A two-dimensional sign pattern is an infinite array \((b_{m,n} : m, n = 0, 1, 2, \ldots)\) of 1's and -1's. Suppose \(B\) is an open subset of \(\mathbb{R}^2\), \(g : B \rightarrow \mathbb{R}\) is infinitely differentiable, and its partial derivatives are nonzero and never change sign on \(B\). We say that \((b_{m,n})\) is the derivative sign pattern of \(g\) on \(B\) if, for all \(m, n\),
\[
b_{m,n} = \text{sgn} \left( \frac{\partial^m + n g}{\partial x^m \partial y^n} \right).
\]

A one-dimensional sign pattern \((c_n : n = 0, 1, 2, \ldots)\) is CTA (constant, then alternating) if there exists \(n\) such that \(c_0 = c_1 = \cdots = c_n = -c_{n+1} = c_{n+2} = -c_{n+3} = \cdots\). We call \(n\) the CTA type of \((c_n)\).

Theorem: A sign pattern is a DSP on \(\mathbb{R}^+ \times (0, 1)\) if, and only if, every column is CTA, and whenever a column is of CTA type \(c\), subsequent columns are of CTA type at most \(c+1\).

That the condition is necessary is proved in [1]. Conversely, suppose the sign pattern \((b_{m,n})\) satisfies the given condition. We construct a function \(g\) whose derivative sign pattern on \(\mathbb{R}^+ \times (0, 1)\) is \((b_{m,n})\).

Let \(c_n\) be the CTA type of column \(n\). Partition \((c_n)\) into a finite number of blocks of consecutive terms so that the first block consists of all terms whose value is \(\infty\) (if any), the terms in each subsequent block take on at most two consecutive values, and the block values are decreasing, in that if a block takes the values \(t, t+1\) and a subsequent block takes the values \(u, u+1\), then \(t > u\). For \(i = 0, 1, \ldots\), let \(t_i\) be the smaller of the values taken on the block to which \(c_i\) belongs.

Let \(T\) be the largest finite CTA type of a column. Define the functions \(a_k : \mathbb{R}^+ \rightarrow \mathbb{R}\) by
\[
a_k(x) = \begin{cases} 
b_{0,k}e^{Tx} & \text{if } t_k = \infty \\
b_{0,k}(1 + x)t_k + \frac{1}{3}(-1)^{t_k+1}b_{t_k+1,k}e^{-x} & \text{if } t_k < \infty
\end{cases}
\]

Then the DSP of
\[
g(x, y) = \sum_{k=0}^{\infty} \frac{a_k(x)}{3^k k!} y^k
\]
on \(\mathbb{R}^+ \times (0, 1)\) is \((b_{m,n})\).

To prove this, suppose that \(t_k = \infty\) for \(k \leq K\), and \(t_k < \infty\) for \(k > K\). Fix \(m, n\); we must show that for \(x > 0\) and \(0 < y < 1\), \(g_{x=y^n}(x, y) = b_{m,n}\).

We consider 3 cases: \(m \leq K\), \(K < m \leq t_n\), and \(m > t_n\).

If \(m \leq K\) (so \(t_m = \infty\)), then
\[ g_{x^m y^n}(x, y) = \frac{b_{0,n} T^m e^{T x}}{3^n} + \sum_{k=n+1}^{\infty} \frac{a_k^{(m)}(x)}{3^k(k-n)!} y^{k-n} \quad (1) \]

The sign of the first term is \( b_{0,n} \), which is the same as \( b_{m,n} \), which will be the sign of \( g_{x^m y^n}(x, y) \) provided this first term has greater magnitude than the sum of all of the other terms. This sum is bounded above by

\[
\sum_{k=n+1}^{K} \frac{T^m e^{T x}}{3^k(k-n)!} + \sum_{k=K+1}^{\infty} \left( t_k(t_k - 1) \cdots (t_k - m + 1)(1 + x)^{t_k-m} + \frac{1}{3} \right) \frac{y^{k-n}}{3^k(k-n)!} 
\leq \left( T^m e^{T x} + \frac{1}{3} \right) \sum_{k=n+1}^{\infty} \frac{1}{3^k(k-n)!} 
\leq \left( T^m e^{T x} + \frac{1}{3} \right) \cdot \frac{e^{1/3} - 1}{3^n} 
\leq T^m e^{T x} \cdot \frac{4}{3} \frac{e^{1/3} - 1}{3^n} \leq \frac{T^m e^{T x}}{3^n} 
\]

which is the magnitude of the first term in (1). (We have used the facts that \( t_k \leq T \), \( (1 + x)^T \leq e^{T x} \), \( 0 < y < 1 \), and \( e^{-x} < 1 \).)

In the last two cases, when \( t_m \) is finite, we have

\[ g_{x^m y^n}(x, y) = \frac{b_{0,n} t_n(t_n - 1) \cdots (t_n - m + 1)(1 + x)^{t_n-m}}{3^n} + \frac{(-1)^{t_n+1+m} b_{t_n+1,n} e^{-x}}{3^{n+1}} 
+ \sum_{k=n+1}^{\infty} \left( b_{0,k} t_k(t_k - 1) \cdots (t_k - m + 1)(1 + x)^{t_k-m} + \frac{1}{3}(-1)^{t_k+1+m} b_{t_k+1,k} e^{-x} \right) \frac{y^{k-n}}{3^k(k-n)!} \quad (2) \]

Now we consider the second case: \( t_n \) is finite, but \( m \leq t_n \). The sign of the first term in (2) is \( b_{0,n} \), which is the same as \( b_{m,n} \), which will be the sign of \( g_{x^m y^n}(x, y) \) provided this first term has greater magnitude than the sum of all of the other terms. This sum is bounded above by

\[
\frac{1}{3^{n+1}} + \sum_{k=n+1}^{\infty} \left( t_k(t_k - 1) \cdots (t_k - m + 1)(1 + x)^{t_k-m} + \frac{1}{3} \right) \frac{1}{3^k(k-n)!} 
\]
\[ \leq \frac{1}{3^{n+1}} + (t_n(t_n - 1) \cdots (t_n - m + 1)(1 + x)^{t_n-m} + \frac{1}{3}) \sum_{k=n+1}^{\infty} \frac{1}{3^k(k-n)!} \]

\[ = \frac{1}{3^{n+1}} + (t_n(t_n - 1) \cdots (t_n - m + 1)(1 + x)^{t_n-m} + \frac{1}{3}) \cdot \frac{e^{1/3} - 1}{3^n} \]

\[ \leq (t_n(t_n - 1) \cdots (t_n - m + 1)(1 + x)^{t_n-m}/3^n \cdot (\frac{4e^{1/3}}{3} - 1) \]

which is less than the magnitude of the first term in (2).

The last case is where \( t_n < m \). In this case, all of the polynomial terms in (2) vanish, leaving \((-1)^{t_n+1+m}b_{t_n+1}e^{-x}/3^{n+1}\) as the first non-zero term. Since column \( n \) in pattern \((b_{m,n})\) alternates after entry \( b_{t_n,n} \), \( b_{m,n} = (-1)^{t_n+1+m}b_{t_n,n} \), and so we must again show that 
\((-1)^{t_n+1+m}b_{t_n+1}e^{-x}/3^{n+1}\) has greater magnitude than the sum of all the terms that follow. In fact,

\[ \sum_{k=n+1}^{\infty} \frac{e^{-x}}{3^{k+1}(k-n)!} y^{k-n} \leq \sum_{k=n+1}^{\infty} \frac{e^{-x}}{3^{k+1}(k-n)!} \]

\[ = \frac{e^{-x}(e^{1/3} - 1)}{3^{n+1}} \]

\[ < \frac{e^{-x}}{3^{n+1}} \]

as required. We are done.