Proof of Fact 1. Looking at right triangle $BDE$, we have $DE = (BD) \tan \alpha$, where $\alpha = \angle DBE$, or

$$x = (1 - y) \tan \alpha.$$  \hfill (A)

From triangle $ACB$ and the law of sines, we have

$$\sin(\theta + \alpha) = \frac{y + 2r}{2r} \sin \theta,$$ \hfill (B)

so that $\alpha = \sin^{-1} \left(\left(1 + \frac{y}{2r}\right) \sin \theta - \theta\right)$. This gives

$$x = (1 - y) \tan \left(\sin^{-1} \left(\left(1 + \frac{y}{2r}\right) \sin \theta - \theta\right)\right).$$

Proof of Fact 2. For this we need just the usual approximations:

$$u \approx \sin u \approx \tan u \approx \sin^{-1} u, \text{ for } u \to 0.$$  

Proof of Fact 3. $\left(\frac{(1 - y)y}{2r} \theta\right)$ is quadratic in $y$ with its maximum at the average of its zeros, namely, $y = 1/2$. 