The Rascal Triangle

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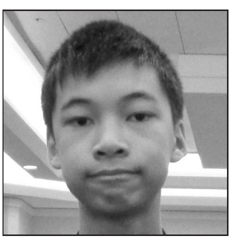
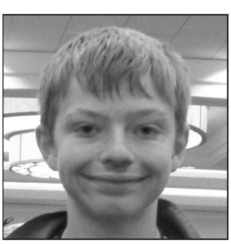
Alif Anggoro is a seventh grader at Al Azhar Junior High School in Java Bekasi, a suburb of Jakarta, Indonesia—the Muslim country with the largest population. He is fairly good at sports, mainly badminton and soccer (called football in Asia). "I like solving puzzles and mathematics problems," Alif writes, "and have participated in a number of mathematics competitions. I have not yet met Angus or Eddy. We work together through email." Just last year Alif won a Bronze medal at the International Mathematics Contest in Singapore.



Eddy Liu is an eighth grader at Washington Middle School in Seattle. He takes chess lessons and has won some prize money in small tournaments. He is also interested in history and plays Diplomacy online. He has met Angus at a Math Camp. "Although I live in a big city," Eddy writes, "I am just as isolated as Angus because both my parents work, my only sister is six years older, and there are very few boys my age in the neighborhood."



Angus Tulloch is an eighth grader at Crestomere School in Rimbey, Alberta—a tiny town no-one's ever heard of in the middle of nowhere, or so he claims. He was home schooled for grades one to seven. He is driven to Math Club on Saturdays in Edmonton, at least an hour away. "Don't get me wrong," Angus writes, "I'm proud of my neighborhood. The big advantage of home schooling and isolation is that there isn't much social interaction. That leaves intellectual stuff only. I also have to thank my parents for their steadfast refusal to buy a gaming system."

An I.Q. test question asked for the next row of numbers in the following triangular array:

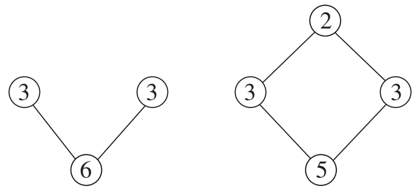
$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
? & & ? & & ? & & ? & & ?
\end{array}
$$

Our answer was "1 4 5 4 1."

"WRONG!" said our teacher. "The answer is 1 4 6 4 1, part of the well-known Pascal triangle."

"The question does not ask for the next row in the Pascal triangle," we complained. "Without telling us what the triangle is, any five numbers should be acceptable."

"Perhaps it was not stated clearly," our teacher conceded, "but the purpose of the question is to find a way of filling in the new row according to some simple rule. For the Pascal triangle, the rule is that the outside numbers on each row are 1s and the inside numbers are determined by the *inverted triangle formula*: $South = West + East$. For example, $6 = 3 + 3$ (see Figure 1a)."



(a) (b)

Figure 1.

"Our rule is that the outside numbers on each row are 1s and the inside numbers are determined by the *diamond formula:* $South = (West \times East + 1) \div North$. For example, $5 = (3 \times 3 + 1) \div 2$ (see Figure 1b)."

"That is not as simple as the inverted triangle formula," our teacher complained. "Moreover, the diamond formula involves division. How do you even know that all numbers in your triangle are integers?"

"We have worked out the next few rows." We produced a piece of paper. "See!"

1

1 1

1 2 1

1 3 3 1

1 4 5 4 1

1 5 7 7 5 1

1 6 9 10 9 6 1

1 7 11 13 13 11 7 1

"That may be, but you do not know for sure. The first non-integer may just be round the corner. In the Pascal triangle, only addition is involved, and we can be sure that all numbers are integers."

We went away, still believing we were right. However, before showing that our triangle was simpler than the Pascal triangle, we had to show that all numbers in our triangle were indeed integers. How could even that be accomplished?

Since our triangle is not quite the Pascal triangle, we call it the *rascal triangle*. We conceive of the two triangles as finished products rather than as expanding structures. We notice that the two triangles are the same for the first two diagonals running from northwest to southeast. The zeroth diagonal consists of all 1s while the first diagonal consists of the positive integers.

The second diagonal in the Pascal triangle is $1, 3, 6, 10, 15, \ldots$. Each term after the first is obtained from the preceding one by adding successively larger integers, namely, $1 + 2 = 3$, $3 + 3 = 6$, $6 + 4 = 10$, $10 + 5 = 15, \ldots$. The second diagonal in the rascal triangle is $1, 3, 5, 7, 9, \ldots,$ which consists of just the odd numbers. Surely, we have a simpler pattern, namely, $1 + 2 = 3, 3 + 2 = 5, 5 + 2 = 7, 7 + 2 = 9, \ldots.$

The third diagonal in the Pascal triangle is $1, 4, 10, 20, 35, \ldots,$ and it is already not quite easy for us to see a pattern. The third diagonal in the rascal triangle is $1, 4, 7, 10, 13, \ldots.$ Each term is obtained from the preceding one by simply adding 3. In fact, the mth diagonal in the rascal triangle starts with 1 as its 0th term, and each subsequent term is obtained from the preceding one by simply adding m. Hence the nth term on this diagonal is $mn + 1$.

If this pattern continues, all numbers in the rascal triangle will be integers. To show this, consider the triangle in which this pattern does continue. All we have to do is show that it is the same as the rascal triangle, in other words, the diamond formula holds. So suppose North is the nth term on the mth diagonal. Then West is the nth term on the $(m + 1)$st diagonal, East is the $(n + 1)$st term on the mth diagonal, and South is the $(n + 1)$st term on the $(m + 1)$st diagonal. The calculation below shows that the diamond formula holds, so that all numbers in the rascal triangle are indeed integers!

$$
\begin{aligned}
\frac{\textit{West} \times \textit{East} + 1}{\textit{North}} &= \frac{(m(n+1)+1)((m+1)n)+1)+1}{mn+1} \\
&= \frac{(mn+m+1)(mn+n+1)+1}{mn+1} \\
&= \frac{m^2n^2+m^2n+mn^2+3mn+m+n+2}{mn+1} \\
&= \frac{mn(mn+1)+m(mn+1)+n(mn+1)+2(mn+1)}{mn+1} \\
&= mn+m+n+2 \\
&= (m+1)(n+1)+1 \\
&= \textit{South}.
\end{aligned}
$$

One of us looks up the formula for the kth number on the rth row of the Pascal triangle. It is

$$
\frac{r!}{k!\,(r-k)!} = \frac{r(r-1)(r-2)\cdots 3\cdot 2\cdot 1}{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1\cdot (r-k)(r-k-1)(r-k-2)\cdots 3\cdot 2\cdot 1},
$$

which has multiplications and divisions galore. In the rascal triangle, the kth number on the rth row is the kth number on the $(r - k)$th diagonal. Hence this number is $k(r - k) + 1$.

Which triangle is simpler now?

Note: The rascal triangle turns out to be sequence A077028 in the Online Encyclopedia of Integer Sequences. The diamond formula appears to be new.

Summary. A number triangle, discovered using a recurrence formula similar to that of Pascal's triangle, yields sequence A077028 from the Online Encyclopedia of Integer Sequences.