing teaching knowledge, future work is needed to revisit the actual Lesson Analysis process itself. Part of that future work will be to understand how to plan ahead to capture the needed student thinking to include in a LAM, or how to make one's own rationales explicit and clear for others to see, or how to reflect carefully on how equitable practices are used in a classroom.

Because of this needed work, we (the book editors) hope to follow this book up with a second volume. Our desire is to organize a second conference of undergraduate mathematics instructors where the focus of our work for this next time would be to work through the Lesson Analysis process beginning to end. This second iteration would allow us to flesh out the Lesson Analysis process, such as describing more clearly what doing Lesson Analysis might look like and what types of records one should make or data one should gather in order to write the most effective LAM. If you are interested in being involved in a future project, please reach out to Doug Corey at corey@mathed.byu.edu.

15.4 Where to Go from Here: Publishing Your Own LAM

We hope that this book will inspire many reflective undergraduate mathematics teachers to share the instructional knowledge they have with the rest of the undergraduate mathematics education community. Moreover, we hope that sharing instructional knowledge in LAMs will allow instructors to not only consume that instructional knowledge but to also contribute back by writing their own LAMs or building on others' LAMs. That is, we hope *you* consider lessons you have thought deeply about that you could share, in which you detail student thinking, class discussions, and the other elements of a written LAM. In this hope, we share here our plans for how one might publish their own LAMs. We see two main avenues for publishing LAMs at this point. The first is for someone to be a part of anticipated future volumes of this book, in which we hope to provide additional collections of example LAMs, just as we did in Chapters 3-14 in this book. Again, if you are interested in participating in this way, please contact Doug Corey at corey@mathed.byu.edu.

The second main avenue for publishing LAMs is in one of the several journals that already encourage manuscripts that discuss classroom teaching. We have been in contact with a few of these journals to ask the editors their disposition toward publishing this genre. Many have reacted favorably, indicating that LAMs would be a welcome addition to the other styles of manuscripts they receive. Of course, some have fairly short page or word limits, but editors have suggested the option of using supplemental materials to still fit a LAM within the journal's pre-existing guidelines. The following is a list of journals that may be natural fits, as well as some useful information about adjusting the LAM format to fit the journal:

- ***PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies***. One of the editors for PRIMUS was part of our conference group and has indicated that PRIMUS is interested in publishing LAMs. PRIMUS was originally created, at least in part, to share the kind of knowledge captured in a LAM with college mathematics teachers. The PRIMUS website is https://primusmath.com.

- ***The International Journal of Mathematics Education in Science and Technology*** (IJMEST). IJMEST has a section in their journal called "Classroom Notes" that provides a space for fairly long articles for sharing instructional knowledge. The editor has agreed that LAMs would be a good fit, particularly those dealing to mathematics lessons connected to science and/or technology in meaningful ways (since that is the focus of the journal). https://www.tandfonline.com/toc/tmes20/current.

- ***The Journal for Inquiry-Based Learning in Mathematics*** (JIBLM). JIBLM is an online publication that was started with the intent of curating inquiry-based learning (IBL) course materials. JIBLM has now also begun to accept instructional materials at the lesson or unit level (called modules). A LAM that uses IBL methods would be a particularly good fit here, under the "modules" category. http://www.jiblm.org/

- ***The MathAMATYC Educator***. The MathAMATYC Educator is a refereed journal from the American Mathematical Association of Two-Year Colleges (AMATYC). It provides resources and research related to topics and issues pertinent to teaching mathematics in two-year colleges, as well as teaching mathematics in the first two years of college in 4 year institutions. The MathAMATYC Educator welcomes LAMs as an accetable genre for publication as long as the focus fits the content of the first two years of college. https://amatyc.org/page/MathAMATYCEducator/

- ***Mathematics Teacher: Learning and Teaching PK-12*** (MTLT). MTLT is a monthly publication by NCTM focused on K-14 education (although the journal title does state that 12th grade is the highest level, NCTM does explicitly state that their focus includes undergraduate mathematics that could be taught in high school). Thus, if the topic and level of the Lesson Analysis is one that could also be taught in a high school classroom (for example, calculus), then this is another option. The word limit for an article focused on a particular grade band is 3500 words, which may be too short for the entire LAM, but there are options to include supplemental material online as well. In this way, several portions of a LAM could be offloaded to the supplemental material. https://www.nctm.org/publications/mathematics-teacher/

If guidance on how to structure a LAM would be helpful, we suggest finding an existing LAM that is close to the structure that might work for your lesson and use it as a model. Finally, we are hoping to curate a list of LAMs that get published in various places, including in this book, in future books, or in any of these journals (or elsewhere). If you submit a LAM to any of these venues, or others, please let us know. We will be working to create a library database that lists the locations of published LAMs. You can let us know the information of your submitted or published LAM by emailing Doug Corey: corey@mathed.byu.edu.

15.5 Future Work: Possible Models for Carrying Out Lesson Analysis

This book has focused on developing a *written* product that provides an avenue for reflective college mathematics instructors to share their knowledge. As we stated earlier, in our planned second book, we will build on the outcomes presented in this book, in terms of different ways one might enact LA in order to produce a strong LAM. In anticipation of that work, we believe it may be beneficial to end this book by briefly explaining four different possible models that might serve as a basis for doing LA. The list is certainly not exhaustive, but one or two may prove to be a good fit for your situation, if you are planning to write and publish a LAM.

1. One possible model is that a mathematics instructor might refine a lesson over time to meet a particular instructional challenge. By having taught the lesson multiple times, the instructor is now attuned to the student thinking in the lesson and can use that information to understand which features are the key features of the lesson. They can also use that information to describe in detail the choices made in refining that lesson and how the lesson actually plays out.

2. A second possible model involves a mathematics education researcher learning important ideas from their research that could be used to overcome one or more instructional challenges. In this model, the researcher tests out the ideas from their research in their own class (or a colleague's class). The researcher can draw on the research to justify instructional decisions in their LAM. It may even be that the enactment of the lesson could raise another important instructional challenge that may not have been noticed previously and the researcher could then turn to crafting a lesson or lesson sequence to overcome the new instructional challenge.

3. A third possible model consists of a group of instructors that work together to design and understand a lesson to address a particular instructional challenge, perhaps addressing a topic that has been difficult to help students understand in the past. This model is becoming more popular because of the success of this kind of professional development in Japan and is called Japanese Lesson Study ([5],[8]). Instructors using this model may do background research on student thinking related to the mathematical goal. They can test out different ideas in their various classrooms and gather information on student mathematical thinking to help their analysis. Thus, the written LAM might not deal with a single teaching of the lesson but might draw on the overall experience to explain the decisions made in the lesson, common student thinking, and how classroom discussions are facilitated.

4. An important fourth model involves an instructor (or set of instructors) studying an existing LAM (like the ones in the book) in order to adapt it to their own classroom. In doing so, they can test the LAM out in an attempt to address the same instructional challenge, but now in their own particular context. The instructor could carefully think about what adaptations they made for their students, whether the lesson in the LAM ended up functioning differently in their environment, and whether they were able to include changes that made the lesson even more successful than reported in the original LAM. The instructor could then write their own LAM detailing these elements. Doing so is a fundamentally important aspect of building a knowledge base for teaching undergraduate mathematics.

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