Dear Reader,

Thank you for your interest in this article supplement that accompanies "The Median Value of a Continuous Function". The exercises in this supplement have been organized into categories as follows:

- "Some Discrete Inquiries"–Exercises 1 & 2 deal with the median value of a finite list of numbers. Exercise 3 is concerned with the fraction of partition midpoints that belong to the union of a finite collection of open intervals. These exercises lay the groundwork for some later exercises.

- "Take it to the Limit"–Exercises 4 and 5 deal with the limit of the fraction of partition midpoints within an open set as the order of the partition goes to infinity. Exercise 6 presents a very clever Cauchy sequence proof suggested by a referee for the existence of the median in the case of a continuous function on a closed interval. Exercises 7 & 8 extend this Cauchy sequence proof to a continuous function on an open interval (a, b).

- "Filling in the Blanks"–Exercises 9–12 lead the reader through a proof of Theorem 1 and omitted portions of Theorem 2. (A proof of Theorem 3 is postponed until Exercise 32 in a collection of exercises that deal with measure theoretic concepts.)

- "Trust but Verify"–Exercises 13–20 verify results of examples in the article or illustrate the theorems. These exercises are appropriate for calculus students.

- "A Median Miscellany"–Exercises 21–29 are "theoretical" exercises of varying levels of difficulty, from beginning calculus to introductory real analysis.

- "Small, Median, and Large"–Exercises 30–39 develop the median within the context of Lebesgue measure and integration.

Happy Solving!
The Authors

Some Discrete Inquiries

1. Let $\mathcal{MED}$ denote the median value of a list A of n real numbers.

(a) Explain why *at most* $\lfloor n/2 \rfloor$ of the numbers in A are greater than (less than) $\mathcal{MED}$ and *at least* $\lfloor n/2 \rfloor$ of the numbers in A are equal to or greater than (equal to or less than) $\mathcal{MED}$.

(b) Prove that if each number in A is replaced by m copies of itself then the median value of the new list will be equal to $\mathcal{MED}$.

(c) Let B denote a sublist of A with more than $n/2$ elements. (That is, the number of elements in the sublist B is at least $\lfloor n/2 \rfloor + 1$.) Prove that if m and M are the minimum and maximum numbers in B respectively, then $m \leq \mathcal{MED} \leq M$.

(d) Suppose that at least k of the numbers in list A lie strictly between $\mathcal{MED}$ and some real number r. Prove that it is impossible to change fewer than k elements of A and produce a new list whose median value is r.

(e) Prove that if each number in A is changed by less than ϵ then the median value of the new list will be less than ϵ away from $\mathcal{MED}$.

2. Let A denote a list of real numbers with median $\mathcal{MED}$. Fix a positive integer k and let B denote a list of real numbers such that every $k+1$ consecutive entries in B differ (pairwise) by less than ϵ. (That is, if $b_i, b_{i+1}, b_{i+2}, \ldots, b_{i+k}$ are any $k+1$ consecutive entries in B then any two of these entries differ by less than ϵ.)

(a) Prove that if the list B is ordered from smallest to largest then every $k+1$ consecutive entries in the ordered list will differ (pairwise) by less than ϵ.. (*Hint:* Induction on the number m of entries in list B.)

(b) Suppose that list B is a sublist of m consecutive entries from A and that more than half the entries in A are contained in B. Assume we change the values of as many as $k-1$ entries that belong to A but that are not in B. Prove that the median value $\mathcal{MED}^*$ of the new list will be less than ϵ away from $\mathcal{MED}$.

For a closed interval $[a, b]$ and a natural number n, let $x_1^*, x_2^*, \ldots, x_n^*$ denote the midpoints of the subintervals that comprise a regular partition of $[a, b]$ of order n. We will refer to the points x_i^* as the midpoints of the partition. If $U \subset (a, b)$ is some fixed open set let $k_U(n)$ denote the number of midpoints from a regular partition of order n that belong to U.

3. Suppose that $U \subset (a, b)$ is a disjoint union $U = \bigcup_{i=1}^{j} U_i$ of j open intervals U_i of length ℓ_i, and that the total measure of U is $\ell = \sum_{i=1}^{j} \ell_i$. Prove that

$$\frac{\ell}{b-a} - \frac{j}{n} \leq \frac{k_U(n)}{n} < \frac{\ell}{b-a} + \frac{j}{n}$$

Take it to the Limit

4. Using the notation, assumptions, and result of Exercise 3, prove that

$$\lim_{n \to \infty} \frac{k_U(n)}{n} = \frac{\ell}{b-a}$$

5. Suppose that $U \subset (a, b)$ is an open set and that the measure of U is ℓ.

(a) Show by way of example that it is possible to have $\ell < b - a$ and $\lim\limits_{n \to \infty} \dfrac{k_U(n)}{n} = 1$.

(b) Show by way of example that $\lim\limits_{n \to \infty} \dfrac{k_U(n)}{n}$ need not exist.

(c) Prove that lim inf $\dfrac{k_U(n)}{n} \geq \dfrac{\ell}{b-a}$.

6. Assume that f is a continuous function on a closed interval $[a, b]$.

(a) Let $\epsilon > 0$ be given. Prove that there exists a natural number N such that if $n \geq N$ then for every natural number m, med(n) and med(mn) differ by less than ϵ. (*Hint:* Use uniform continuity to conclude that there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$ for all x and y in the interval $[a, b]$. Choose N such that $(b - a)/(2N) < \delta$. For $n \geq N$ use the properties in parts (b) and (e) of Exercise 1 to conclude that med(n) and med(mn) differ by less than ϵ for $m = 1, 2, \ldots$.)

(b) Conclude from part (a) that med(n) is a Cauchy sequence and thus convergent. (*Hint:* Apply the result of part (a) with ϵ replaced by $\epsilon/2$. Assume that both n and m are N or greater.)

(c) Can you apply the approach outlined in parts (a) & (b) to the *average* value of a continuous function on a closed interval?

The next exercise uses the notation $\text{med}_f(n)$ and $\text{med}_g(n)$ to distinguish between $\text{med}(n)$ for a function f and $\text{med}(n)$ for a function g respectively.

7. Suppose that f is continuous on (a, b) and that $\epsilon > 0$ is given. Prove that there exist $\delta > 0$ and a natural number N such that for any function g on (a, b) that is equal to f on $[a + \delta, b - \delta]$ and for any regular partition of order $n \geq N$ the difference between $\text{med}_f(n)$ and $\text{med}_g(n)$ will be less than ϵ. (*Hint:* Invoke uniform continuity on a subinterval $[a_1, b_1]$ of (a, b) such that $b_1 - a_1 > (b - a)/2$ to produce $0 < \delta_1$ such that if x and y are in $[a_1, b_1]$ and satisfy $|x - y| < \delta_1$ then $|f(x) - f(y)| < \epsilon$. Take $\delta = \min\{a_1 - a, b - b_1, \delta_1/4\}$. Argue that for n sufficiently large the result from Exercise 2(b) can be applied to the list A of values of f on partition midpoints in (a, b) and the sublist B of values of f on partition midpoints in $[a_1, b_1]$.)

8. Suppose that f is continuous on (a, b). Use the results of Exercises 6 & 7 to prove that $\text{med}(n)$ is a Cauchy sequence. Conclude that f_{med} exists for any function f that is continuous on a bounded interval with endpoints $a < b$. (*Hint:* Let $\epsilon > 0$ be given and choose δ and N_1 such that for any function g that is continuous on (a, b) and equal to f on $[a + \delta, b - \delta]$ and for any partition of order $n \geq N_1$ the difference between $\text{med}_f(n)$ and $\text{med}_g(n)$ will be less than $\epsilon/3$. Take g to be the function that is equal to f on $[a + \delta, b - \delta]$ and constant on $[a, a + \delta]$ and on $[b - \delta, b]$. Since g is continuous on $[a, b]$ the argument in Exercise 6 shows that there exists N_2 such that for all $n, m \geq N_2$ we have $|\text{med}_g(n) - \text{med}_g(m)| < \epsilon/3$. Let $N = \max\{N_1, N_2\}$ and apply a triangle inequalities argument to show that for all $n, m \geq N$ we have $|\text{med}_f(n) - \text{med}_f(m)| < \epsilon$.)

Filling in the Blanks

Exercise 9 provides a proof of Proposition 1. Exercises 10–12 give some details of the proof of Theorem 2 that were omitted in our article. (A proof of Theorem 3 requires some concepts from measure theory and appears later in Exercise 32.)

9. Parts (a)–(c) of this exercise ask you to prove that $A(t)$ is a convex function. Parts (d)–(f)

ask you to show that there is a unique real number t_m such that $A(t_m)$ is the minimum value m of the area function.

Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is *convex* provided the inequality

$$h[(1-\lambda)t_0 + \lambda t_1] \leq (1-\lambda)h(t_0) + \lambda h(t_1)$$

is satisfied for all real numbers $t_0 < t_1$ and for all $0 \leq \lambda \leq 1$.

(a) Suppose that f is a continuous function on an interval $[a, b]$ and we let $g(t, x) = |f(x)-t|$. Use the triangle inequality to show that

$$g[(1-\lambda)t_0 + \lambda t_1, x] \leq (1-\lambda)g(t_0, x) + \lambda g(t_1, x)$$

for $t_0 < t_1$, $0 \leq \lambda \leq 1$, and $a \leq x \leq b$.

(b) Explain why the inequality in part (a) is strict if $t_0 < f(x) < t_1$ and $0 < \lambda < 1$.

(c) Use the inequality in part (a) along with the linearity and order properties of the definite integral to show that

$$A[(1-\lambda)t_0 + \lambda t_1] \leq (1-\lambda)A(t_0) + \lambda A(t_1)$$

for all real numbers $t_0 < t_1$ and for all $0 \leq \lambda \leq 1$. Conclude that $A(t)$ is a convex function.

(d) Explain why the inequality in part (c) is strict if $0 < \lambda < 1$ and if there exists $a \leq x \leq b$ such that $t_0 < f(x) < t_1$.

In the next two parts suppose, by way of contradiction, that $A(t_0) = A(t_1) = m$ for $t_0 < t_1$.

(e) Explain why there must be a point on the graph of f that is between the horizontal lines $y = t_0$ and $y = t_1$. (*Hint:* Argue that if not then one of the two lines $y = t_0$ or $y = t_1$ will not be area minimizing.)

(f) Use the inequality in part (c) with $\lambda = 1/2$ and the result of part (d) to derive a contradiction. Conclude that there is a unique value $t = t_m$ that satisfies $A(t_m) = m$.

10. In this exercise you will justify the statement after Equation (9) in our article: "In fact, N can be chosen *independently* of t. " It assumes familiarity with the definition of the Riemann integral in terms of upper and lower sums. If g is a bounded function defined on

(a, b) and $\mathcal{P}$ is a partition of $[a, b]$ we let $U(g, \mathcal{P})$ and $L(g, \mathcal{P})$ denote the upper and lower sums of g over $\mathcal{P}$.

(a) If f is a bounded function on (a, b) explain why

$$U(f - t, \mathcal{P}) - L(f - t, \mathcal{P}) = U(f, \mathcal{P}) - L(f, \mathcal{P})$$

for any real number t.

(b) Use the reverse triangle inequality to prove that

$$U(|g|, \mathcal{P}) - L(|g|, \mathcal{P}) \leq U(g, \mathcal{P}) - L(g, \mathcal{P})$$

for any bounded function g on (a, b).

(c) Conclude from parts (a) & (b) that

$$U(|f - t|, \mathcal{P}) - L(|f - t|, \mathcal{P}) \leq U(f, \mathcal{P}) - L(f, \mathcal{P})$$

for every bounded function f on (a, b) and for every real number t.

(d) Assume now that f is a bounded continuous function on (a, b). Prove that for any $\epsilon > 0$ there exists a natural number N such that for any regular partition $\mathcal{P}$ of order $n \geq N$ we have

$$U(|f - t|, \mathcal{P}) - L(|f - t|, \mathcal{P}) < \epsilon/2$$

for all real numbers t. Conclude that $|A(t) - \mathcal{M}_n(t)| < \epsilon/2$ for $n \geq N$ no matter what the value of t. (*Hint:* Since f is a bounded continuous function on (a, b), f is integrable on (a, b). Hence there exists a natural number N such that for any regular partition $\mathcal{P}$ of order $n \geq N$ we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon/2$$

Combine this with the inequality in part (c) to conclude that for all real numbers t

$$U(|f - t|, \mathcal{P}) - L(|f - t|, \mathcal{P}) < \epsilon/2$$

Finally, use the inequalities

$$L(|f - t|, \mathcal{P}) \leq A(t) \leq U(|f - t|, \mathcal{P}) \text{ and } L(|f - t|, \mathcal{P}) \leq \mathcal{M}_n(t) \leq U(|f - t|, \mathcal{P})$$

that are valid for all regular partitions.)

11. In our article we proved the existence of f_{med} for f a *bounded* continuous function on (a, b). In this exercise you will extend the existence proof to *unbounded* functions. Suppose that f is an unbounded continuous function on an open interval (a, b). For any $0 < \delta \leq (b - a)/8$ let M_δ denote a number greater than the maximum value of $|f(x)|$ on $[a+\delta, b-\delta]$. (Readers concerned about the axiom of choice can take M_δ to be 1 unit greater than the maximum value of $|f(x)|$ on $[a + \delta, b - \delta]$.) Define the truncated function f_δ by

$$f_\delta(x) = \begin{cases} M_\delta \ , & f(x) > M_\delta \\ f(x) \ , & -M_\delta \leq f(x) \leq M_\delta \\ -M_\delta \ , & f(x) < -M_\delta \end{cases}$$

(a) Prove that f_δ is continuous and bounded on (a, b).

(b) Given a regular partition let $\text{med}_\delta(n)$ denote the median of the list of truncated function values $f_\delta(x_1^*),\ f_\delta(x_2^*), \ldots, f_\delta(x_n^*)$. Show that $-M_\delta < \text{med}(n) = \text{med}_\delta(n) < M_\delta$ for every natural number n.

(c) Conclude that f_{med} exists (and is equal to the median value of any f_δ).

12. Suppose that f is continuous and unbounded on (a, b) and that the improper integral $\int_a^b |f(x)|\ dx$ converges. For any $0 < \delta \leq (b - a)/8$ let

$$A_\delta(t) = \int_a^b |f_\delta(x) - t|\ dx$$

where $f_\delta(x)$ is defined as in Exercise 11.

(a) Show that

$$A(t) = \lim_{\delta \to 0^+} A_\delta(t) \tag{1}$$

for all real numbers t.

(b) Conclude that $A(f_{\text{med}}) \leq A(t)$ for all real numbers t. (*Hint:* Use the result in part (c) of Exercise 11.)

Trust but Verify

13. Verify the solution to Example 2 by finding an algebraic formula for $E[|Y - t|]$.

14. Verify the result of Example 3 by finding an algebraic formula for the area between the line $y = tx$ and the parabola $y = x^2$.

15. Verify Theorem 1 by direct computation of $A(t)$ for the function f defined piecewise on $[0,1]$ by

$$f(x) = \begin{cases} 2x\ , & 0 \le x < 1/4 \\ 1/2\ , & 1/4 \le x \le 3/4 \\ 2x-1\ , & 3/4 < x \le 1 \end{cases}$$

16. (a) Use Theorem 1 and the property of central symmetry to give an alternative solution to Example 1.

(b) Use Theorem 3 and the property of central symmetry to give an alternative solution to Example 1.

17. Consider the area minimization problem of Example 3 with the parabola $y = x^2$ replaced by: (a) $y = f(x) = x^n$ for $n \ge 3$ an integer, (b) the graph of $y = f(x) = \sin \pi x$, (c) $y = f(x)$ for f a positive–valued, differentiable function that satisfies $f'(x) < 0$ and (d) $y = f(x)$ for f a nonnegative differentiable function that satisfies $f(0) = 0$ and $f''(0) < 0$. In each part verify that the area minimizing line and the graph of f cross at $x = 1/\sqrt{2}$.

18. Justify the comments made about Figures 8 and 9 in our article. (*Hint:* First make the change of variables $u = x^2/2$. Apply Theorem 2 where appropriate.)

19. Use Theorem 2 and a change of variables to find the parabolic arc $y = ax^2$ such that the area between the arc and the parabola $y = 1 - x^2$ over $[0,1]$ is a minimum. Verify the answer by finding an algebraic formula for the relevant area.

20. Use Theorem 3 to find the (nonvertical) line through the origin such that the area between the line and the parabola $y = (1-x)^2$ over $[0,2]$ is a minimum. With the help of a CAS verify your solution by direct computation.

A Median Miscellany

21. Assume f is a strictly increasing continuous function on $[a,b]$ that has a differentiable inverse function f^{-1}. Prove that if t belongs to $(f(a), f(b))$ then

$$A(t) = \int_a^b |f(x) - t|\ dx = 2tf^{-1}(t) - (a+b)t - \int_a^{f^{-1}(t)} f(x)\ dx - \int_b^{f^{-1}(t)} f(x)\ dx$$

and $A'(t) = 2f^{-1}(t) - (a+b)$. (*Hint:* Use the Fundamental Theorem of Calculus and the Chain Rule.) Conclude that the area minimizing horizontal line is $y = f_{\text{med}}$. Draw the same conclusion for a strictly decreasing function.

22. Assume that f is continuous and nondecreasing on $[a, b]$, but not strictly increasing. For each t in the range of f choose an element $a \le x_t \le b$ such that $f(x_t) = t$. Derive the formula

$$A(t) - A(f_{\text{med}}) = 2\int_{x_t}^{(a+b)/2} [f(x) - t]\ dx$$

and use it to conclude that the area minimizing horizontal line is $y = f_{\text{med}}$. Draw the same conclusion for a continuous and nonincreasing function on $[a, b]$.

23. Let f denote a continuous function on $[a, b]$ whose graph is on or below the horizontal line $y = f[(a+b)/2]$ to the left of the vertical line $x = (a+b)/2$, and is on or above this horizontal line to the right of the line $x = (a+b)/2$.

(a) Express $A(t) - A(f_{\text{med}})$ as a single integral.

(b) If $t < f_{\text{med}}$ prove that $A(t) > A(f_{\text{med}})$. (*Hint:* Write the integral from part (a) as the sum of two integrals, one over $[a, (a+b)/2]$ and the other over $[(a+b)/2, b]$. Show that the latter integral is equal to $[f_{\text{med}} - t](b-a)/2$ while the former integral is strictly greater than $-[f_{\text{med}} - t](b-a)/2$.)

(c) Prove that $A(t) > A(f_{\text{med}})$ if $t > f_{\text{med}}$.

(d) Conclude that the area minimizing horizontal line is $y = f_{\text{med}}$. Draw the same conclusion for a continuous function whose graph is on or above the horizontal line $y = f[(a+b)/2]$ to the left of the vertical line $x = (a+b)/2$, while the reverse is the case to the right of the vertical line $x = (a+b)/2$.

24. Assume that f is a bounded continuous function on an open interval (a, b). Use the result of Exercise 10, the continuity of $A(t)$, and a triangle inequalities argument to show that if $t_n \to L$ as $n \to +\infty$ then

$$\lim_{n\to+\infty} \mathcal{M}_n(t_n) = A(L)$$

25. Combine the existence proof in Exercise 6, inequality (6) in the article, and the result of Exercise 24 to give an alternate proof that $t = f_{\text{med}}$ minimizes $A(t)$ in the special case of a continuous function f on a closed interval.

26. Let f_{med} denote the median value of a continuous function on a bounded interval with endpoints $a < b$. Prove that, like f_{ave}, there exists $a \le c \le b$ such that $f(c) = f_{\text{med}}$.

27. Give examples of continuous functions f and g on a closed interval $[a, b]$ such that $(f + g)_{\text{med}} \neq f_{\text{med}} + g_{\text{med}}$.

28. Our definition of f_{med} is based upon regular partitions. Suppose we remove this restriction and replace $\lim\limits_{n\to+\infty}$ in Definition (3) of our article by $\lim\limits_{\max\,\Delta x_k\to 0}$. Show by example that this approach fails to yield a viable alternative definition of the median.

29. Let f and g denote continuous functions on $[a, b]$ and assume that $g(x) > 0$ on (a, b). With the help of the median, study the problem of minimizing the area between the graph of f and a curve in the family $y = tg(x), t \in \mathbb{R}$.

Small, Median, and Large

This collection of exercises assumes familiarity with Lebesgue measure and the Lebesgue integral. In Exercises 30–38 we will assume that f is a measurable function on a bounded interval with endpoints $a < b$. If U is a measurable set then $m[U]$ will denote the Lebesgue measure of U. If f is absolutely (Lebesgue) integrable then we define the "area function" $A(t)$ by

$$A(t) = \int_a^b |f(x) - t|\ dx$$

30. (a) Use an argument similar to that in Exercise 9 to show that $A(t)$ is a convex (hence continuous) function of t that satisfies

$$\lim_{t\to-\infty} A(t) = +\infty \quad \text{and} \quad \lim_{t\to+\infty} A(t) = +\infty$$

Conclude that $A(t)$ has an absolute minimum value m.

(b) Use convexity to argue that there exist real numbers $t_1 \le t_2$ such that $A(t)$ is strictly decreasing on the interval $(-\infty, t_1]$, is strictly increasing on the interval $[t_2, +\infty)$, and satisfies $A(t) = m$ for $t_1 \le t \le t_2$.

The next sequence of exercises will tease out more fully the nature of the points $t_1 \le t_2$ in Exercise 30(b). Define a "measure theoretic median" of f to be any real number t such that

$$m[\text{Below}(t)] \leq \frac{b-a}{2} \qquad \text{and} \qquad m[\text{Above}(t)] \leq \frac{b-a}{2}$$

31. Define

$$t_1 = \inf\ \{t \mid m[\text{Above}(t)] \leq \frac{b-a}{2}\} \text{ and } t_2 = \sup\ \{t \mid m[\text{Below}(t)] \leq \frac{b-a}{2}\}$$

(The choice of notation for the points t_1 and t_2 is deliberate. We will see in Exercise 33 that for an absolutely integrable function these points coincide with $t_1 \leq t_2$ in Exercise 30(b).)

(a) Prove that t_1 and t_2 exist, are finite, and satisfy $t_1 \leq t_2$.

(b) Prove that t is a measure theoretic median of f if and only if $t_1 \leq t \leq t_2$.

(c) Assume that $t_1 < t_2$. Prove that

(i) $m[\text{Above}(t_1)] = m[\text{Below}(t_2)] = (b-a)/2$

(ii) On (t_1, t_2): $m[\text{Above}(t)] = m[\text{Below}(t)] = \dfrac{b-a}{2}$

We will refer to t_1 as the smallest median value of f on $[a, b]$ and to t_2 as the largest median value of f on $[a, b]$. If $t_1 = t_2$ then we will denote their common value by $f_{\mathcal{MED}}$ which we will refer to as the measure theoretic median of f.

32. Assume that f is a continuous function on (a, b).

(a) Prove that $t_1 = t_2 = f_{\mathcal{MED}}$. (*Hint:* Show that continuity disallows the conclusions of Exercise 31(c).)

(b) Use Exercises 1 and 4 to give a proof that $\lim_{n\to+\infty} \text{med}(n)$ exists and is equal to the measure theoretic median $f_{\mathcal{MED}}$ of f on (a, b). Explain why this result implies Theorem 3 in our article. (*Hint:* Let $\epsilon > 0$ be given. Use the fact that $f_{\mathcal{MED}} = t_1$ to argue that Above$(f_{\mathcal{MED}} - \epsilon)$ is an open set of measure greater than $(b-a)/2$. Conclude that there exists a finite collection of disjoint open intervals that are subsets of Above$(f_{\mathcal{MED}} - \epsilon)$ and have total measure greater than $(b-a)/2$. Use Exercise 4 to argue that for large n more than half the midpoints of a regular partition will belong to Above$(f_{\mathcal{MED}} - \epsilon)$. Then apply the result of Exercise 1(c). Repeat the argument using the open set Below$(f_{\mathcal{MED}} + \epsilon)$.)

33. Assume that f is absolutely integrable on (a, b).

(a) Fix a t–value t_0 and let $\Delta t = t - t_0$ for $t > t_0$. Prove that

$$|f(x) - t| - |f(x) - t_0| = \begin{cases} \Delta t\ , & f(x) \le t_0 \\ -\Delta t + 2(t - f(x))\ , & t_0 < f(x) < t \\ -\Delta t\ , & t \le f(x) \end{cases}$$

(b) Let $\Delta A = A(t) - A(t_0)$. Using (Lebesgue) integration and the result of part (a) show that for $t > t_0$

$$\Delta A = 2\left[\left(\frac{b-a}{2} - m[\text{Above}(t_0)]\right)\Delta t + \int_{\text{Above}(t_0)\cap\text{Below}(t)} (t - f(x))\ dx\right]$$

(c) Fix a t–value t_0 and let $\Delta t = t - t_0$ for $t < t_0$. Use an argument similar to that in part (b) to show that $\Delta A = A(t) - A(t_0)$ is given by

$$\Delta A = 2\left[\left(m[\text{Below}(t_0)] - \frac{b-a}{2}\right)\Delta t + \int_{\text{Below}(t_0)\cap\text{Above}(t)} (f(x) - t)\ dx\right]$$

(d) Conclude from parts (b) and (c) that $A(t)$ is minimized by any measure theoretic median t_0 of f on $[a, b]$.

(e) Let t_1 and t_2 denote respectively the smallest and largest median values of f on $[a, b]$. Use parts (b) and (c) to prove that $A(t)$ is strictly decreasing on $(-\infty, t_1]$ and is strictly increasing on $[t_2, +\infty)$. Conclude that t_1 and t_2 are the values described in Exercise 30(b) and thus that $t = t_0$ minimizes $A(t)$ if and only if t_0 is a median value of f on $[a, b]$.

34. (a) It is well–known that a convex function has one–sided derivatives everywhere. Derive the formulas,

$$A'_+(t_0) = 2\left[\frac{b-a}{2} - m[\text{Above}(t_0)]\right] \quad \text{and} \quad A'_-(t_0) = 2\left[m[\text{Below}(t_0)] - \frac{b-a}{2}\right]$$

(*Hint:* For the right–hand derivative use the fact that

$$\bigcap_{t_0 < t} \text{Above}(t_0) \cap \text{Below}(t) = \emptyset$$

and

$$0 < t - f(x) < \Delta t \quad \text{for } x \in \text{Above}(t_0) \cap \text{Below}(t)$$

to conclude that $\displaystyle\lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{\text{Above}(t_0)\cap\text{Below}(t)} (t - f(x))\ dx = 0.$)

(b) Verify the formulas in part (a) for the function $A(t)$ in Exercise 15.

(c) Prove that $A'_-(t) \leq A'_+(t)$, that both $A'_-(t)$ and $A'_+(t)$ are monotonically increasing (that is, nondecreasing) functions, and that if $t < s$ then $A'_+(t) \leq A'_-(s)$.

(d) Let t_1 and t_2 denote respectively the smallest and largest median values of f on $[a, b]$ and suppose that $t_1 < t_2$. In this case we saw in Exercise 30(b) that $A(t)$ is constant on the interval $[t_1, t_2]$. Show that (as would be expected) the formulas in part (a) yield $A'(t) = 0$ for $t_1 < t < t_2$.

35. Use the formulas in Exercise 34(a) to give another proof that if $t = t_0$ minimizes $A(t)$ then t_0 is a median value of f.

36. It is well–known that a convex function is differentiable except at most countably many points. Use the inequalities in Exercise 34(c) to give a direct proof of this result for $A(t)$. (*Hint:* Consider the function $\phi(t) = (A'_-(t), A'_+(t))$ whose domain is the set of points at which $A(t)$ is not differentiable and whose codomain is the set of open intervals of real numbers. Argue that ϕ takes distinct points to disjoint open intervals.)

37. Let $\text{Equal}(t) = \{x \in (a, b) \bigm| f(x) = t\}$.

(a) Use the formulas in Exercise 34(a) to prove that $A(t)$ is differentiable at $t = t_0$ if and only if $m[\text{Equal}(t_0)] = 0$.

(b) Use the result of part (a) to give another proof (distinct from that in Exercise 36) that $A'(t)$ is differentiable except at most countably many points. (*Hint:* For each natural number n argue that the set of points t such that $m[\text{Equal}(t)] > 1/n$ is finite.)

(c) Prove that if $A(t)$ is differentiable at $t = t_0$ then $A'(t_0) = m[\text{Below}(t_0)] - m[\text{Above}(t_0)]$.

(d) Show that the formula in part (c) agrees with that of Exercise 21.

(e) Explain how the formula for $A'(t_0)$ in part (c) can be interpreted as the result of "differentiation under the integral sign". (The authors thank Jim Hartman, College of Wooster, for suggesting parts (c) and (e) of this exercise.)

38. Assume that $m[\text{Equal}(t)] = 0$ for all t. (See Exercise 37.)

(a) Prove that $A(t)$ is continuously differentiable.

(b) Assume that f is continuous. Prove that f has a unique critical point $t = f_{\mathrm{med}}$ and that the graph of $A(t)$ is concave up in a neighborhood of f_{med}.

39. Suppose that f is a measurable absolutely integrable function on an interval with endpoints $a < b$. Let X denote a random variable that is uniformly distributed over the interval and let $Y = f(X)$ denote the associated composite random variable. Derive the formula $D_+ (E[|Y - t|]) = 2P[Y \leq t] - 1$ for the right–hand derivative of the average absolute deviation function. Conclude that the probability distribution function $P[Y \leq t]$ is determined uniquely by $E[|Y - t|]$.