Addendum to *Derivative Sign Patterns in Two Dimensions*

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A *two-dimensional sign pattern* is an infinite array $(b_{m,n} : m, n = 0, 1, 2, \ldots)$ of 1's and -1's. Suppose B is an open subset of $\mathbf{R}^2$, $g : B \to \mathbf{R}$ is infinitely differentiable, and its partial derivatives are nonzero and never change sign on B . We say that $(b_{m,n})$ is the *derivative sign pattern* of g on B if, for all m, n, $b_{m,n} = \operatorname{sgn}\left(\frac{\partial^{m+n} g}{\partial x^m \partial y^n}\right)$.

A one-dimensional sign pattern $(c_n : n = 0, 1, 2, \ldots)$ is CTA (constant, then alternating) if there exists n such that $c_0 = c_1 = \cdots = c_n = -c_{n+1} = c_{n+2} = -c_{n+3} = \cdots$. We call n the CTA type of (c_n).

Theorem: A sign pattern is a DSP on $\mathbf{R}^+ \times (0, 1)$ if, and only if, every column is CTA, and whenever a column is of CTA type c, subsequent columns are of CTA type at most $c+1$.

That the condition is necessary is proved in [1]. Conversely, suppose the sign pattern $(b_{m,n})$ satisfies the given condition. We construct a function g whose derivative sign pattern on $\mathbf{R}^+ \times (0, 1)$ is $(b_{m,n})$.

Let c_n be the CTA type of column n. Partition (c_n) into a finite number of blocks of consecutive terms so that the first block consists of all terms whose value is ∞ (if any), the terms in each subsequent block take on at most two consecutive values, and the block values are decreasing, in that if a block takes the values $t, t+1$ and a subsequent block takes the values $u, u+1$, then $t > u$. For $i = 0, 1, \ldots$, let t_i be the smaller of the values taken on the block to which c_i belongs.

Let T be the largest finite CTA type of a column. Define the functions $a_k : \mathbf{R}^+ \to \mathbf{R}$ by

$$a_k(x) = \begin{cases} b_{0,k} e^{Tx} & \text{if } t_k = \infty \\ b_{0,k}(1+x)^{t_k} + \frac{1}{3}(-1)^{t_k+1} b_{t_k+1,k} e^{-x} & \text{if } t_k < \infty \end{cases}$$

Then the DSP of

$$g(x, y) = \sum_{k=0}^{\infty} \frac{a_k(x)}{3^k k!} y^k$$

on $\mathbf{R}^+ \times (0, 1)$ is $(b_{m,n})$.

To prove this, suppose that $t_k = \infty$ for $k \le K$, and $t_k < \infty$ for $k > K$. Fix m, n; we must show that for $x > 0$ and $0 < y < 1$, $g_{x^m y^n}(x, y) = b_{m,n}$.

We consider 3 cases: $m \le K$, $K < m \le t_n$, and $m > t_n$.

If $m \le K$ (so $t_m = \infty$), then

$$g_{x^m y^n}(x,y) = \frac{b_{0,n} T^m e^{Tx}}{3^n} + \sum_{k=n+1}^{\infty} \frac{a_k^{(m)}(x)}{3^k (k-n)!} y^{k-n} \qquad (1)$$

The sign of the first term is $b_{0,n}$, which is the same as $b_{m,n}$, which will be the sign of $g_{x^m y^n}(x,y)$ provided this first term has greater magnitude than the sum of all of the other terms. This sum is bounded above by

$$\sum_{k=n+1}^{K} \frac{T^m e^{Tx}}{3^k (k-n)!} + \sum_{k=K+1}^{\infty} \left(t_k(t_k-1)\cdots(t_k-m+1)(1+x)^{t_k-m} + \frac{1}{3} \right) \frac{y^{k-n}}{3^k (k-n)!}$$

$$\begin{aligned} &\leq \left(T^m e^{Tx} + \frac{1}{3} \right) \sum_{k=n+1}^{\infty} \frac{1}{3^k (k-n)!} \\ &\leq \left(T^m e^{Tx} + \frac{1}{3} \right) \cdot \frac{e^{1/3}-1}{3^n} \\ &\leq T^m e^{Tx} \cdot \frac{4}{3} \cdot \frac{e^{1/3}-1}{3^n} < \frac{T^m e^{Tx}}{3^n} \end{aligned}$$

which is the magnitude of the first term in (1). (We have used the facts that $t_k \leq T$, $(1+x)^T \leq e^{Tx}$, $0 < y < 1$, and $e^{-x} < 1$.)

In the last two cases, when t_m is finite, we have

$$g_{x^m y^n}(x,y) = \frac{b_{0,n} t_n (t_n-1)\cdots(t_n-m+1)(1+x)^{t_n-m}}{3^n} + \frac{(-1)^{t_n+1+m} b_{t_n+1,n} e^{-x}}{3^{n+1}}$$

$$+ \sum_{k=n+1}^{\infty} \left(b_{0,k} t_k (t_k-1)\cdots(t_k-m+1)(1+x)^{t_k-m} + \frac{1}{3}(-1)^{t_k+1+m} b_{t_k+1,k} e^{-x} \right) \frac{y^{k-n}}{3^k (k-n)!} \qquad (2)$$

Now we consider the second case: t_n is finite, but $m \leq t_n$. The sign of the first term in (2) is $b_{0,n}$, which is the same as $b_{m,n}$, which will be the sign of $g_{x^m y^n}(x,y)$ provided this first term has greater magnitude than the sum of all of the other terms. This sum is bounded above by

$$\frac{1}{3^{n+1}} + \sum_{k=n+1}^{\infty} \left(t_k(t_k-1)\cdots(t_k-m+1)(1+x)^{t_k-m} + \frac{1}{3} \right) \frac{1}{3^k (k-n)!}$$

$$
\begin{aligned}
&\leq \frac{1}{3^{n+1}} + (t_n(t_n-1)\cdots(t_n-m+1)(1+x)^{t_n-m} + \frac{1}{3}) \sum_{k=n+1}^{\infty} \frac{1}{3^k(k-n)!} \\
&= \frac{1}{3^{n+1}} + (t_n(t_n-1)\cdots(t_n-m+1)(1+x)^{t_n-m} + \frac{1}{3}) \cdot \frac{e^{1/3}-1}{3^n} \\
&\leq \frac{(t_n(t_n-1)\cdots(t_n-m+1)(1+x)^{t_n-m}}{3^n} \cdot \left(\frac{4e^{1/3}}{3} - 1\right)
\end{aligned}
$$

which is less than the magnitude of the first term in (2).

The last case is where $t_n < m$. In this case, all of the polynomial terms in (2) vanish, leaving $(-1)^{t_n+1+m} b_{t_n+1} e^{-x}/3^{n+1}$ as the first non-zero term. Since column n in pattern $(b_{m,n})$ alternates after entry $b_{t_n,n}$, $b_{m,n} = (-1)^{t_n+1+m} b_{t_n,n}$, and so we must again show that $(-1)^{t_n+1+m} b_{t_n+1} e^{-x}/3^{n+1}$ has greater magnitude than the sum of all the terms that follow. In fact,

$$
\begin{aligned}
\sum_{k=n+1}^{\infty} \frac{e^{-x}}{3^{k+1}(k-n)!} y^{k-n} &\leq \sum_{k=n+1}^{\infty} \frac{e^{-x}}{3^{k+1}(k-n)!} \\
&= \frac{e^{-x}(e^{1/3}-1)}{3^{n+1}} \\
&< \frac{e^{-x}}{3^{n+1}}
\end{aligned}
$$

as required. We are done.

[1] K. Schilling, Derivative sign patterns in two dimensions, to appear in *College Math. J.*, March 2012.