This is a translation of sections 30 and 31 of Lagrange's *Réflexions sur la Résolution Algébrique des Équations*, contained in his collected works (*Oeuvres vol. 3*, 205-421) I am at work on a translation of the entire work, but this goes slowly since it can only be done in my "free time". I do not claim to be an expert at French, and I welcome comments and corrections. I have tried to be as literal as I could be without the English sounding stilted, but in a number of places I have cut the sentences into two or three parts, since Lagrange's sentences, in the classical French manner, tend to go on for a long time. Although I append a copyright notice you are free to copy this work for any non-commercial purpose, including distribution to classes. [1]

— Greg (St. George)

Note from previous sections: Lagrange is discussing the quartic, in the form

$$x^4 + mx^3 + nx^2 + px + q$$

The references to m, n, &c. in the sequel are references to this equation.

30. One can therefore deduce form this remark a direct way to arrive at the reduced equation [réduite] of the fourth degree and by its use to a general solution for this degree. Since the combination $ab + cd$ of the four roots a, b, c, d, is such that is only admits three variations, that is

$$ad + cd \quad ac + bd \quad ad + cd$$

it follows that if at first we let

$$ab + cd = u$$

one will have an equation in u of the third degree, which will have the roots

$$ab + cd \quad ac + bd \quad ad + bc$$

and which will consequently be of the form

$$u^3 - Au^2 + Bu - C = 0$$

where one will have, by the nature of the equations

$$\begin{aligned} A &= ab + cd + ac + bd + ad + cb \\ B &= (ab + cd)(ac + bd) + (ab + cd)(ad + cb) + (ac + bd)(ad + cb) \\ C &= (ab + cd)(ac + bd)(ad + cb) \end{aligned}$$

that is

$$\begin{aligned} A &= ab + ac + ad + bc + bd + cd \\ B &= a^2(bc + bd + cd) + b^2(ac + ad + cd) + c^2(ab + ad + bd) + d^2(ab + ac + bc) \\ C &= abcd(a^2 + b^2 + c^2 + d^2) + a^2b^2c^2 + a^2b^2d^2 + a^2c^2d^2 + b^2c^2d^2. \end{aligned}$$

And it is easy to see that the values A, B, c must be given the the coefficients m, n, p q of the proposed equation, without any extraction by roots, because they remain the same under any permutation that one makes between the roots a, b, c, and d. From this it follows that each cannot have but one and the same value. Because, having

$$\begin{aligned}-m &= a+b+c+d\\ n &= ab+ac+ad+bc+bd+cd\\ -p &= abc+abd+acd+bcd\\ q &= abcd\end{aligned}$$

one has right away

$$A = n.$$

Then, to find B, one observes that

$$a(bc+bd+cd) = -p - bcd$$

[2] and in the same way

$$b(ac+ad+cd) = -p - acd$$

and similarly with the others, from which one will have

$$B = (a+b+c+d)(-p) - 4abcd$$

that is to say

$$B = mp - 4q$$

Finally to get C we notice that

$$a^2+b^2+c^2+d^2 = m^2 - 2n$$

so that the part of the equation with the term $abcd(a^2+b^2+c^2+d^2)$ becomes $(m^2-2n)q$. To get the other part, one calculates the square of p and from this deduces that

$$\begin{aligned}a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2 &= p^2 - 2abcd(ab+ac+bc+ad+bd+cd)\\ &= p^2 - 2nq,\end{aligned}$$

so that one has

$$C = (m^2-4n)q + p^2$$

By these means we see that our reduced equation will be

$$u^3 - nu^2 + (mp-4q)u - (m^2-4n)q - p^2 = 0$$

which is the same that we arrived at in section **27**, if we let $u = 2y$.

31. We will now see how, by knowing one of the values of u, one can find the four roots a, b, c, d. Since

$$u = ab+cd \quad \text{and} \quad abcd = q$$

[2]Translators footnote: so that $a^2(bc+bd+cd) = -ap - abcd$

it is clear that the two quantities ab and cd will be the roots of this second degree equation:

$$t^2 - ut + q = 0$$

so that by naming these roots t' and t'' one will know the two products

$$ab = t' \quad \text{and} \quad cd = t''.$$

Furthermore, one has

$$-p = ab(c+d) + cd(a+b) = t'(c+d) + t''(a+b)$$

and as

$$a + b + c + d = -m$$

one will have

$$a + b = \frac{p - mt'}{t' - t''}, \quad c + d = \frac{p - mt''}{t'' - t'}$$

so that since

$$ab = t' \quad \text{and} \quad cd = t''$$

it is clear that a and b will be the roots of the equation

$$x^2 - \frac{p - mt'}{t' - t''} x + t' = 0$$

and c and d will be the roots of

$$x^2 - \frac{p - mt''}{t'' - t} x + t'' = 0.$$

So one sees that it suffices to know one of the roots of the reduced equation in u in order to have the four roots a, b, c, d of the proposed equation. Also note that each of the roots of this reduced equation will always give the same four roots a, b, c, d. Because if in place of taking $u = ab + cd$ one had taken $u = ac + bd$ or $u = ad + bc$, there would be no other change in our formulas except that b would be changed into c or into d and *vice versâ*.