Analyses of Mad Vet Scenarios #3–#7 in "The Graph Menagerie"

by Gene Abrams and Jessica K. Sklar

We here provide computations which allow us to identify the Mad Vet semigroups of Scenarios #3–#7. These Scenarios appear in the article "The Graph Menagerie", *Mathematics Magazine* **83**(3), June 2010, pp. 168 - 179.

Let $\mathbb{Z}^+$ be the set of positive integers and $\mathbb{N} = \{0, 1, 2, \ldots\}$. Then let $S = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \{(0,0,0)\}$. The elements $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ generate S as a semigroup; for notational convenience, we denote these elements by A, B, and C, respectively. We use $\sim$ to indicate the use of a machine, and use $=$ to indicate that a given expression in a Mad Vet semigroup is equal to another due simply to semigroup properties (for example, $(A+B)+(A+C) = 2A+B+C$).

Mad Vet Scenario #3.

Machine 1 turns one ant into one beaver and one cougar;
Machine 2 turns one beaver into one ant and one cougar;
Machine 3 turns one cougar into one ant and one beaver.

The semigroup W of this Mad Vet scenario is the Klein 4-group $V = \mathbb{Z}_2 \times \mathbb{Z}_2$. We first show that any element of S is equivalent to A, B, C, or $A + B + C$ in S. Note that

$$A + (A + B + C) = (A + B) + (A + C)$$

$$\sim C + B \text{ (using Machines 3 and 2 in reverse)}$$

$$\sim A \text{ (using Machine 1 in reverse).}$$

Similarly $B+(A+B+C) \sim B$ and $C+(A+B+C) \sim C$, so that $I = [A+B+C]$ acts as an identity element in W. (It's sufficient to show that $A+B+C$ acts as the identity on the three generators $[A]$, $[B]$, and $[C]$ of W.) Next,

$$[2A] = [A + A] = [A + (B + C)] = I,$$

and, similarly, $[2B] = [2C] = I$. Thus, for any $n \in \mathbb{Z}^+$, we have $[nA] = [A]$ (resp., $[nB] = [B]$, $[nC] = [C]$) in case n is odd, and $[nA] = I$ (resp., $[nB] = I$, $[nC] = I$) in case n is even. Thus, any element $aA + bB + cC$ of S is equivalent to $a'A + b'B + c'C$ where each a', b', c' is in $\{0, 1\}$, and not all are zero. The three equivalences $A \sim B + C, B \sim A + C$, and $C \sim A + B$ then yield the result.

We next show that $[A]$, $[B]$, $[C]$ and $[A + B + C]$ are distinct in W. First, suppose that each ant is worth \$2, each beaver \$1, and each cougar \$1. Then the total value of a collection changes by \$2 with the application of a machine (forwards or backwards). So $A \not\sim B$ and $A \not\sim C$. Similarly, B and C are not equivalent to $A + B + C$. To show that other pairs elements in $\{A, B, C, A+B+C\}$ are not equivalent one to the other, we can use a similar invariant. (For example: suppose each ant and each cougar is worth \$1, while each beaver is worth \$2; then, again, any use of the machines changes the collection's total value by a multiple of \$2, so B is not equivalent to C.)[†]

Since W has identity element I and we've seen that each element in W has itself as an inverse, we have $W \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. ∎

Mad Vet Scenario #4.

Machine 1 turns one ant into two ants;

Machine 2 turns one beaver into two beavers;

Machine 3 turns one cougar two cougars.

The semigroup W of this Mad Vet scenario is a nonmonoid with seven elements. It's clear from the nature of the machines that every element of S is equivalent to an element of the form $aA + bB + cC$ with $a, b, c \in \{0, 1\}$, not all zero. Moreover, such expressions are not equivalent to one another, since no machine takes an input an animal of one species and outputs any animals of another species. So W consists of seven elements. Finally, W has no identity element: if I were an identity element of W then we'd have

[†]This argument is a version of the argument used to show that the elements $[(1, 0, 0)]$, $[(2, 0, 0)]$, and $[(3, 0, 0)]$ are distinct in the Mad Vet semigroup of Scenario #1.

$[A] + I = [A]$, so any representative of I would consist solely of ants; but then we'd have $[B] + I \neq [B]$. Thus, W is not a monoid. ■

Mad Vet Scenario #5.

Machine 1 turns one ant into one beaver and one cougar;
Machine 2 turns one beaver into one ant and one beaver;
Machine 3 turns one cougar into one ant and one cougar.

The Mad Vet semigroup W of this Mad Vet scenario is $\mathbb{Z}$. First, note that

$$A \sim B + C \sim (A + B) + C = A + (B + C) \sim 2A.$$

So we have $nA \sim A$ for any $n \in \mathbb{Z}^+$. Next, for any $m, n \in \mathbb{Z}^+$, applying Machine 2 (resp., Machine 3) in reverse m times yields that $mA + nB \sim nB$ (resp., $mA+nC \sim nC$). So $[A]$ acts as an the identity for W, since $A+A \sim A$, $A + B \sim B$, and $A + C \sim C$. Also, B and C are inverses, since

$$B + C \sim A.$$

We claim that all elements of W are equivalent to one of these:

$$nB \quad (\text{some } n \in \mathbb{Z}^+), \quad nC \ (\text{some } n \in \mathbb{Z}^+), \quad \text{or } A.$$

Indeed, start with any menagerie $aA + bB + cC$. If $b = c$, then

$$aA + b(B + C) \sim aA + bA = (a + b)A \sim A.$$

Next, if $b > c$, then

$$aA + bB + cC = aA + (b - c)B + c(B + C) \sim aA + (b - c)B + cA \sim (b - c)B.$$

Similarly, if $c > b$, then $aA + bB + cC$ is equivalent to a positive multiple of C.

We now show that the classes $[A]$, $[nB]$, and $[nC]$ ($n \in \mathbb{Z}^+$) are distinct.

First, for each $n \in \mathbb{Z}^+$ neither nB nor nC is equivalent to A. To show this, for a given such n, value each ant at \$$n$+1, each beaver at \$1, and each cougar at \$$n$. Then applying the machines either keeps the total value of a collection the same, or changes it by some multiple of \$$n$+1. Since n beavers is worth \$$n$, and we can't obtain \$$n$ by starting with \$$n$+1 and adding or subtracting multiples of \$$n$+1, we have $A \not\sim nB$. Next, if $nC \sim A$, then

$$nB \sim nB + A \sim nB + nC = n(B + C) \sim nA \sim A,$$

which is a contradiction.

Second, we show that $nB \not\sim mB$ for positive integers $n \neq m$. Suppose without loss of generality that $n > m$. If $nB \sim mB$, then

$$(n - m)B \sim (n - m)B + mA \sim (n - m)B + m(B + C) = nB + mC$$

$$\sim mB + mC = m(B + C) \sim mA \sim A,$$

a contradiction. In a completely analogous way, $nC \not\sim mC$ whenever $n \neq m$.

Finally, we claim that $mB \not\sim nC$ for any pair of positive integers m, n. Let $m, n \in \mathbb{Z}^+$ and suppose we had $mB \sim nC$. Then we would have

$$mB + nB \sim nC + nB$$

(adding nB to both sides), so that

$$(m + n)B \sim n(B + C) \sim nA \sim A,$$

which is a contradiction to what was shown above.

Thus, the indicated classes are distinct. But then mapping $[A]$ to 0, $[B]$ to 1, and $[C]$ to -1 clearly induces an isomorphism from W to $\mathbb{Z}$. ■

Mad Vet Scenario #6.

Machine 1 turns one ant into one beaver;
Machine 2 turns one beaver into one cougar;
Machine 3 turns one cougar into one cougar.

The Mad Vet semigroup W in this case is $\mathbb{Z}^+$. Clearly, any menagerie (a, b, c) is equivalent to $(0, 0, a + b + c)$. Since S doesn't contain $(0, 0, 0)$, we have $a + b + c \in \mathbb{Z}^+$. Finally, since all the machines output exactly one animal, the size of a menagerie is invariant, so that $(0, 0, i) \not\sim (0, 0, j)$ for $i \neq j$. ■

Mad Vet Scenario #7.

Machine 1 turns one ant into one ant, one beaver and one cougar;
Machine 2 turns one beaver into one ant and one cougar;
Machine 3 turns one cougar into one ant and one beaver.

The Mad Vet semigroup W for this scenario is $\mathbb{Z}_4$. Note first that

$$\begin{aligned} A &\sim A + B + C \text{ (using Machine 1)} \\ &\sim 2B \text{ (using Machine 2 in reverse),} \end{aligned}$$

and

$$C \sim A + B \text{ (using Machine 3) } \sim (2B) + B \sim 3B.$$

So every element of W is of the form $[nB]$ for some $n \in \mathbb{Z}^+$. Next,

$$B \sim A + C \sim 2B + 3B = 5B = B + 4B,$$

so that $I = [4B]$ is the identity of the semigroup, and all elements of W are of the form $[B]$, $[2B]$, $[3B]$, or $[4B]$. To show that these elements of W are distinct, value ants at \$2, beavers at \$1, and cougars at \$3. Then each of the three machines changes the value of a collection by a multiple of \$4, so since none of the values of B, $2B$, $3B$ and $4B$ are congruent modulo \$4, their equivalence classes in W must all be distinct. Clearly, $[B]$ is a generator for W, so $W \cong \mathbb{Z}_4$. ■