A proof of the Graph Semigroup Group Test in "The Graph Menagerie"

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We present a proof of the Graph Semigroup Group Test. This Test plays a key role in the article "The Graph Menagerie", which appeared in *Mathematics Magazine* **83**(3), June 2010, pp. 168 - 179. The elegant proof we give here is due to Enrique Pardo of the Universidad de Cádiz, Spain. Although the proof is somewhat long, most of the tools used can be found in undergraduate-level texts on graph theory and/or abstract algebra.

Indeed, the one portion of the proof of the Graph Semigroup Group Test for which we do not know a proof using undergraduate-level tools involves showing that if the graph semigroup W_Γ is a group, then the graph is necessarily sink-free. Since the Mad Vet graphs which arise in our discussion are always sink-free, the proof given here in fact yields a complete proof of the Mad Vet Group Test using only undergraduate-level tools.

Here as a reminder are the statements of the two results of interest.

The Graph Semigroup Group Test. Let Γ be a finite directed graph. Then the graph semigroup W_Γ is a group if and only if Γ has the following three properties:

(1) Γ is cofinal;

(2) Every cycle in Γ has an exit; and

(3) Γ contains no sinks.

The Mad Vet Group Test. The Mad Vet semigroup W of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph Γ has the following two properties.

(1) Γ is cofinal; and

(2) Every cycle in Γ has an exit.

We start by providing some definitions.

Definition 1. Let $\Gamma = (V, E)$ be a finite directed graph, and let S denote a subset of the vertex set V.

(1) S is called *hereditary* if whenever we have a vertex v in S and there is a directed path in Γ from v to some other vertex w, then w is also in S. Formally, S is hereditary if whenever $v \in S$ and there is a path μ in Γ for which $i(\mu) = v$ and $t(\mu) = w$, then $w \in S$.
(2) S is called *saturated* if whenever we have a vertex w in V for which *all* of the edges emanating from w have the property that their terminal vertices are in S, then w is in S as well. More formally: S is saturated if whenever $w \in S$ has the property that if $t(e) \in S$ for all edges in E for which $i(e) = w$, then w is also in S.
(3) If X is any subset of V then the *hereditary saturated closure of* X is defined to be the smallest hereditary and saturated subset of V which includes X. (There is an algorithmic method to produce the hereditary saturated closure of any subset of V.)
(4) For a vertex $v \in V$, a *closed simple path based at* v is a path $\mu = e_1 e_2 \cdots e_n$ for which $i(\mu) = t(\mu) = v$, and for which $i(e_j) \neq v$ for all $2 \leq j \leq n$. A *cycle based at* v is a closed simple path based at v with the added property that $i(e_j) \neq i(e_k)$ for all $1 \leq j \neq k \leq n$.
(5) If v is a vertex, then the *tree of* v is the subset of V consisting of those vertices w for which there is a path μ having $i(\mu) = v$ and $t(\mu) = w$. We sometimes denote the tree of v by $\text{Tree}(v)$. (This is standard notation, but perhaps a misleading choice of verbiage, because $\text{Tree}(v)$ need not necessarily be a tree in the usual graph-theoretic sense. In particular, $\text{Tree}(v)$ might contain closed paths.) If S is a subset of V then we denote by $\text{Tree}(S)$ the union $\cup_{v \in S} \text{Tree}(v)$.

Recall that the finite directed graph Γ is called *cofinal* in case every vertex of Γ connects to every sink and every cycle of Γ. By [2, Lemma 2.8],

Lemma 2. Γ *is cofinal if and only if the only hereditary saturated subsets of* V *are* V *and* $\emptyset$.

(The proof of the lemma is essentially a proof by induction on the minimum length of appropriately defined paths in Γ. The proof uses only undergraduate-level graph-theoretic tools.)

We will eventually be interested in analyzing the hereditary saturated closure of a single vertex $\{v\}$. In that situation we have the following useful property of the vertices in this closure.

Lemma 3. *Let v, w be vertices in the graph Γ. If w is in the hereditary saturated closure of the set $\{v\}$, then there is a positive integer n for which: every path p of length n having $i(p) = w$ has the property that $t(p) \in \mathrm{Tree}(v)$.*

The proof of the lemma follows from the construction presented in the proof of [1, Lemma 7]. As with the previously referenced results, the tools used in this proof are accessible to the reader who is familiar with undergraduate level graph theory ideas.

Our goal now is to determine conditions under which the graph semigroup W_Γ is actually a group. We start by introducing some convenient semigroup notation.

Definition 4. Let $(S, +)$ be a semigroup, and let $a, b \in S$. We write

$$a < b \text{ in case there exists } x \in S \text{ such that } b = a + x.$$

We write $a \leq b$ in case either $a = b$ or $a < b$.

This notation makes intuitive sense, since it agrees with the usual notion of $<$ when the semigroup is $(\mathbb{Z}^+, +)$. However, a word of caution is warranted here: it is possible to have $a < b$ and $b < a$, even if $a \neq b$. In a commutative semigroup S we often use the symbol $+$ to denote the binary operation. In particular, if $a \in S$, then the element $a + a$ is often denoted $2a$, the element $a + a + a$ by $3a$, and so on. Note that for every element $a \in S$ and every $n \in \mathbb{N}$ with $n \geq 2$ we have $a < na$.

However, note that the relation $<$ is definitely NOT reflexive in general (in other words, we need not have $a < a$, as seen in the $S = \mathbb{Z}^+$ example), but $<$ is reflexive in case S is a monoid (i.e., when S has a zero element). In contrast, $<$ is always transitive. For suppose that $a < b$ and $b < c$ in S; then there exist elements $d, e \in S$ for which $b = a + d$ and $c = b + e$, so by substituting (and associativity) we get $c = a + (d + e)$. Finally, note that in a commutative semigroup the relation $<$ is consistent with the $+$ operation; that is, if $a < b$ and $c < d$ then $a + c < b + d$. This follows easily: if $b = a + x$ and $d = c + y$ for $x, y \in S$, then $b + d = (a + x) + (c + y) = (a + c) + (x + y)$ (by commutativity).

We are now in position to begin the proof of the "if" part of the Graph Semigroup Group Test.

The elements of W_Γ are equivalence classes of expressions taken from the free commutative semigroup $S(V)$. However, to make the notation less cumbersome, we eliminate the brackets around the elements of W_Γ throughout this note.

We assume throughout the following series of claims that Γ satisfies conditions (1), (2), and (3) given in the theorem. Before we start, it is worth pointing out that the elements of V are (of course) vertices in Γ, but also play a second role as the generators of the semigroup W_Γ. The proofs we will give below about properties of W_Γ will first focus on these special elements (i.e., the generators of W_Γ), and then proceed to an analysis of all the elements of W_Γ.

Claim 1. If $w \in \mathrm{Tree}(v)$ then $w \leq v$ in W_Γ.

Proof of Claim 1. Let $p = e_1 e_2 \cdots e_n$ denote a path having $i(p) = v$ and $t(p) = w$. By applying the relation at v, and denoting $i(e_1)$ by v_1, we have

$$v = v_1 \quad \text{or} \quad v = v_1 + \sum_{e \neq e_1 \in E, i(e) = v} t(e)$$

in W_Γ. Now denote $t(e_2)$ by v_2, and apply the relation at v_1 to get

$$v_1 = v_2 \quad \text{or} \quad v_1 = v_2 + \sum_{e \neq e_2 \in E, i(e) = v_1} t(e),$$

which, when combining with the first displayed equation, gives an equation of the form

$$v = v_2 \quad \text{or} \quad v = v_2 + T_2$$

for some $T_2 \in W_\Gamma$. Continuing inductively in this way, and keeping in mind that $v_n = t(e_n) = w$, gives

$$v = w \quad \text{or} \quad v = w + T_n$$

for some $T_n \in W_\Gamma$, and Claim 1 is established. ∎

Claim 2. Γ has the following property: If $v \in V$ is the base of some closed simple path in Γ, then v is the base of at least two distinct closed simple paths in Γ. (This is property is sometimes called *Condition (K)* in the literature.)

Proof of Claim 2. It is shown in [1, Lemma 7] that if Γ is any graph which satisfies hypotheses (1) and (2) of the Graph Semigroup Group Test, then Γ satisfies the property given in Claim 2. Here's an outline of the proof. Suppose there is a closed simple path based at v. We argue by contradiction that it is not the unique such path. If it is, then this path must be a cycle. But every cycle has an exit by hypotheses; call the exit f. Then $t(f)$ cannot be in the cycle by hypothesis (since this would give a second closed simple path based at v, contradiction). By induction and the fact that Γ is cofinal one can then show that there must exist a path from $t(f)$ to v, which would again give a second closed simple path based at v, contradiction.) ∎

Claim 3. For every $v \in V$, $2v < v$ in W_Γ.

Proof of Claim 3. Suppose $|V| = n$. Then every path of length at least n must contain at least one vertex u for which u is the base of a closed simple path based at u. (This is because any path of length at least n must contain repeated vertices.)

We first consider vertices v for which v is the base of some closed simple path of Γ. We showed in Claim 2 that there are at least two distinct closed simple paths in Γ for which v is the base. Let α and β denote two such distinct closed simple paths based at v. Write $\alpha = e_1 e_2 \cdots e_n$ and $\beta = f_1 f_2 \cdots f_m$ where $i(e_1) = i(f_1) = v = t(e_n) = t(f_m)$. Since $\alpha \neq \beta$ by hypothesis, there is an integer j for which $v_j = i(e_j) = i(f_j)$, but $e_j \neq f_j$. (Here's why: If $e_1 \neq f_1$ we are done, otherwise, continue by comparing e_2 to f_2, and so on. There are two possibilities. One possibility is that we arrive at a situation in which $e_j \neq f_j$ for some $j \in \mathbb{Z}^+$, in which case we are done. The second possibility is that we run out of edges in one of the paths prior to completing the other path. (In other words, that one of the paths is an initial subpath of the other.) *But this situation cannot occur here*, because α and β are assumed to be closed simple paths based at v, and so it is not possible for an initial subpath of either path to contain v as a vertex.)

We apply the semigroup relations, in order, at the vertices v through $i(e_{j-1})$. After substituting each equation into the previous one, we get an equation of the form

$$v = t(e_{j-1}) \quad \text{or} \quad v = t(e_{j-1}) + T_j$$

for some $T_j \in W_\Gamma$. But $t(e_{j-1}) = i(e_j)$, so we have

$$v = i(e_j) = v_j \quad \text{or} \quad v = i(e_j) + T_j = v_j + T_j.$$

We now apply the semigroup relation at $v_j = i(e_j)$. Because $e_j \neq f_j$ and each has initial vertex v_j, this gives either

$$v_j = t(e_j) + t(f_j) \quad \text{or} \quad v_j = t(e_j) + t(f_j) + Z$$

for some $Z \in W_\Gamma$. By substituting we now have an equation of the form

$$v = t(e_j) + t(f_j) \quad \text{or} \quad v = t(e_j) + t(f_j) + X$$

for some $X \in W_\Gamma$. But there is a path from $t(e_j)$ (resp., $t(f_j)$) to v, namely, $e_{j+1} \cdots e_n$ (resp., $f_{j+1} \cdots f_m$). (The path might just be the trivial path v, but that's OK.) Rephrased, $v \in \text{Tree}(t(e_i))$ and $v \in \text{Tree}(t(f_i))$. In particular, by Claim 1, we have equations of the form

$$t(e_j) = v \quad \text{or} \quad t(e_j) = v + T_e \quad (\text{resp., } t(f_j) = v \quad \text{or} \quad t(f_j) = v + T_f),$$

for some $T_e, T_f \in W_\Gamma$. Again by substituting, what we can conclude so far is that, for any vertex v which is the base of some closed simple path in Γ, that there is an equation of the form

$$v = 2v \quad \text{or} \quad v = 2v + T$$

for some $T \in W_\Gamma$.

We now consider those $v \in V$ with the property that v is *not* the base vertex for any closed simple path in Γ. Here is an algorithm. Since by hypothesis v is not a sink, we can apply the semigroup relation at v to get

$$v = \sum_{e \in E, i(e) = v} t(e)$$

in W_Γ. Identify all vertices $t(e)$ in this sum having the property that $t(e)$ is not the base of a closed simple path in Γ. If there are no such vertices, then stop the process. If there are such vertices, then apply the semigroup relation at each of these vertices (this is possible, since by hypothesis Γ contains no sinks). For each of the vertices appearing in the new expression, again identify those having the property that the vertex is not the base of a closed simple path in Γ. We continue in this way. By the observation made at the start of the proof of this claim, after n steps (at most) the process stops. Now let Bases(Γ) denote the set of vertices $u \in V$ for which u is the base of (at least one) closed simple path in Γ. The algorithm then gives, upon stopping, an equation of the form

$$v = \sum_{b_j \in B} n_j b_j$$

for some nonempty subset B of Bases(Γ), and some positive integers $n_1, ..., n_{|B|}$.

But using what we already proved in the first part of the proof of Claim 3, we know that for each $b_j \in B$ we have either

$$b_j = 2b_j \quad \text{or } b_j = 2b_j + Z_j$$

for some $Z_j \in W_\Gamma$. So we have, on substituting, either

$$v = \sum_{b_j \in B} n_j(2b_j) = 2 \sum_{b_j \in B} n_j b_j$$

or

$$v = \sum_{b_j \in B} n_j(2b_j + Z_j) = 2 \sum_{b_j \in B} n_j b_j + T \quad \text{for some } T \in W_\Gamma,$$

which in turn gives that either

$$v = 2v \quad \text{or} \quad v = 2v + T$$

for some $T \in W_\Gamma$.

The upshot is this: For any $v \in V$ we have established that, inside W_Γ, there is either an equation of the form $v = 2v$ or $v = 2v + T$ for some $T \in W_\Gamma$.

But in the case $v = 2v$ we also get $2v = v + v = 2v + v = 3v$, which gives $v = 3v = 2v + v$, which then gives by definition that $2v < v$. On the other hand, an equation of the form $v = 2v + T$ for $T \in W_\Gamma$ gives $2v < v$ by definition. So we are done with the proof of Claim 3. ■

We note here that we have indeed shown $2v < v$, which is a stronger statement than the statement $2v \leq v$. We will definitely need the more powerful version later.

Claim 4. For every $v \in V$ and every $n \in \mathbb{Z}^+$ we have $nv < v$ in W_Γ.

Proof of Claim 4. By induction. The case $n = 1$ is nontrivial! (Remember, in general the relation $<$ is not reflexive ...) It is established as follows: We have $v < 2v$ (this holds for any element in any semigroup). But $2v < v$ by Claim 3. So by transitivity we get $v < v$. Now assume $kv < v$ for $k \geq 1$, and show $(k+1)v < v$. But $(k+1)v = kv + v < v + v$ (by the induction hypothesis and the fact that $<$ preserves $+$). But $v + v = 2v < v$ by Claim 3, so that $(k+1)v < v$ by transitivity of $<$. ■

We note here that establishing the statement $v < v$ required some serious work! In particular, we have shown that the relation $<$ on W_Γ is reflexive.

Claim 5. For all $v, w \in V$, $w < v$ in W_Γ.

Proof of Claim 5. First, suppose $v = w$ in W_Γ; the result then follows from the $n = 1$ case of Claim 4.

We now establish that $w < v$ for every pair $v, w \in V$ for which $v \neq w$ in W_Γ, which will give the result. First, suppose $w \in \text{Tree}(v)$ (i.e., there exists a path α such that $i(\alpha) = v$ and $t(\alpha) = w$). Then by Claim 1 we have $w \leq v$. Thus either $v = w$ in W_Γ (so we are done by the previous paragraph), or $w < v$, which is what we needed to prove.

Suppose now that $w \notin \text{Tree}(v)$. We consider the hereditary saturated closure of the subset $\{v\}$ of V; we denote this set by Y. Since Y contains v, $Y \neq \emptyset$. But Γ is cofinal, so by Lemma 2 the only nonempty hereditary saturated subset of Γ is all of V. So $Y = V$, which in particular gives $w \in Y$. By Lemma 3, this means that there exists $n \in \mathbb{Z}^+$ with the property that every path of length n having initial vertex w ends in $\text{Tree}(v)$. By applying the semigroup relation at each vertex along the way, and using the fact that there are no sinks in Γ, we can eventually write

$$w = \sum_{j=1}^{q} m_j w_j$$

where each $w_j \in \text{Tree}(v)$ and $m_j \in \mathbb{Z}^+$. In particular, by Claim 1 we have $w_j \leq v$ for each $1 \leq j \leq q$. Let Q denote the positive integer $\sum_{j=1}^{q} m_j$. This then gives, along with Claim 4, that

$$w = \sum_{j=1}^{q} m_j w_j \leq \sum_{j=1}^{q} m_j v = Qv < v.$$

Thus we have shown $w < v$, as desired. (Note that since v, w were arbitrary, we can write the result as: $v < w$ for all $v, w \in W_\Gamma$.) ∎

Claim 6. For every $x, y \in W_\Gamma$ we have $x < y$.

Proof of Claim 6. Since the set V of vertices is a generating set for the semigroup W_Γ, we can write

$$x = \sum_{i \in I} w_i, \quad y = \sum_{j \in J} v_j$$

in W_Γ, where w_i, v_j are (not necessarily distinct) elements of V. Let v be any element of V. Let N_I (resp., N_J) denote $|I|$ (resp., $|J|$). But then Claims 4 and 5, along with the transitivity of $<$ and the compatibility of $+$ with $<$ give

$$x = \sum_{i \in I} w_i < N_I \cdot v < v < N_J \cdot v < \sum_{j \in J} w_j = y,$$

so that, for all $x, y \in W_\Gamma$, $x < y$. ∎

We are now poised to finish the proof of the "if" direction of the Graph Semigroup Group Test. To do so, we need only use the following result.

Lemma 5. *Let $(S, +)$ be a commutative semigroup. Then the following statements are equivalent.*

(1) S is an (abelian) group.

(2) S has the property that $x < y$ for every pair of elements $x, y \in S$.

The proof of this lemma is left to the interested reader. (In fact, it's likely you will find this lemma in your abstract algebra book, either as a proven result or as an exercise. It will probably be the case that "commutative semigroup" will appear as "set with a commutative binary operation", and that the $x < y$ notation will appear as "there exists z for which $x + z = y$". It also might be the case that the non-commutative version of this lemma appears in your book, in which case the symmetric version of property (2) will be added as a hypothesis.)

And now we are in a position to finish the proof.

Proof of the "if" direction of the Graph Semigroup Group Test. If Γ has the indicated properties, then we established in Claim 6 that $x < y$ for each $x, y \in W_\Gamma$. W_Γ is commutative by definition. But by Lemma 5, any commutative semigroup with this property is a group. And so we are done. ∎

We note that the proof given above yields automatically a proof of the "if" direction of Mad Vet Group Test as well: Since all Mad Vet graphs are necessarily sink-free, the three indicated hypotheses on Γ given in the Graph Semigroup Group Test are satisfied in this case.

We complete the proof of the Graph Semigroup Group Test by sketching a proof of the "only if" direction. It is probably the case that some of the tools and much of the terminology used here will be unfamiliar to most of our readers. We would be *very* interested if any of our readers could produce a proof of this direction of the Graph Semigroup Group Test which uses only elementary concepts!

Step 1. Show that if W_Γ is a group, then Γ is cofinal. (This can be done with undergraduate-level tools.)

Step 2. Show that if W_Γ is a group, then every cycle in Γ has an exit. (This can also be done with undergraduate-level tools.)

So we are only one step from being finished! We need only show that Γ has no sinks. We do not know how to do that using only undergraduate-level tools. But in the remaining steps we show how to do that, using some more advanced tools.

Step 3. As mentioned above, the two properties already established for Γ in Steps 1 and 2 imply that Γ has Condition (K).

Step 4. So by Step 3 the Leavitt path algebra $L(\Gamma)$ has the exchange property. (See e.g. [3].)

Step 5. Every Leavitt path algebra is semiprimitive (i.e., the Jacobson radical is zero). Using this, with Step 4, one can show that $L(\Gamma)$ is an I-ring. (That is, idempotents lift mod $J(R)$, and every nonzero right ideal of $L(\Gamma)$ contains a nonzero idempotent.)

Step 6. For an associative ring R we denote by $\mathcal{V}(R)^*$ the semigroup of nonzero finitely generated projective right R-modules (with operation $\oplus$). One then proves a ring-theoretic result: Suppose R is a semiprimitive I-ring. Then R is purely infinite simple if and only if $\mathcal{V}(R)^*$ is a group.

Step 7. For any graph Γ, the graph semigroup W_Γ is isomorphic to $\mathcal{V}(L(\Gamma))^*$ by [3, Theorem 3.5]. So the hypothesis that W_Γ is a group implies that $\mathcal{V}(L(\Gamma))^*$ is a group. So by Steps 5 and 6, $L(\Gamma)$ is purely infinite simple.

Step 8. Now apply [1, Theorem 12] to conclude that Γ in fact satisfies all three of the desired properties. $\blacksquare$.

We conclude by noting that the "only if" direction of the Mad Vet Group Test follows immediately, since we need only show for a Mad Vet graph Γ that the first two conditions on Γ are satisfied (as the sink-free condition is automatically satisfied for any Mad Vet graph). More importantly, we note that the first two conditions in fact can be shown using only undergraduate-level tools (see Steps 1 and 2 in the above sketch). Therefore, this implies that we have provided a proof of the Mad Vet Group Test which uses only undergraduate-level tools.

Update, March 2010: A method to show that Γ has no sinks that does not use the ring theoretic tools of Steps 3 through 8 above, has been provided by Enrique Pardo. This more elementary method is based on ideas related to the "refinement" property of W_Γ as established in Section 4 of [3]. The interested reader is encouraged to recreate this approach for her/himself. Perhaps there is yet another, even more direct proof of the nonexistence of sinks?

References

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