Maximizing the Chances of a Color Match—Web Supplement

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Proofs of several theorems from this paper were thought to be too technical to interest many readers of the MAGAZINE. We offer them her to benefit those who would like to learn more about this subject.

Proofs of the theorems

THEOREM 2. *Every complete graph is n-monophilic for all $n \geq 2$.*

Proof. Let K_m denote the complete graph on m vertices. Given an n-color list for each vertex of K_m, let p denote the probability of obtaining a match. We will prove that having identical lists minimizes $\bar{p} = 1 - p$.

To compute $\bar{p}$, we divide the number of proper colorings by the total number of ways to color the vertices. The latter is just n^m, regardless of whether or not the lists are identical. So it's enough to show that the number of proper colorings is minimized when all the lists are identical.

If $n < m$, i.e., there are more vertices than colors, then a match cannot be avoided when all the lists are identical, since all the vertices are adjacent to each other. So the number of proper colorings is zero, and we are done. So we'll assume $n \geq m$.

Let's try to count the number of proper colorings for an arbitrary set of n-color lists. Denote the vertices of K_m by $v_1, \cdots, v_m$. There are n ways to choose a color for v_1. Having picked colors for $v_1, \cdots, v_k$, where $k \in \{1, \cdots, n-1\}$, there are *at least* $n - k$ ways to pick a color for v_{k+1}. If all the lists are identical, then this lower bound of $n - k$ is in fact achieved, because: (1) all the vertices are adjacent; so (2) $v_1, \cdots, v_k$ must have k distinct colors; and (3) these k colors appear in the list for v_{k+1}. Therefore there are always at least $n(n-1)\cdots(n-m+1)$ proper colorings for K_m, and this lower bound is achieved if all the lists are identical.

THEOREM 3. *Every tree is n-monophilic for all $n \geq 2$.*

Proof. This follows immediately from Lemma 7 by using induction on the number of edges of the tree.

To prove the next theorem, we consider a path with $k \geq 1$ edges where each vertex of degree two has an n-color list and each of the two terminal vertices (of degree one) has an $(n-1)$-color list; we call this a (k, n)**-path**. Denote the color list of each vertex v by L_v.

LEMMA 1. *If a (k, n)-path has the smallest possible number of proper colorings among all (k, n) paths, then its n-color lists are all identical and contain its $(n-1)$-color lists as subsets. For any (k, n)-path whose n-color lists are identical and contain its $(n-1)$-color lists as subsets, let A_k denote the number of proper colorings when the $(n-1)$-color lists are identical, and let B_k denote the corresponding number when they are not identical. Then, for $k \geq 1$, $A_k - B_k = (-1)^k$.*

Proof. Let α be a (k, n)-path, v one of its terminal vertices, and w the vertex adjacent to v. For each color c in the $(n-1)$-color list L_v, let P_c denote the number of proper colorings of α with c assigned to v. Suppose there is a color c that is in L_v but not in L_w. Then by replacing c with a color that is in L_w but not in L_v, we can reduce P_c without changing P_d for any color $d \neq c$ in L_v. Thus, if α has the smallest possible number of proper colorings then each of the $(n-1)$-color lists is a subset of the n-color lists.

Next we show that if α has the smallest possible number of proper colorings then its n-color lists must be identical. For $k = 1$ and $k = 2$ there is nothing to show, so suppose $k \geq 3$. L_v contains at least two colors, c_1 and c_2. From above, for each c_i, P_{c_i} is minimized only if $L_w - \{c_i\}$ is a subset of the n-color list of the vertex $w' \neq v$ adjacent to w; this implies $L_w = L'_w$. Thus, by induction on k, all the n-color lists of α must be identical.

Now suppose α has identical n-color lists and identical $(n-1)$-color lists which are subsets of the n-color lists. Then, for each of the $n-1$ colors assigned to v, there remains an $(n-1)$-color list of possibilities for w, and this list is not the same as the $(n-1)$-color list of the other terminal vertex of α. Thus we get

$$A_k = (n-1)B_{k-1}$$

By a similar reasoning, we see that

$$B_k = A_{k-1} + (n-2)B_{k-1}$$

Subtracting the second equation from the first gives

$$A_k - B_k = (-1)(A_{k-1} - B_{k-1})$$

This, together with the equation $A_2 - B_2 = (n-1)(n-2) - [(n-1) + (n-2)^2] = 1$ (which is verified by direct computation), inductively yield $A_k - B_k = (-1)^k$.

THEOREM 4. *Every cycle is n-monophilic for all $n \geq 2$.*

Proof. Let C be a cycle of length k with an n-color list for each vertex. Let α be the path of length $k-2$ obtained by removing one vertex v and its two incident edges from C. Assigning a color $c \in L_v$ to v leaves $n-1$ or n possible colors to pick from for each of the two vertices u and w adjacent to v. If we think of α as a $(k-2, n)$-path (by throwing out one color from each of $L_u - \{c\}$ and $L_w - \{c\}$ if it contains n colors), we see from Lemma 1 that the smallest possible number of proper colorings for α is A_{k-2} or B_{k-2}, depending on the parity of k.

Case 1: k is odd. Then $A_{k-2} < B_{k-2}$. Now, as a $(k-2, n)$-path, α will have A_{k-2} proper colorings if its n-color lists are identical and its $(n-1)$-color lists for u and w are also identical and are subsets of the n-color lists. It follows that if all the lists for C are identical, its number of proper colorings will be minimized.

Case 2: k is even. (We assume $k \geq 4$, since a cycle cannot have only two edges.) Then $B_{k-2} < A_{k-2}$. Now, as a $(k-2, n)$-path, α will have B_{k-2} proper colorings iff its n-color lists are identical and its $(n-1)$-color lists for u and w are distinct and are subsets of the n-color lists. In this case, without loss of generality, $L_v \neq L_w$; so there is a color c in L_v that is not in L_w. Hence after assigning color c to v there will still be n possible choices for coloring w. Thus the number of proper colorings for α (after assigning c to v) is at least $B_{k-2} + A_{k-3}$ (since when k is even $A_{k-3} < B_{k-3}$).

We show as follows that this results in C having a larger number of proper colorings than if all its lists were identical. For, with all lists identical, for each color assigned to v there would be $A_{k-2} = B_{k-2} + 1$ proper colorings for α, which would give a total of

$$n(B_{k-2} + 1)$$

proper colorings for C. On the other hand, without all lists being identical, there are at least $B_{k-2} + A_{k-3}$ proper colorings of α when c is assigned to v, and at least B_{k-2} proper colorings for each of the $n-1$ colors of $L_v - \{c\}$, giving a total of

$$B_{k-2} + A_{k-3} + (n-1)B_{k-2}$$

Thus it's enough to check that $A_{k-3} > n$ for $k \geq 4$, which is easily verified from the the recursive formulas given for A_k and B_k in the proof of Lemma 1.

THEOREM 5. *A connected graph is* not *2-monophilic iff all its cycles are even and it contains at least two cycles whose union is not* $K_{2,3}$.

Proof. We give the proof in two parts, the "if direction" and the "only if direction."

Part 1: If G is a connected graph such that all its cycles are even and it contains at least two cycles whose union is not $K_{2,3}$, then it is not 2-monophilic.

By Lemma 2 below, if all vertices of G have identical lists, then G has exactly two proper colorings. Therefore, to show G is not 2-monophilic, it's enough to show that for some set of 2-color lists, G has fewer than two proper colorings. Now, for any subgraph $G' \subseteq G$, where G' inherits color lists from G, each proper coloring of G gives a proper coloring of G'. In other words, if, for some set of 2-color lists for G, G' has fewer than two proper colorings, then so does G. So our goal will be to show that for some set of 2-color lists, some subgraph $G' \subseteq G$ has fewer than two proper colorings.

By hypothesis, there exist two cycles C and D in G, such that $C \cup D$ is not $K_{2,3}$. Also, as all cycles in G have even length, $|C|$ and $|D|$ must each be greater than or equal to four (there are no cycles of length two since we are considering simple graphs only). Depending on how many vertices C and D share, we have three cases, and a few subcases:

Case 1: C and D are disjoint. Since G is connected, there exists a path P from some vertex $v \in C$ to some vertex $w \in D$ such that $P - \{v, w\}$ is disjoint from C and D.

Case 1.1: P has even length. We assign color lists to the vertices of $G' = C \cup D \cup P$ as in FIGURE 1. Each of the three ellipses in the figure represents an *odd* number of edges. All the missing vertices in P are given the list $\{1, 2\}$. And all the missing vertices in C and D are given the list $\{2, 3\}$.

We will show G' has no proper colorings with the given lists. Suppose we are trying to find a proper coloring. If we assign color 1 to v, then its two adjacent vertices in C will have to be colored 2 and 3. Then, regardless of what colors are assigned to the remaining vertices, there will be a match involving one of the vertices in C that has $\{2, 3\}$ as its list. If, on the other hand, we assign color 2 to v, then, to avoid a match in P, w will have to be colored 2 also. This will cause a match in D, for a similar reason as for C. So there is no proper coloring of G' with the given lists, as we set out to show.

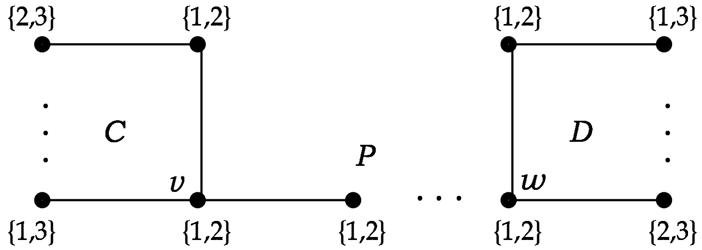

FIGURE 1: Disjoint even cycles connected by an even path

Case 1.2: P has odd length. Then assign color lists to $G' = C \cup D \cup P$ as in FIGURE 2, where each ellipsis again represents an odd number of edges. The proof that G' has no proper colorings is very similar to the case when P has even length.

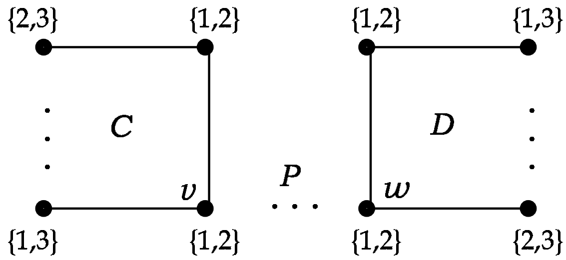

FIGURE 2: Disjoint even cycles connected by an odd path

Case 2: C and D share exactly one vertex. We treat this case just like Case 1.1 above, except that now $v = w$, so P has length zero, which is still even.

Case 3: C and D share two or more vertices. If $C \cup D = K_{2,4}$, then we're done, because, as we saw in the proof of Theorem **??**, $K_{2,4}$ has no proper colorings if given lists as in FIGURE **??**. So assume $C \cup D \neq K_{2,4}$. Also, by hypothesis, $C \cup D \neq K_{2,3}$. So, by Lemmas 3 and 4, there exist two vertices v and w connected by three paths α, β, γ in $C \cup D$ such that the three paths mutually intersect in v and w only, and α, β, γ have lengths that are, respectively, odd and greater than or equal to $1, 3, 3$, or even and greater than or equal to $2, 2, 4$. In each of these two cases, we will construct a set of 2-color lists for a subgraph $G' \subseteq C \cup D$ such that G' has no proper colorings.

Case 3.1: α, β, γ have lengths that are, respectively, odd and greater than or equal to $1, 3, 3$. First suppose $|\alpha| = 2$, $|\beta| = 2$, and $|\gamma| = 4$. Then $G' = \alpha \cup \beta \cup \gamma$ looks exactly as in FIGURE **??**, where b plays the role of v, e the role of w, $\alpha = be$, $\beta = bade$, and $\gamma = bcfe$. Then, as was proved in the section titled "A counterexample," there is no proper coloring of G' with lists as in FIGURE **??**.

When one or more of α, β, γ are, respectively, larger than $1, 3, 3$, we add an even number of vertices and edges to them, and give each vertex the same list as one of its adjacent vertices. (For example, when $|\beta| > 3$, we add an even number of vertices and edges to ad, thereby keeping $|\beta|$ odd, and give each new vertex say the list $\{1, 3\}$.) Then, by the same reasoning as before, G' has no proper colorings.

Case 3.2: α, β, γ have lengths that are, respectively, even and greater than or equal to $2, 2, 4$. First suppose $|\alpha| = 2$, $|\beta| = 2$, and $|\gamma| = 4$. Then $G' = \alpha \cup \beta \cup \gamma$ looks as in FIGURE 3, with v and w labeled. Given the lists in FIGURE 3, it can be verified that G' has exactly one proper coloring.

If one or more of α, β, γ are, respectively, larger than $2, 2, 4$, then, as in Case 3.1, we add an even number of vertices and edges to them, and give each vertex the same

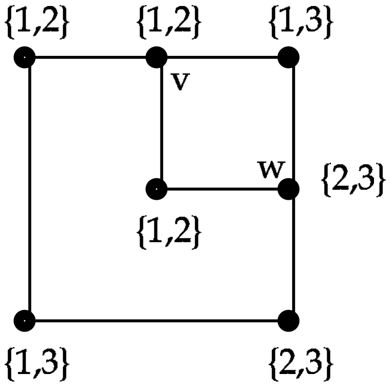

FIGURE 3: $|\alpha| = 2$, $|\beta| = 2$, and $|\gamma| = 4$

list as one of its adjacent vertices. Then G' will still have exactly one proper coloring, which is less than two, as desired. This finishes the proof of Part 1.

Remark. *Every non-n-monophilic graph we constructed before Case 3.2 above was not n-choosable. Recall that a graph is n-choosable if it has a proper coloring for any given set of n-color lists assigned to its vertices. Case 3.2, however, shows examples of graphs that are 2-choosable but not 2-monophilic.*

Part 2: If G is not 2-monophilic, then all its cycles are even, and it contains at least two cycles whose union is not $K_{2,3}$.

Assume G is not 2-monophilic. We first show G has no odd cycles. By Lemma 2, if G contains an odd cycle, then it has no proper colorings when identical 2-color lists are assigned to its vertices. So the probability of obtaining a match is one; i.e., G is 2-monophilic, which is a contradiction.

Since G is not 2-monophilic, by Theorem **??**, it is not a tree. So it must have at least one cycle. Then, by Theorem **??** and Lemma 6, G contains at least two cycles C and D. If $C \cup D \neq K_{2,3}$, then we are done. So assume $C \cup D = K_{2,3}$, which by Lemma 8 is 2-monophilic. This implies that G must contain a cycle $E \not\subset C \cup D$, for otherwise, by Lemma 6, G would be 2-monophilic. Now we can verify, case by case (details omitted), that $C \cup D \cup E$ contains two cycles whose union is not $K_{2,3}$, as desired. This completes the proof of Part 2 and Theorem **??**.

LEMMA 2. *Let G be a connected graph with identical 2-color lists assigned to its vertices. Then: (1) G has no proper colorings if it contains an odd cycle. (2) G has exactly two proper colorings if it contains no odd cycles.*

Proof. The proof is short and elementary, and is left as an exercise for the reader.

LEMMA 3. *Let G be a graph with no odd cycles. Suppose G contains two cycles C and D that share at least two vertices and whose union is neither $K_{2,3}$ nor $K_{2,4}$. Then $C \cup D$ contains a cycle of length greater than or equal to six.*

Proof. Since G has no odd cycles, all its cycles must have length at least four. If C or D has length six or greater, there is nothing to show. So assume they both have length four. Now we have several cases, depending on the number of vertices and edges that C and D share. A case by case examination (details omitted) shows that there is in fact only one case—when C and D share exactly one edge and two vertices—that satisfies all the conditions in the hypotheses: no odd cycles, and $C \cup D$ does not equal

$K_{2,3}$ or $K_{2,4}$. Then $C \cup D$ must look as in FIGURE **??**, so it contains a cycle of length six, as desired.

LEMMA 4. *Let G be a graph with no odd cycles. Suppose G contains two cycles C and D that share at least two vertices, with $|D| \geq 6$. Then there exist two vertices v and w connected by three paths α, β, γ in $C \cup D$ such that the three paths mutually intersect in v and w only, and α, β, γ have lengths that are, respectively, odd and greater than or equal to $1, 3, 3$, or even and greater than or equal to $2, 2, 4$.*

Proof. Let α be a path $v_1, \cdots, v_k$ in C such that v_1 and v_k are in D, but for $1 < i < k$, $v_i \notin D$. Such a path exists because C and D share at least two vertices and $C \neq D$.

Let $v = v_1$, $w = v_k$. Then v and w split D into two paths β and γ whose interiors are disjoint from each other, as well as from α. By hypothesis, $|\beta| + |\gamma| = |D| \geq 6$. Since all cycles in G are even, α, β, and γ must have the same parity. If they are odd, then $|\beta|$ and $|\gamma|$ must each be greater than or equal to 3. And if they are even, then the larger one of $|\beta|$ and $|\gamma|$, which we will rename, if necessary, to be $|\gamma|$, must be greater than or equal to 4.

LEMMA 5. *Every even cycle is 2-monophilic.*

Proof. Let C be an even cycle, with a 2-color list L_v assigned to each vertex $v \in C$. If all the lists are identical, then, by Lemma 2, there are exactly two proper colorings. So it's enough to show when the lists are not all identical there are always at least two proper colorings.

Let v and w be adjacent vertices with distinct lists. Let P be the path obtained by removing the edge vw from C. We will show there are two proper colorings for P that remain proper after adding back the edge vw.

First we need a definition. Let x and y be any two vertices in P. Suppose when we assign a certain color $c \in L_x$ to x, there is a proper coloring of P only for one of the colors $d \in L_y$ assigned to y. Then we say c **forces** d.

If each color in L_v forces a color in L_w, then all the lists in P are identical. To see why, denote the vertices of P by $v = v_1, \cdots, v_k = w$. Then each color in L_{v_i} forces a color in $L_{v_{i+1}}$. So $L_{v_i} = L_{v_{i+1}}$, which implies all the lists in P are identical.

But v and w have distinct lists. So at least one of the colors in L_v forces neither color in L_w. Let $L_v = \{a, b\}$ and $L_w = \{c, d\}$. Then, without loss of generality, a forces neither c nor d. So there are two proper colorings of P with a assigned to v, and at least one with b assigned to v.

If a is different from both c and d, then both proper colorings that assign a to v remain proper after adding back the edge vw, as desired. On the other hand, suppose a is not different from both c and d. Then, without loss of generality, $a = c$. So the proper coloring with a assigned to v and d to w remains proper after adding back the edge vw. Also, $b \neq c$, since $b \neq a$. And $b \neq d$, since $\{a, b\} \neq \{c, d\}$. So the proper coloring with b assigned to v, too, remains proper after adding back the edge vw.

Let G be a graph. To **add a terminal edge** to G means to add a new vertex and connect it by a new edge to a vertex in G.

LEMMA 6. *Let G' be a connected subgraph of a connected graph G such that G' contains every cycle of G. Then G is n-monophilic iff G' is.*

Proof. Since every cycle of G is contained in G', there is a finite sequence of connected graphs $G' = G_1, G_2, \cdots, G_k = G$ such that, for each i, G_{i+1} is obtained from G_i by adding a terminal edge. By Lemma 7, G_i is n-monophilic iff G_{i+1} is. So G is n-monophilic iff G' is.

LEMMA 7. *Suppose G is obtained from a connected graph G' by adding a terminal edge to G'. Then G is n-monophilic iff G' is.*

Proof. Let v and w be the endpoints of the terminal edge, with w the vertex in G that's not in G'. Given a set of n-color lists for the vertices of G, let G' inherit color lists from G. Each proper coloring of G' can be extended to either $n-1$ or n different proper colorings of G, according to whether or not the color assigned to v in the given proper coloring of G' is in the list for w. Therefore, the number of proper colorings of G is minimized iff the number of proper colorings of G' is minimized and v and w have identical lists. So G is n-monophilic iff G' is.

LEMMA 8. *$K_{2,3}$ is 2-monophilic.*

Proof. In FIGURE 4 we have labeled the five vertices of $K_{2,3}$ as u, v, w, x, and y. Given 2-color lists L_u, L_v, L_w, L_x, and L_y, if they are all identical, then by Lemma 2 there are exactly two proper colorings. So, to show $K_{2,3}$ is 2-monophilic, it's enough to show that when the lists are not all identical, there are at least two proper colorings. Now, L_x and L_y share zero, one, or two colors. We'll show in each of these three cases there are always at least two proper colorings.

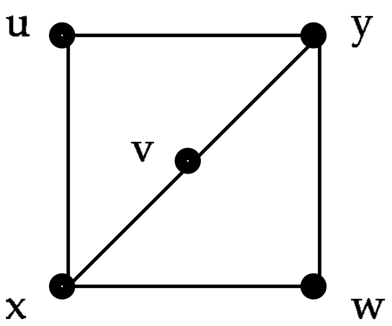

FIGURE 4: $K_{2,3}$

Case 1: L_x and L_y share two colors. So we can, without loss of generality, assume $L_x = L_y = \{1, 2\}$. If we assign color 1 to both x and y, then there exists a proper coloring, since each of u, v, w will have at least one color other than 1 in its list. Similarly, we can obtain a second proper coloring by assigning color 2 to both x and y.

Case 2: L_x and L_y share one color. So we can assume $L_x = \{1, 2\}$ and $L_y = \{1, 3\}$. If at least one of the vertices u, v, w, does not contain color 1 in its list, then we can obtain two proper colorings by assigning color 1 to both x and y. On the other hand, if all three vertices u, v, w contain color 1 in their lists, then we can obtain one proper coloring by assigning color 2 to x, 3 to y, and 1 to u, v, w, and another proper coloring by assigning color 1 to both x, y and using the second color in each of the lists for u, v, w.

Case 3: L_x and L_y share no colors.

Case 3.1: L_u, L_v, or L_w is disjoint from L_x or L_y. Without loss of generality, assume L_u is disjoint from L_x. Let G' be the graph obtained by removing the edge xu. Then, by Theorem **??** and Lemma 6, G' is 2-monophilic, so it has at least two proper colorings. Each of these is also a proper coloring for $K_{2,3}$, because x and u have disjoint lists and adding back the edge between them will not introduce a match.

Case 3.2: Each of L_u, L_v, and L_w intersects both L_x and L_y. Since L_x and L_y share no colors, we can assume $L_x = \{1, 2\}$ and $L_y = \{3, 4\}$. So L_u, L_v, and L_w are in the set $S = \{\,\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\,\}$.

If L_u, L_v, and L_w are all distinct, then, without loss of generality, we can assume $L_u = \{1, 3\}$, $L_v = \{1, 4\}$, and $L_w = \{2, 3\}$. So $\{2, 4\}$ is disjoint from L_u. Therefore we can obtain two proper colorings by assigning color 2 to x, 4 to y, and each of the two colors in L_u to u.

On the other hand, suppose two or more of L_u, L_v, and L_w are the same. Then at least two of the four lists in S are not among L_u, L_v, and L_w. And for each such list we can obtain at least one proper coloring.

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