A Conversation with Manjul Bhargava

Stephen Abbott

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On my way to the MathFest 2011 meeting in Lexington, Kentucky, I had the pleasure of meeting Manjul Bhargava, this year's invited Hedrick lecturer, on my shuttle bus from the airport. Although I knew something of his reputation, I did a quick Internet search at my hotel when I arrived and was inundated with anecdotes about the extraordinary talents of this young mathematician.

A math prodigy, he graduated second in his class from Harvard, received his doctorate at Princeton working under Andrew Wiles, and was invited to return to Princeton two years later as a full professor. His dissertation extended, in a dramatic and surprising way, some classical results of Gauss on quadratic forms, and he has since gone on to do groundbreaking work in algebraic number theory and combinatorics. In 2005, he won a Clay Research Award, and in 2008 he was awarded the Cole Prize.

Just last year, he announced a proof of a special case of the Birch and Swinnerton-Dyer conjecture, one of the seven Millennium Prize Problems. (These are deemed so important and so challenging that a complete proof would earn its author $1 million from the Clay Mathematics Institute.)

This dazzling list of accolades formed a curious contrast with the soft-spoken and humble friend I had just made on the shuttle bus. There was no hint of arrogance, no aloofness one might understandably expect from someone who does mathematics at such an elite level. I was intrigued and made a careful note of when Manjul was scheduled to speak.

The Earle Raymond Hedrick Lecture Series has been a fixture of the MAA's summer meeting. The list of speakers over the decades reads like a Who's Who in contemporary mathematics—Persi Diaconis, William Thurston, John Conway, Timothy Gowers. The invited speaker gives a series of three lectures during the meeting. The first lecture is always quite popular, but the typical pattern is for attendance to wane as the week goes on and as the complexity of the ideas goes up. Not so this year.

Manjul's finely tuned teaching instincts were on display from the beginning, and the positive buzz from the first lecture quickly went viral. By day two, napkins with scribbled diagrams of rational points on algebraic graphs started appearing in the various restaurants near the conference center, and at the conclusion of Manjul's third lecture, his now-expanded audience of several hundred erupted with applause. For the moment, we were all experts in elliptic curves—or at least Manjul had made us feel that way. [See inset article, next page.]

In the audience, several thoughts went through my mind. The first was simply being impressed that someone who spends the bulk of his time at the frontiers of the hardest problems in mathematics could so gracefully adapt his thinking to the diverse viewpoints of this crowd of nonspecialists. But I was also struck by the warmth and generosity of our speaker. Our questions mattered, our understanding mattered, we mattered. All week Manjul had been ubiquitous at the gamut of MathFest events—he was even spotted

A Quick Tour of Manjul Bhargava's 2011 Hedrick Lectures on Elliptic Curves

Nathan Carter and Eleanor Farrington

Doesn't it seem strange that Fermat's Last Theorem, one of the most famous questions of number theory, was proven using elliptic curves? Why is a theorem about discrete things, like integers, answered using smooth things, like curves?

The answer is that the geometry of a curve says a lot about the rational (or integral) points on that curve. For example, in the story "Harvey Plotter and the Circle of Irrationality" [see page 10], the characters use the unit circle (a curve) to generate Pythagorean triples (of integers). The technique they use applies just as well to any second-degree polynomial equation in two variables, which is called a *conic*.

Beginning with one rational point on the conic with rational coefficients, consider each line of rational slope through it, and find the other point of intersection between the line and the conic. This process gives us all the rational points on the conic. Thus, any conic containing at least one rational point contains infinitely many. That means there are two possibilities for conics: zero rational points or infinitely many.

Algebraic curves with rational coefficients are more general; they are arbitrary-degree polynomial equations in two variables. Consider first only degree-one algebraic curves; i.e., straight lines. You can easily find a method yielding all rational points on such a curve.

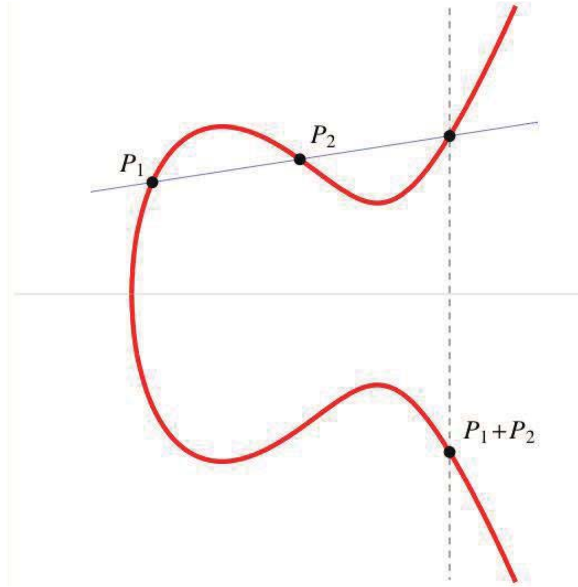

Figure 1. How to add two rational points on an elliptic curve.

(Try it!) Degree-two algebraic curves were just discussed. For higher degrees, our degree-two method does not apply because a line may intersect such a curve at more than two points. We need a new approach.

Graphing the algebraic curve over $\mathbb{C}^2$ (rather than $\mathbb{R}^2$) gives an object in four dimensions, called a *Riemann surface*. Faltings's theorem is the surprising fact that the topology of this surface (specifically its genus) in $\mathbb{C}^2$ tells us something about rational solutions in $\mathbb{R}^2$. It says that

at the business meeting. Watching him converse with the posse of students gathered round him in the wake of his final talk, it became clear to me that this wasn't a dignitary paying a cordial visit to the MAA community, but a member of the community intent on engaging it at every level.

I approached him after the lecture. "If you have time later, I would love to do an informal interview with you for *Math Horizons*," I asked, already feeling guilty about adding to Manjul's busy schedule. Despite having a host of valid excuses for politely declining, Manjul agreed and was waiting for me in the hotel lobby that afternoon.

Math Horizons: Congratulations on an extraordinary series of lectures this week—the response has been overwhelming.

Manjul Bhargava: Thank you.

MH: Tell me about your interest in teaching. Teaching is usually considered a gift—something you have or you don't have.

MB: I think teaching is also something you can learn.

MH: But your enjoyment of it seems very sincere, very much a natural part of who you are.

MB: Sure, to do anything well, you have to enjoy it. You need a passion to want to get better—to try different things, and see what works.

MH: When did this passion start for you? Did you teach in graduate school?

MB: I never taught in graduate school, but I taught a lot as an undergraduate.

MH: Really?

MB: Yeah, that happens a lot. At Harvard, the undergraduates are allowed to teach.

MH: What sorts of things?

MB: I was a TA for first-year calculus and the algebra sequence—things like that. That's when I realized I loved teaching. It was my favorite any generic algebraic curve (with rational coefficients) of degree greater than three contains only finitely many rational points! But it does not say how to find them.

So, cubic (degree-three) algebraic curves are special; only they exhibit all three possibilities—zero rational points, finitely many, or infinitely many. Therefore, cubics are heavily studied, particularly ones with no self-intersections or spikes; these are called elliptic curves and can be written in the canonical form $y^2 = x^3 + Ax + B$. Despite this attention, there is no known method for determining which of the above three possibilities a given cubic falls into. The Birch and Swinnerton-Dyer conjecture is a famous open problem that proposes such a method. In 2000, the Clay Mathematics Institute listed it as one of seven Millennium Prize Problems; a proof is awarded a million dollars!

We can still use a variant of the degree-two method for elliptic curves. Pick at least two rational points on the curve, and connect them with a line. It will have rational slope and thus intersect the curve in exactly one other rational point. Reflecting this point about the x-axis gives another new rational point (note the y^2 in the canonical form). Applying this technique again and again, connecting old points to new ones, generates more and more rational points. Mordell's theorem states that for every elliptic curve, there exists a finite initial set of points from which this technique will find all rational points on the curve! The size of that initial set is called the rank of the curve (discounting a few points we'll explain below). Many unsolved questions about rank are being actively studied today, including finding an algorithm that gives that initial finite set.

This process gives us a way to reveal an important algebraic structure on elliptic curves. We define addition of two points on an elliptic curve as follows: Draw the line between them as before, find the third point of intersection with the curve, and take its x-axis reflection. (See figure 1.) This operation satisfies all the properties of an Abelian group, with the zero (or identity) being a "point at infinity," touched by all vertical lines. The points not counted in the rank are those with finite order in this Abelian group (called torsion points). (For an exercise, try figuring out what the inverse of a point is.)

A weaker question than the Birch and Swinnerton-Dyer conjecture asks how many elliptic curves have infinitely many rational points. Goldfeld's conjecture states that the average rank of an elliptic curve is $\frac{1}{2}$ and, furthermore, that an elliptic curve chosen at random has probability $\frac{1}{2}$ of having rank 0, probability $\frac{1}{2}$ of having rank 1, and a negligible probability of having rank 2 or higher. And that's where things really get interesting. To test Goldfeld's conjecture, people have gathered a lot of data on the ranks of elliptic curves and always found an average rank larger than $\frac{1}{2}$! In fact, no one was able to say for sure that the average rank was even finite! Until last year.

Manjul Bhargava and his Ph.D. student Arul Shankar showed not only that the average rank of elliptic curves is finite, but also that it's less than one. What's more, they proved that at least 10 percent of all elliptic curves have no rational points (except for the identity point at infinity). That means at least 10 percent of elliptic curves satisfy the Birch and Swinnerton-Dyer conjecture. Did this incredible accomplishment earn Manjul and Arul 10 percent of the million-dollar prize? Sadly no, but perhaps they are on their way. ■

extracurricular activity.

MH: It was the same for me, actually, although I was just a tutor.

MB: Of course, my mom is a teacher, and I used to go and attend her classes.

MH: She's a mathematics professor at Hofstra [University], right? She let you do this?

MB: All the time when I was younger I would say, you know, I don't want to go to school today, can I just come with you. She was very cool about that.

MH: What part did this play in your decision to become a mathematician?

MB: Actually, as far back as I can remember I've loved math.

MH: What was it that you loved so much?

MB: I just liked playing around; trying to understand why something was happening. One of my earliest memories was stacking oranges in a pyramid and trying to figure out that if you had n oranges on a side, how many did you need to make the whole pyramid? That was one of the first problems I solved: n times $(n+1)$ times $(n+2)$ over 6. [*laughs*] It took me many months to figure it out.

MH: Do you remember how old you were?

MB: I must've been around eight or something. My mother was pretty busy; she was not usually teaching me directly, but she was always there as a resource. And she let me skip my school to go to hers.

MH: Number theory, with its very primitive questions, is a common introduction to mathematics—but you have stayed with it.

MB: Hmm. This $n(n+1)(n+2)/6$ thing is a number theory problem—I just liked these kinds of questions from the beginning.

MH: That's never gone away?

MB: [*shrugs*] That's never gone away.

MH: Do you remember where you were when Andrew Wiles announced

his proof of Fermat's Last Theorem?
MB: That was late in college for me. The math students were very excited—emails were flying around.
MH: And then you ended up working with Wiles. That must've been exciting.
MB: It was wonderful. He has such a great sense of the global picture of number theory. Even though I worked in an area that was different from what he does, he stayed interested in my work and was able to say when an idea represented a promising direction.
MH: Were you star struck at all when you first met him?
MB: I think I met him at the Joint Meetings. I had joined Princeton as a graduate student, but I hadn't met him yet. [*dawning on him*] He was getting the Cole Prize and I was getting the Morgan Prize, and so we were on stage together. [*laughs*]
MH: And you leaned over and said, "Hey, I'm Manjul . . . "?
MB: Hey, I'm at Princeton too! The next time I met him—or the next time we really talked—was my oral generals exam.
MH: Was that scary?
MB: Oh yeah. Three professors—Conway was the chair, Wiles, and [Charles] Fefferman—all sitting in the audience while I am at the chalkboard answering questions.
MH: Ouch. That's what you get for going to Princeton.

[*At this moment, an unknown MathFest participant (UMP) walks up to our table and addresses Manjul, who is unfazed and perfectly cordial.*]
UMP: Can I interrupt? To the youngest full professor at Princeton, I just want to say I found your talk enjoyable and understandable. So many times these Ph.D.s just want to talk up here [*gestures above his head with his hand*], but you put it right here—and I really appreciated it.
MB: Well, I appreciate it.


Laura McHugh (MAA)

Manjul Bhargava, center, talks with members of the audience after one of his lectures at MathFest 2011.

UMP: Keep up the good work.
MB: Thank you. What's your name?
UMP: My name is Joe. I've got a master's in statistics, and I teach high school mathematics. I like to poke around at the meetings, I saw your [talk], and I really like the way you broke it down. Keep doing that.
MB: Okay, thanks.
UMP: The best mathematicians can break it down for the lowest common denominator, and I like to say that I'm the lowest common denominator in the room. Bye-bye now.
[*He leaves.*]
MH: So, tell me about your liberal arts math course that Joe Gallian mentioned in his introduction. Is this something you currently teach at Princeton?
MB: Yes, the first time I had about 100 students—the next year demand nearly tripled.
MH: And how is the course structured? Do you use a particular text?
MB: I don't use a text. The inspiration came from a similar course developed by Dick Gross and Joe Harris at Harvard, and they wrote a text that I considered using, but the thing that makes these courses is that they have to be individual to the instructor. There has to be a certain passion and interest coming from the teacher about the particular subject being taught. It's not so much which mathematics is included as much as what can the teacher be really enthusiastic about, what topics have they thought about from every possible angle.
MH: So what do you like to do?
MB: I start with music.
MH: Really?
MB: The first third of the course is mathematics and music—the mathematics of rhythm, the mathematics of pitch. I'm a tabla player . . .
MH: I've heard—word is that you are very accomplished.
MB: A lot of the examples I use come from ancient India—mathematics that was written by percussionists and poets of ancient times. Much of this is not really known by mathematicians; in fact, many of the people who sit in on my class are mathematicians and math majors who aren't actually allowed to take it.
MH: You make a really good point. A lot of these courses for nonmajors cover fascinating topics . . .
MB: Right, that our regular students never get a chance to learn! There is also a lot of history of mathematics

that is not known to mathematicians, because, for instance, it was written in Sanskrit.

MH: Which you also know something about.

MB: My grandfather was a scholar of Sanskrit. One way I got interested in mathematics when I was young was through what he taught me about the rhythms of Sanskrit poetry.

MH: Does that end up being number theory too?

MB: Number theory and combinatorics. So here was my grandfather, a Sanskrit scholar, teaching me all this amazing combinatorics, and then when I went to study mathematics I thought, gee, I've already seen these ideas, from Sanskrit. That connection was amazing to me, and I like to talk about those experiences from my life in the courses I teach.

MH: Do you play the tabla in class?

MB: Sure, and I often bring guest musicians so we can show some of the ways these rhythmic techniques are actually implemented in practice.

MH: That sounds fantastic. Do you play a lot outside of class?

MB: I try to—it gets harder every year. The mathematics keeps me busier and busier.

MH: Do you consciously seek out connections between music and mathematics, or Sanskrit and mathematics?

MB: Yeah, but I love it when I'm not looking for them. When I was learning Sanskrit poetry, I had no idea I was encountering mainstream mathematics—rediscovering those ideas later in a math class, under a totally different name, and in a totally different guise—wow! That's what makes me so excited, the unexpected unity of two very different subjects.

MH: You made a comment in your first lecture this week that I wanted to discuss with you. You said that "geometry was for intuition and algebra was for proofs."

"When I was learning Sanskrit poetry, I had no idea I was encountering mainstream mathematics."

MB: Well, I just meant in that particular area.

MH: Of course, but I wanted to be provocative and quote you out of context so that I could ask whether you think geometry—in modern mathematics—is being shortchanged in some way.

MB: What do you mean?

MH: The human brain, it seems to me, thinks geometrically much more naturally than it thinks algebraically. In your lecture today, when you were explaining the group structure on the rational points along elliptic curves, you explained it by drawing chords along the graphs.

MB: Yes, it's a geometric construction. But then you have to ask what kind of structure do you get, and that's where you have to do the algebra.

MH: Your point is well taken, but there are some subjects like, say, complex analysis that have divorced themselves from their geometric roots and are frequently taught as largely algebraic subjects, as though casting something in purely algebraic terms makes it more true.

MB: Well, it doesn't make it more true. Sometimes geometry allows you to visualize and get a feeling for what should be true, and those kinds of discoveries couldn't have been obtained simply by manipulating formulas—you would just get a mess, you wouldn't know where to go. So the geometry is very critical.

MH: But sometimes I would like the geometry to be more than that—for the visualization to be enough on its own.

MB: That was, in a large part, what Grothendieck was trying to do.

MH: Tell me more.

MB: A lot of the work of [Alexander] Grothendieck was taking all of the geometric intuition that people had collected over the years, putting it on a very solid algebraic foundation, and making an equivalence between certain geometric language and algebraic language—making that connection so strong that in the future people could just think geometrically and know that that geometry had an algebraic translation. Nowadays, algebraic geometers can just proceed geometrically and be totally confident that what they are visualizing really is proved. That's one way of thinking about Grothendieck's contributions—making geometric intuition broadly usable, in a rigorous way.

MH: So that it becomes a properly justified way of doing mathematics.

MB: Exactly.

MH: I know we are talking about research mathematics here, but watching my kids' journey through secondary school, I would say that geometry is buried amid a lot of symbolic manipulation techniques. I think that makes mathematics harder than it needs to be.

MB: You're right; the human mind is very visual. In number theory, when you first state the problems, you don't see geometry right away. But somehow, the way people go about solving these problems often requires some way of visualizing the question.

At this moment we are joined by another guest in the hotel lobby—a dignified older gentleman whom Manjul knows well. It is Richard Guy, the distinguished British number theorist, and it becomes apparent that he and Manjul have dinner plans. Well into his 90s, Professor Guy is still sharp as a tack and has come equipped with pen, paper, and several mathematical queries for his young friend. Manjul Bhargava, to no one's surprise, is happy to oblige. ■

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