Proof of Fact 9. For each fixed $r \in (0, 1/2)$ and $y \in (0, 1)$, the maximum value of x is

$$x(y, \sin^{-1} \frac{2r}{1+2r}, r) = (1-y) \tan\left(\sin^{-1}\left(\frac{y+2r}{1+2r}\right) - \sin^{-1}\left(\frac{2r}{1+2r}\right)\right).$$

Letting $r \to 0$ in the above gives

$$(1-y)\frac{y}{\sqrt{1-y^2}},$$

a quantity which is zero when $y = 0$ and for $y \to 1$, and which is otherwise positive. The derivative of this quantity is

$$\frac{-(1-y)(y^2+y-1)}{(1-y^2)^{3/2}},$$

which has as its single zero in $(0, 1)$ the number we desire.

Our work here is done. Shoot.