**Proof of Fact 6.** $f(y)$ has a unique maximum for $0 \leq y \leq 1$, since $f(0) = f(1) = 0$ and $f''(y) < 0$ on this interval. By showing $f'(1/2) > 0$, it will follow that this maximum occurs for $1/2 < y < 1$.

We’ve already noted (implicitly) the dependence of $\alpha$ on $y$, but let’s set $y = 1/2$ in (B) to get

$$\sin(\alpha + \theta) = (\sin \theta) \left(1 + \frac{1}{4r}\right),$$

and continue to write $\alpha$ for the specific value of $\alpha$ so obtained (which still depends on the fixed values of $\theta$ and $r$). Using the fact that $\sec^2 \alpha > \sec \alpha$, our formula for $f'(y)$ from a previous napkin gives

$$f'(1/2) > -\tan \alpha + \frac{1}{4r} \sec \alpha \sin \theta \sec(\alpha + \theta)$$

$$= \sec \alpha (-\sin \alpha + (\sin(\alpha + \theta) - \sin \theta) \sec(\alpha + \theta))$$

$$= \sec \alpha \sec(\alpha + \theta) \cdot L,$$

where $L = \sin(\alpha + \theta) - \sin \alpha \cos(\alpha + \theta) - \sin \theta$. We’re done if $L > 0$.

We note that $L = 0$ for $\alpha = 0$, so we’ll be done if $\frac{\partial L}{\partial \alpha} > 0$. And indeed,

$$\frac{\partial L}{\partial \alpha} = \cos(\alpha + \theta) - \cos \alpha \cos(\alpha + \theta) + \sin \alpha \sin(\alpha + \theta)$$

$$> \cos(\alpha + \theta) - \cos(\alpha + \theta) + \sin \alpha \sin(\alpha + \theta)$$

$$= \sin \alpha \sin(\alpha + \theta)$$

$$> 0.$$