NEW MATHEMATICAL DIVERSIONS

REVISED EDITION

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MATHEMATICAL ASSOCIATION OF AMERICA

CHAPTER THREE

Eight Problems

1. ACUTE DISSECTION

GIVEN A TRIANGLE with one obtuse angle, is it possible to cut the triangle into smaller triangles, all of them acute? (An acute triangle is a triangle with three acute angles. A right angle is of course neither acute nor obtuse.) If this cannot be done, give a proof of impossibility. If it can be done, what is the smallest number of acute triangles into which any obtuse triangle can be dissected?

Figure 10 shows a typical attempt that leads nowhere. The triangle has been divided into three acute triangles, but the fourth is obtuse, so nothing has been gained by the preceding cuts.

The problem (which came to me by way of Mel Stover of Winnipeg) is amusing because even the best mathematician is likely to be led astray by it and come to a wrong conclusion. My pleasure

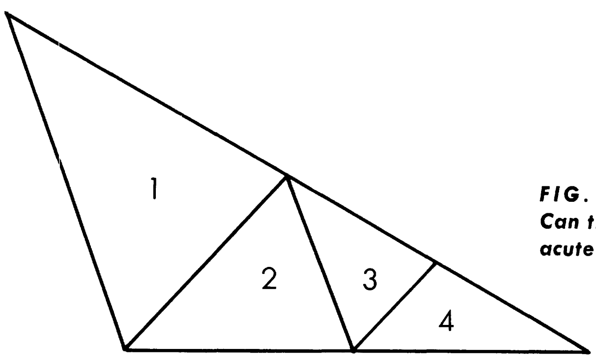

FIG. 10
Can this triangle be cut into acute ones?

in working on it led me to ask myself a related question: "What is the smallest number of acute triangles into which a *square* can be dissected?" For days I was convinced that nine was the answer; then suddenly I saw how to reduce it to eight. I wonder how many readers can discover an eight-triangle solution, or perhaps an even better one. I am unable to prove that eight is the minimum, though I strongly suspect that it is.

2. HOW LONG IS A "LUNAR"?

IN H. G. WELLS'S NOVEL *The First Men in the Moon* our natural satellite is found to be inhabited by intelligent insect creatures who live in caverns below the surface. These creatures, let us assume, have a unit of distance that we shall call a "lunar." It was adopted because the moon's surface area, if expressed in square lunars, exactly equals the moon's volume in cubic lunars. The moon's diameter is 2,160 miles. How many miles long is a lunar?

3. THE GAME OF GOOGOL

IN 1958 JOHN H. FOX, JR., of the Minneapolis-Honeywell Regulator Company, and L. Gerald Marnie, of the Massachusetts Institute of Technology, devised an unusual betting game which they call Googol. It is played as follows: Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of 1 to a number the size of a "googol" (1 followed by a hundred 0's) or even larger. These slips are turned face down and shuffled over the top of a table. One at a time you turn the slips face up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick a previously turned slip. If you turn over all the slips, then of course you must pick the last one turned.

Most people will suppose the odds against your finding the highest number to be at least five to one. Actually if you adopt the best strategy, your chances are a little better than one in three. Two questions arise. First, what is the best strategy? (Note that this is not the same as asking for a strategy that will maximize the *value* of the selected number.) Second, if you follow this strategy, how can you calculate your chances of winning?

When there are only two slips, your chance of winning is obviously 1/2, regardless of which slip you pick. As the slips increase in number, the probability of winning (assuming that you use the best strategy) decreases, but the curve flattens quickly, and there is very little change beyond ten slips. The probability never drops below 1/3. Many players will suppose that they can make the task more difficult by choosing very large numbers, but a little reflection will show that the sizes of the numbers are irrelevant. It is only necessary that the slips bear numbers that can be arranged in increasing order.

The game has many interesting applications. For example, a girl decides to marry before the end of the year. She estimates that she will meet ten men who can be persuaded to propose, but once she has rejected a proposal, the man will not try again. What strategy should she follow to maximize her chances of accepting the top man of the ten, and what is the probability that she will succeed?

The strategy consists of rejecting a certain number of slips of paper (or proposals), then picking the next number that exceeds the highest number among the rejected slips. What is needed is a formula for determining how many slips to reject, depending on the total number of slips.

4. MARCHING CADETS AND A TROTTING DOG

A SQUARE FORMATION of Army cadets, 50 feet on the side, is marching forward at a constant pace [*see Fig. 11*]. The company mascot, a small terrier, starts at the center of the rear rank [*position A in the illustration*], trots forward in a straight line to

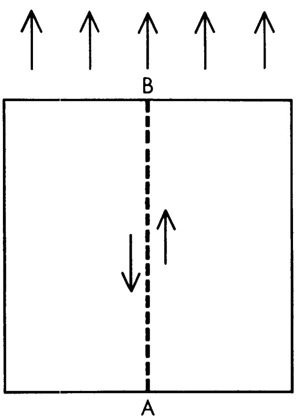

FIG. 11
How far does the dog trot?

the center of the front rank [*position B*], then trots back again in a straight line to the center of the rear. At the instant he returns to position A, the cadets have advanced exactly 50 feet. Assuming that the dog trots at a constant speed and loses no time in turning, how many feet does he travel?

If you solve this problem, which calls for no more than a knowledge of elementary algebra, you may wish to tackle a much more difficult version proposed by the famous puzzlist Sam Loyd. (See *Mathematical Puzzles of Sam Loyd,* Vol. 2, Dover paperback, 1960, page 103.) Instead of moving forward and back through the marching cadets, the mascot trots with constant speed around the *outside* of the square, keeping as close as possible to the square at all times. (For the problem we assume that he trots along the perimeter of the square.) As before, the formation has marched 50 feet by the time the dog returns to point A. How long is the dog's path?

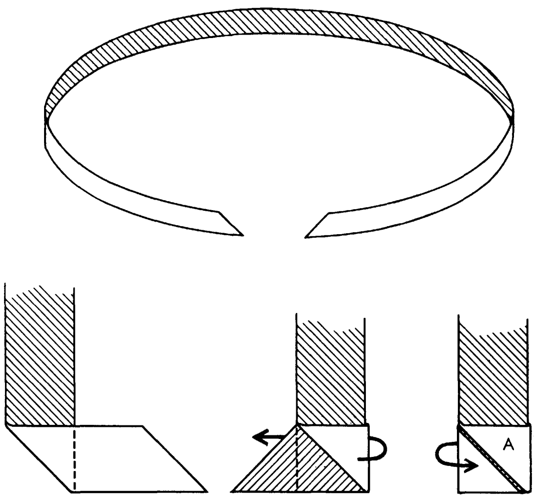

FIG. 12
***Barr's belt* (top) *and an unsatisfactory way to fold it* (bottom).**

5. BARR'S BELT

STEPHEN BARR of Woodstock, New York, says that his dressing gown has a long cloth belt, the ends of which are cut at 45-degree angles as shown in Figure 12. When he packs the belt for a trip, he likes to roll it up as neatly as possible, beginning at one end, but the slanting ends disturb his sense of symmetry. On the other hand, if he folds over an end to make it square off, then the uneven thicknesses of cloth put lumps in the roll. He experimented with more complicated folds, but try as he would, he could not achieve a rectangle of uniform thickness. For example, the fold shown in the illustration produces a rectangle with three thicknesses in section A and two in section B.

"Nothing is perfect," says one of the Philosophers in James Stephens' *The Crock of Gold.* "There are lumps in it." Nonetheless, Barr finally managed to fold his belt so that each end was straight across and part of a rectangle of uniform thickness throughout. The belt could then be folded into a neat roll, free of lumps. How did Barr fold his belt? A long strip of paper, properly cut at the ends, can be used for working on the problem.

6. WHITE, BLACK AND BROWN

PROFESSOR MERLE WHITE of the mathematics department, Professor Leslie Black of philosophy, and Jean Brown, a young stenographer who worked in the university's office of admissions, were lunching together.

"Isn't it remarkable," observed the lady, "that our last names are Black, Brown and White and that one of us has black hair, one brown hair and one white."

"It is indeed," replied the person with black hair, "and have you noticed that not one of us has hair that matches his or her name?"

"By golly, you're right!" exclaimed Professor White.

If the lady's hair isn't brown, what is the color of Professor Black's hair?

7. THE PLANE IN THE WIND

AN AIRPLANE FLIES in a straight line from airport A to airport B, then back in a straight line from B to A. It travels with a constant engine speed and there is no wind. Will its travel time for the same round trip be greater, less or the same if, throughout both flights, at the same engine speed, a constant wind blows from A to B?

8. WHAT PRICE PETS?

THE OWNER of a pet shop bought a certain number of hamsters and half that many pairs of parakeets. He paid $2 each for the hamsters and $1 for each parakeet. On every pet he placed a retail price that was an advance of 10 per cent over what he paid for it.

After all but seven of the creatures had been sold, the owner found that he had taken in for them an amount of money exactly equal to what he had originally paid for all of them. His potential profit, therefore, was represented by the combined retail value of the seven remaining animals. What was this value?

ANSWERS

1. A number of readers sent "proofs" that an obtuse triangle cannot be dissected into acute triangles, but of course it can. Figure 13 shows a seven-piece pattern that applies to any obtuse triangle.

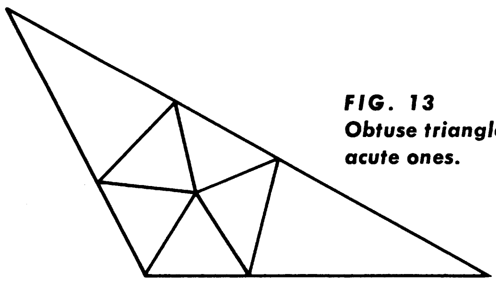

FIG. 13
Obtuse triangle cut into seven acute ones.

It is easy to see that seven is minimal. The obtuse angle must be divided by a line. This line cannot go all the way to the other side, for then it would form another obtuse triangle, which in turn would have to be dissected, consequently the pattern for the large triangle would not be minimal. The line dividing the obtuse angle must, therefore, terminate at a point *inside* the triangle. At this vertex, at least five lines must meet, otherwise the angles at this vertex would not all be acute. This creates the inner pentagon of five triangles, making a total of seven triangles in all. Wallace Manheimer, a Brooklyn high school teacher at the time, gave this proof as his solution to problem E1406 in *American Mathematical Monthly,* November 1960, page 923. He also showed how to construct the pattern for any obtuse triangle.

The question arises: Can any obtuse triangle be dissected into seven acute *isosceles* triangles? The answer is no. Verner E. Hoggatt, Jr., and Russ Denman (*American Mathematical Monthly,* November 1961, pages 912–913) proved that eight such triangles are sufficient for all obtuse triangles, and Free Jamison (*ibid.,* June-July 1962, pages 550–552) proved that eight are also necessary. These articles can be consulted for details as to conditions under which less than eight-piece patterns are possible. A right triangle and an acute nonisosceles triangle can each be cut into nine acute isosceles triangles, and an acute isosceles triangle can be cut into four congruent acute isosceles triangles similar to the original.

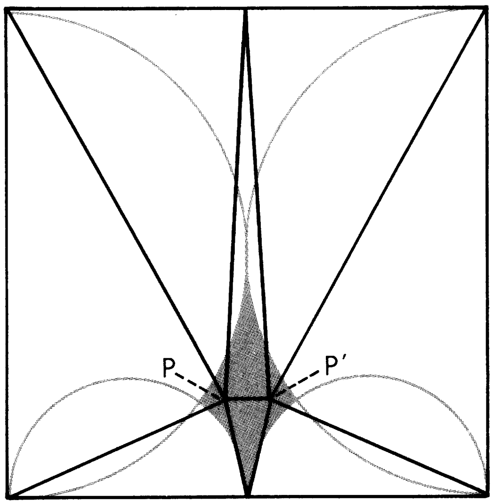

FIG. 14
Square cut into eight acute triangles.

A square can be cut into eight acute triangles as shown in Figure 14. If the dissection has bilateral symmetry, points P and P′ must lie within the shaded area determined by the four semicircles. Donald L. Vanderpool pointed out in a letter that asymmetric distortions of the pattern are possible with point P anywhere outside the shaded area provided it remains outside the two large semicircles.

About 25 readers sent proofs, with varying degrees of formality, that the eight-piece dissection is minimal. One, by Harry Lindgren, appeared in *Australian Mathematics Teacher,* Vol. 18, pages 14–15, 1962. His proof also shows that the pattern, aside from the shifting of points P and P′ as noted above, is unique.

H. S. M. Coxeter pointed out the surprising fact that for any rectangle, even though its sides differ in length by an arbitrarily small amount, the line segment PP′ can be shifted to the center to give the pattern both horizontal and vertical symmetry.

Free Jamison found in 1968 that a square can be divided into ten acute isosceles triangles. See *The Fibonacci Quarterly* (December 1968) for a proof that a square can be dissected into any number of acute isosceles triangles equal or greater than 10.

Figure 15 shows how the pentagram (regular five-pointed star) and the Greek cross can each be dissected into the smallest possible number of acute triangles.

2. The volume of a sphere is $4\pi/3$ times the cube of the radius. Its surface is 4π times the square of the radius. If we express the moon's radius in "lunars" and assume that its surface in square lunars equals its volume in cubic lunars, we can determine the length of the radius simply by equating the two formulas and solving for the value of the radius. Pi cancels out on both sides, and we find that the radius is three lunars. The moon's radius is 1,080 miles, so a lunar must be 360 miles.

3. Regardless of the number of slips involved in the game of Googol, the probability of picking the slip with the largest number (assuming that the best strategy is used) never drops below .367879. This is the reciprocal of *e*, and the limit of the probability of winning as the number of slips approaches infinity.

If there are ten slips (a convenient number to use in playing the game), the probability of picking the top number is .398. The strategy is to turn three slips, note the largest number among them, then pick the next slip that exceeds this number. In the long run you stand to win about two out of every five games.

What follows is a compressed account of a complete analysis of the game by Leo Moser and J. R. Pounder of the University of Alberta. Let n be the number of slips and p the number rejected before picking a number larger than any on the p slips. Number the slips serially from 1 to n. Let $k + 1$ be the number of the slip bearing the largest number. The top number will not be chosen unless k is equal to or greater than p (otherwise it will be rejected among the first p slips), and then only if the highest number from 1 to k is also the highest number from 1 to p (otherwise *this* number will be chosen before the top number is reached). The prob-

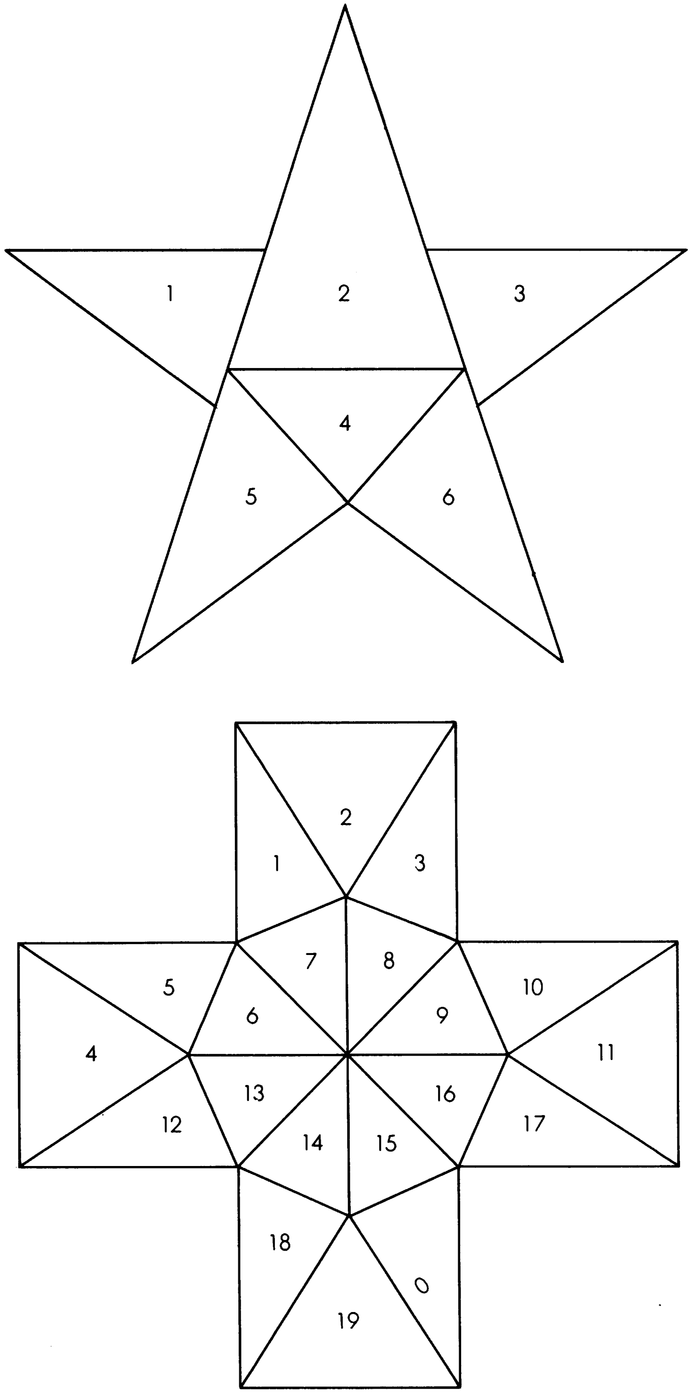

FIG. 15
Minimum dissections for the pentagram and Greek cross.

ability of finding the top number in case it is on the $k + 1$ slip is p/k, and the probability that the top number actually is on the $k + 1$ slip is $1/n$. Since the largest number can be on only one slip, we can write the following formula for the probability of finding it:

$$\frac{p}{n}\left(\frac{1}{p} + \frac{1}{p+1} + \frac{1}{p+2} \cdots + \frac{1}{n-1}\right)$$

Given a value for n (the number of slips) we can determine p (the number to reject) by picking a value for p that gives the greatest value to the above expression. As n approaches infinity, p/n approaches $1/e$, so a good estimate of p is simply the nearest integer to n/e. The general strategy, therefore, when the game is played with n slips, is to let n/e numbers go by, then pick the next number larger than the largest number on the n/e slips passed up.

This assumes, of course, that a player has no knowledge of the range of the numbers on the slips and therefore no basis for knowing whether a single number is high or low within the range. If one *has* such knowledge, the analysis does not apply. For example, if the game is played with the numbers on ten one-dollar bills, and your first draw is a bill with a number that begins with 9, your best strategy is to keep the bill. For similar reasons, the strategy in playing Googol is not strictly applicable to the unmarried girl problem, as many readers pointed out, because the girl presumably has a fair knowledge of the range in value of her suitors, and has certain standards in mind. If the first man who proposes comes very close to her ideal, wrote Joseph P. Robinson, "she would have rocks in her head if she did not accept at once."

Fox and Marnie apparently hit independently on a problem that had occurred to others a few years before. A number of readers said they had heard the problem before 1958 — one recalled working on it in 1955 — but I was unable to find any published reference to it. The problem of maximizing the *value* of the selected object (rather than the chance of getting the object of highest value) seems first to have been proposed by the famous mathematician Arthur Cayley in 1875. (See Leo Moser, "On a Problem of Cayley," in *Scripta Mathematica,* September-December 1956, pages 289–292.)

4. Let 1 be the length of the square of cadets and also the time it takes them to march this length. Their speed will also be 1. Let x be the total distance traveled by the dog and also its speed. On the dog's forward trip his speed relative to the cadets will be $x - 1$. On the return trip his speed relative to the cadets will be $x + 1$. Each trip is a distance of 1 (relative to the cadets), and the two trips are completed in unit time, so the following equation can be written:

$$\frac{1}{x-1} + \frac{1}{x+1} = 1$$

This can be expressed as the quadratic: $x^2 - 2x - 1 = 0$, for which x has the positive value of $1 + \sqrt{2}$. Multiply this by 50 to get the final answer: 120.7+ feet. In other words, the dog travels a total distance equal to the length of the square of cadets plus that same length times the square root of 2.

Loyd's version of the problem, in which the dog trots *around* the moving square, can be approached in exactly the same way. I paraphrase a clear, brief solution sent by Robert F. Jackson of the Computing Center at the University of Delaware.

As before, let 1 be the side of the square and also the time it takes the cadets to go 50 feet. Their speed will then also be 1. Let x be the distance traveled by the dog and also his speed. The dog's speed with respect to the speed of the square will be $x - 1$ on his forward trip, $\sqrt{x^2 - 1}$ on each of his two transverse trips, and $x + 1$ on his backward trip. The circuit is completed in unit time, so we can write this equation:

$$\frac{1}{x-1} + \frac{2}{\sqrt{x^2-1}} + \frac{1}{x+1} = 1$$

This can be expressed as the quartic equation: $x^4 - 4x^3 - 2x^2 + 4x + 5 = 0$. Only one positive real root is not extraneous: 4.18112+. We multiply this by 50 to get the desired answer: 209.056+feet.

Theodore W. Gibson, of the University of Virginia, found that the first form of the above equation can be written as follows, simply by taking the square root of each side:

$$\frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x+1}} = 1$$

which is remarkably similar to the equation for the first version of the problem.

Many readers sent analyses of variations of this problem: a square formation marching in a direction parallel to the square's diagonal, formations of regular polygons with more than four sides, circular formations, rotating formations, and so on. Thomas J. Meehan and David Salsburg each pointed out that the problem is the same as that of a destroyer making a square search pattern around a moving ship, and showed how easily it could be solved by vector diagrams on what the Navy calls a "maneuvering board."

5. The simplest way to fold Barr's belt so that each end is straight across and part of a rectangle of uniform thickness is shown in Figure 16. This permits the neatest roll (the seams balance the long fold) and works regardless of the belt's length or the angles at which the ends are cut.

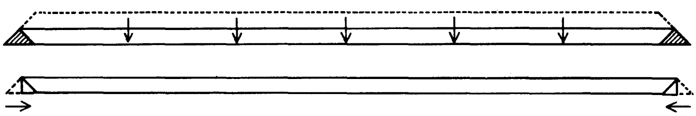

FIG. 16
How Barr folds his belt.

6. The assumption that the "lady" is Jean Brown, the stenographer, quickly leads to a contradiction. Her opening remark brings forth a reply from the person with black hair, therefore Brown's hair cannot be black. It also cannot be brown, for then it would match her name. Therefore it must be white. This leaves brown for the color of Professor Black's hair and black for Professor White. But a statement by the person with black hair prompts an exclamation from White, so they cannot be the same person.

It is necessary to assume, therefore, that Jean Brown is a man. Professor White's hair can't be white (for then it would match his or her name), nor can it be black because he (or she) replies to the black-haired person. Therefore it must be brown. If the lady's hair isn't brown, then Professor White is not a lady. Brown is a man, so Professor Black must be the lady. Her hair can't be black or brown, so she must be a platinum blonde.

7. Since the wind boosts the plane's speed from A to B and retards it from B to A, one is tempted to suppose that these forces balance each other so that total travel time for the combined flights will remain the same. This is not the case, because the time during which the plane's speed is boosted is shorter than the time during which it is retarded, so the over-all effect is one of retardation. The total travel time in a wind of constant speed and direction, regardless of the speed or direction, is always greater than if there were no wind.

8. Let x be the number of hamsters originally purchased and also the number of parakeets. Let y be the number of hamsters among the seven unsold pets. The number of parakeets among the seven will then be $7 - y$. The number of hamsters sold (at a price of \$2.20 each, which is a markup of 10 per cent over cost) will be $x - y$, and the number of parakeets sold (at \$1.10 each) will be $x - 7 + y$.

The cost of the pets is therefore $2x$ dollars for the hamsters and x dollars for the parakeets — a total of $3x$ dollars. The hamsters that were sold brought $2.2\ (x - y)$ dollars and the parakeets sold brought $1.1\ (x - 7 + y)$ dollars — a total of $3.3x - 1.1y - 7.7$ dollars.

We are told that these two totals are equal, so we equate them and simplify to obtain the following Diophantine equation with two integral unknowns:

$$3x = 11y + 77$$

Since x and y are positive integers and y is not more than 7, it is a simple matter to try each of the eight possible values (including zero) for y to determine which of them makes x also integral. There are only two such values: 5 and 2. Each would lead to a solution of the problem were it not for the fact that the parakeets were bought in pairs. This eliminates 2 as a value for y because it would give x (the number of parakeets purchased) the odd value of 33. We conclude therefore that y is 5.

A complete picture can now be drawn. The shop owner bought 44 hamsters and 22 pairs of parakeets, paying altogether \$132 for them. He sold 39 hamsters and 21 pairs of parakeets for a total of \$132. There remained five hamsters worth \$11 retail and two parakeets worth \$2.20 retail — a combined value of \$13.20, which is the answer to the problem.