**Proof of Fact 1.** Looking at right triangle $BDE$, we have $DE = (BD) \tan \alpha$, where $\alpha = \angle DBE$, or

$$x = (1 - y) \tan \alpha.$$  

From triangle ACB and the law of sines, we have

$$\sin(\theta + \alpha) = \frac{y + 2r}{2r} \sin \theta,$$  

so that $\alpha = \sin^{-1} \left( \left( 1 + \frac{y}{2r} \right) \sin \theta - \theta \right)$. This gives

$$x = (1 - y) \tan \left( \sin^{-1} \left( \left( 1 + \frac{y}{2r} \right) \sin \theta - \theta \right) \right).$$

**Proof of Fact 2.** For this we need just the usual approximations:

$$u \approx \sin u \approx \tan u \approx \sin^{-1} u, \text{ for } u \to 0.$$  

**Proof of Fact 3.** $\frac{(1 - y)y}{2r} \theta$ is quadratic in $y$ with its maximum at the average of its zeros, namely, $y = 1/2$.  

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Proof of Fact 4. Let $\alpha_k$ denote the angle of deflection of the $k$’th ball after it is contacted by the $(k-1)$’st, where the cue ball is the 0’th ball. The following are easy to verify:

\[
\frac{x}{1-y_n} = \tan \alpha_n \\
\sin(\alpha_n + \alpha_{n-1}) = (y_n - y_{n-1})\frac{\sin \alpha_{n-1}}{2r} \\
\alpha_n + \alpha_{n-1} \approx \frac{y_n - y_{n-1}}{2r} \alpha_{n-1} \\
\vdots \\
\sin(\alpha_k + \alpha_{k-1}) = (y_k - y_{k-1})\frac{\sin \alpha_{k-1}}{2r} \\
\alpha_k + \alpha_{k-1} \approx \frac{y_k - y_{k-1}}{2r} \alpha_{k-1} \\
\vdots \\
\sin(\alpha_2 + \alpha_1) = (y_2 - y_1)\frac{\sin \alpha_1}{2r} \\
\alpha_2 + \alpha_1 \approx \frac{y_2 - y_1}{2r} \alpha_1 \\
\sin(\alpha_1 + \theta) = (y_1 + 2r)\frac{\sin \theta}{2r} \\
\alpha_1 + \theta \approx \frac{y_1 + 2r}{2r} \theta.
\]

Solving gives $x \approx (1 - y_n) \left( \frac{y_n - y_{n-1} - 2r}{2r} \right) \cdots \left( \frac{y_2 - y_1 - 2r}{2r} \right) \left( \frac{y_1}{2r} \right) \theta$. Letting $z_{n+1} = 1 - y_n$, $z_k = y_k - y_{k-1} - 2r$ (for $1 < k < n$) and $z_1 = y_1$, we want to maximize $z_1z_2 \cdots z_{n+1}$ subject to $z_1 + z_2 + \cdots + z_{n+1} = 1 - 2r$. This is standard (by Lagrange multipliers, say) resulting in

\[
z_1 = z_2 = \cdots = z_{n+1} = \frac{1 - 2r}{n + 1}.
\]
Proof of Fact 5. Holding $\theta$ and $r$ constant, let $f(y) = x(y, \theta, r)$ and differentiate (3) and (4) with respect to $y$ to get

$$f'(y) = -\tan \alpha + (1 - y) \sec^2 \alpha \frac{d\alpha}{dy}$$

and

$$\frac{d\alpha}{dy} = \frac{\sin \theta}{2r} \sec(\theta + \alpha).$$

Differentiating each once more gives

$$f''(y) = -2 \sec^2 \alpha \frac{d\alpha}{dy} + (1 - y) \sec^2 \alpha \left( 2 \tan \alpha \left( \frac{d\alpha}{dy} \right)^2 + \frac{d^2\alpha}{dy^2} \right)$$

and

$$\frac{d^2\alpha}{dy^2} = \frac{\sin \theta}{2r} \sec(\alpha + \theta) \tan(\alpha + \theta) \frac{d\alpha}{dy}.$$

Combining these gives

$$\frac{d^2x}{dy^2} = \frac{d\alpha}{dy} (\sec^2 \alpha)(-2 + J),$$

where

$$J = (1 - y) \frac{\sin \theta}{2r} \sec(\alpha + \theta)(2 \tan \alpha + \tan(\alpha + \theta)).$$
We'll show that $J < 3/2$, which will prove that $f''(y) < 0$.
Since $\tan \alpha < \tan(\alpha + \theta)$, we have

$$\sec(\alpha + \theta)(2 \tan \alpha + \tan(\alpha + \theta)) < 3 \sec(\alpha + \theta) \tan(\alpha + \theta)$$

$$= \frac{3 \sin(\alpha + \theta)}{1 - \sin^2(\alpha + \theta)}$$

$$= \frac{3(\sin \theta)(1 + \frac{y}{2r})}{1 - (\sin^2 \theta)(1 + \frac{y}{2r})^2}.$$  

This implies that

$$J < \frac{(1 - y)}{2r} \frac{3 \left( \frac{2r}{1 + 2r} \right)^2 \left( 1 + \frac{y}{2r} \right)}{1 - \left( \frac{2r}{1 + 2r} \right)^2 \left( 1 + \frac{y}{2r} \right)^2}$$

$$= \frac{3(2r + y)}{1 + 4r + y}$$

$$< \frac{3}{2}.$$  

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Proof of Fact 6. \( f(y) \) has a unique maximum for \( 0 \leq y \leq 1 \), since \( f(0) = f(1) = 0 \) and \( f''(y) < 0 \) on this interval. By showing \( f'(1/2) > 0 \), it will follow that this maximum occurs for \( 1/2 < y < 1 \).

We’ve already noted (implicitly) the dependence of \( \alpha \) on \( y \), but let’s set \( y = 1/2 \) in (4) to get

\[
\sin(\alpha + \theta) = (\sin \theta) \left( 1 + \frac{1}{4r} \right),
\]

and continue to write \( \alpha \) for the specific value of \( \alpha \) so obtained (which still depends on the fixed values of \( \theta \) and \( r \)). Using the fact that \( \sec^2 \alpha > \sec \alpha \), our formula for \( f'(y) \) from a previous napkin gives

\[
f'(1/2) > -\tan \alpha + \frac{1}{4r} \sec \alpha \sin \theta \sec(\alpha + \theta)
= \sec \alpha (- \sin \alpha + (\sin (\alpha + \theta) - \sin \theta) \sec (\alpha + \theta))
= \sec \alpha \sec(\alpha + \theta) \cdot L,
\]

where \( L = \sin(\alpha + \theta) - \sin \alpha \cos(\alpha + \theta) - \sin \theta \). We’re done if \( L > 0 \).

We note that \( L = 0 \) for \( \alpha = 0 \), so we’ll be done if \( \frac{\partial L}{\partial \alpha} > 0 \). And indeed,

\[
\frac{\partial L}{\partial \alpha} = \cos(\alpha + \theta) - \cos \alpha \cos(\alpha + \theta) + \sin \alpha \sin(\alpha + \theta)
> \cos(\alpha + \theta) - \cos(\alpha + \theta) + \sin \alpha \sin(\alpha + \theta)
= \sin \alpha \sin(\alpha + \theta)
> 0.
\]
Proof of Fact 7. We want to show that \( f'(0) < -f'(1) \). Letting \( a = 1/(2r) \), this translates into proving that \( a \tan \theta < \tan((\sin^{-1}((a+1)\sin \theta) - \theta) \), or

\[
\tan^{-1}(a \tan \theta) + \theta < \sin^{-1}((a+1)\sin \theta). \tag{5}
\]

Note that \( a > 1 \) and that \( (a+1)\sin \theta > 1 \) (our condition of the maximum angle). Both sides of the inequality in (5) are zero for \( \theta = 0 \), so we are finished if the inequality holds when differentiated. That is, we are done if we can show that

\[
1 + \frac{a \sec^2 \theta}{1 + a^2 \tan^2 \theta} < \frac{(a + 1) \cos \theta}{\sqrt{1 - (a + 1)^2 \sin^2 \theta}}. \tag{6}
\]

Squaring both sides, cross-multiplying, then gathering everything to the right (brute-force here; I won’t say if I had any electronic assistance), our inequality in (6) is true if

\[
a^2 \sin^2 \theta (3 - (a^2 + 2a + 3) \sin^2 \theta) > 0.
\]

Again using the fact that \( \sin \theta < \frac{1}{a+1} \), we have

\[
3 - (a^2 + 2a + 3) \sin^2 \theta > 3 - \frac{a^2 + 2a + 3}{(a+1)^2} = \frac{2a(a+2)}{(a+1)^2},
\]

which is positive, so we’re done.
Proof of Fact 8. Since $f(1-y) > f(y) \iff \frac{f(1-y)}{y(1-y)} > \frac{f(y)}{y(1-y)}$, by defining $g(y) = \frac{f(y)}{y(1-y)}$, it suffices then to show that $g$ is increasing on $[0, 1/2]$. Using $\sin(\theta + \alpha) = (1 + \frac{y}{2r}) \sin \theta$, or $\sin(\theta + \alpha) - \sin \theta = \frac{y}{2r} \sin \theta$, and $\sin \theta \leq \frac{2r}{y+2r}$, we have,

$$g'(y) = \frac{f'(y)}{y(1-y)} + \frac{f(y)(2y-1)}{y^2(1-y)^2}$$

$$= \frac{-y \tan \alpha + y(1-y)(\sec^2 \alpha) \sin \theta \sec(\theta + \alpha) + (2y-1) \tan \alpha}{y^2(1-y)}$$

$$= \frac{y \sec^2 \alpha \sin \theta \sec(\theta + \alpha) - \tan \alpha}{y^2}$$

$$= \frac{\sec^2 \alpha (\sin(\theta + \alpha) - \sin \theta) \sec(\theta + \alpha) - \tan \alpha}{y^2}$$

$$> \frac{(\sin(\theta + \alpha) - \sin \theta) \sec(\theta + \alpha) - \tan \alpha}{y^2}$$

$$= \frac{\sec \alpha \sec(\theta + \alpha)}{y^2} ((\sin(\theta + \alpha) - \sin \theta) \cos \alpha - \cos(\theta + \alpha) \sin \alpha)$$

$$= \frac{\sec \alpha \sec(\theta + \alpha)}{y^2} \sin \theta (1 - \cos \alpha)$$

$$> 0.$$
Proof of Fact 9. For each fixed $r \in (0, 1/2)$ and $y \in (0, 1)$, the maximum value of $x$ is

$$x(y, \sin^{-1} \frac{2r}{1 + 2r}, r) = (1 - y) \tan \left( \sin^{-1} \left( \frac{y + 2r}{1 + 2r} \right) - \sin^{-1} \left( \frac{2r}{1 + 2r} \right) \right).$$

Letting $r \to 0$ in the above gives

$$\frac{1 - y}{\sqrt{1 - y^2}},$$

a quantity which is zero when $y = 0$ and for $y \to 1$, and which is otherwise positive. The derivative of this quantity is

$$-(1 - y)(y^2 + y - 1) \quad \frac{1}{(1 - y^2)^{3/2}},$$

which has as its single zero in $(0, 1)$ the number we desire.

Our work here is done. Shoot.