**Proof of Fact 9.** For each fixed \( r \in (0, 1/2) \) and \( y \in (0, 1) \), the maximum value of \( x \) is

\[
x(y, \sin^{-1} \frac{2r}{1+2r}, r) = (1 - y) \tan \left( \sin^{-1} \left( \frac{y + 2r}{1+2r} \right) - \sin^{-1} \left( \frac{2r}{1+2r} \right) \right).
\]

Letting \( r \to 0 \) in the above gives

\[
(1 - y) \frac{y}{\sqrt{1 - y^2}},
\]

a quantity which is zero when \( y = 0 \) and for \( y \to 1 \), and which is otherwise positive. The derivative of this quantity is

\[
-(1 - y)(y^2 + y - 1)
\]

\[
(1 - y^2)^{3/2},
\]

which has as its single zero in \((0, 1)\) the number we desire.

Our work here is done. Shoot.