Proof of Fact 7. We want to show that \( f'(0) < -f'(1) \). Letting \( a = 1/(2r) \), this translates into proving that 
\[
\tan^{-1}(a \tan \theta) + \theta < \sin^{-1}((a + 1) \sin \theta).
\]

Note that \( a > 1 \) and that \( (a + 1) \sin \theta > 1 \) (our condition of the maximum angle). Both sides of the inequality in (C) are zero for \( \theta = 0 \), so we are finished if the inequality holds when differentiated. That is, we are done if we can show that
\[
1 + \frac{a \sec^2 \theta}{1 + a^2 \tan^2 \theta} < \frac{(a + 1) \cos \theta}{\sqrt{1 - (a + 1)^2 \sin^2 \theta}}.
\]

Squaring both sides, cross-multiplying, then gathering everything to the right (brute-force here; I won’t say if I had any electronic assistance), our inequality in (D) is true if
\[
a^2 \sin^2 \theta(3 - (a^2 + 2a + 3) \sin^2 \theta) > 0.
\]

Again using the fact that \( \sin \theta < \frac{1}{a+1} \), we have
\[
3 - (a^2 + 2a + 3) \sin^2 \theta > 3 - \frac{a^2 + 2a + 3}{(a + 1)^2} = \frac{2a(a + 2)}{(a + 1)^2},
\]
which is positive, so we’re done.