ONLINE SUPPLEMENT TO “THE JAMES FUNCTION”

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We consider several technical questions relating to Jamesian functions, including the construction of a discontinuous Jamesian function and a function that satisfies the proto-James conditions but not the James conditions. We also obtain a more concrete representation for some of the hyper-James functions.

1. Introduction

When studying the James function

$$P(a,b)=\frac{a(1-b)}{a(1-b)+b(1-a)},$$

it is helpful to consider certain sets of axiomatic conditions. The following set, which we refer to as the *proto-James conditions*, was introduced by Bill James in his original treatment of the so-called “log5 method” [4]:

(1) $P(a,a)=\frac{1}{2}$.
(2) $P(a,\frac{1}{2})=a$.
(3) If $a>b$ then $P(a,b)>\frac{1}{2}$, and if $a<b$ then $P(a,b)<\frac{1}{2}$.
(4) Let $0<a<1$. If $b<\frac{1}{2}$ then $P(a,b)>a$, and if $b>\frac{1}{2}$ then $P(a,b)<a$.
(5) $0\le P(a,b)\le 1$, and if $0<a<1$ then $P(a,0)=1$ and $P(a,1)=0$.
(6) $P(a,b)+P(b,a)=1$.

In our primary article on the James function [3], we introduced a set of conditions called the *James conditions*:

(a) $P(a,\frac{1}{2})=a$.
(b) $P(a,0)=1$ for $0<a\le 1$.
(c) $P(b,a)=1-P(a,b)$.
(d) $P(1-b,1-a)=P(a,b)$.
(e) $P(a,b)$ is a non-decreasing function of a for $0\le b\le 1$ and a strictly increasing function of a for $0<b<1$.

These conditions apply to functions defined on the set $\overline{S}\setminus\{(0,0)\cup(1,1)\}$, where $S=(0,1)\times(0,1)$. Any function that satisfies the James conditions must also satisfy the proto-James conditions. We refer to any function that satisfies the James conditions, including the James function itself, as a *Jamesian function*. In [3], we identified a particular class of Jamesian functions that we designated *hyper-James functions.*

This note has two primary objectives:

- To construct an example of a Jamesian function with discontinuities in S.
- To obtain a more concrete representation for some of the hyper-James functions.

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In the process of considering the first objective, we will also identify a function that satisfies the proto-James conditions but not the James conditions.

2. Discontinuous Jamesian functions

It is possible to use any Jamesian function $J(a,b)$ to construct a discontinuous Jamesian function. Take $0 < \gamma < 1$ and define the function $\widetilde{J}_\gamma(a,b)$ as follows:

$$\widetilde{J}_\gamma(a,b) = \begin{cases} J(a,b), & (a,b) \in A = \left([0,\tfrac{1}{2}] \times [0,\tfrac{1}{2}]\right) \setminus \{(0,0)\} \\ \gamma J(a,b), & (a,b) \in B = [0,\tfrac{1}{2}) \times (\tfrac{1}{2},1] \\ \gamma J(a,b) + 1 - \gamma, & (a,b) \in C = (\tfrac{1}{2},1] \times [0,\tfrac{1}{2}) \\ J(a,b), & (a,b) \in D = \left([\tfrac{1}{2},1] \times [\tfrac{1}{2},1]\right) \setminus \{(1,1)\} \end{cases},$$

as illustrated in Figure 1.

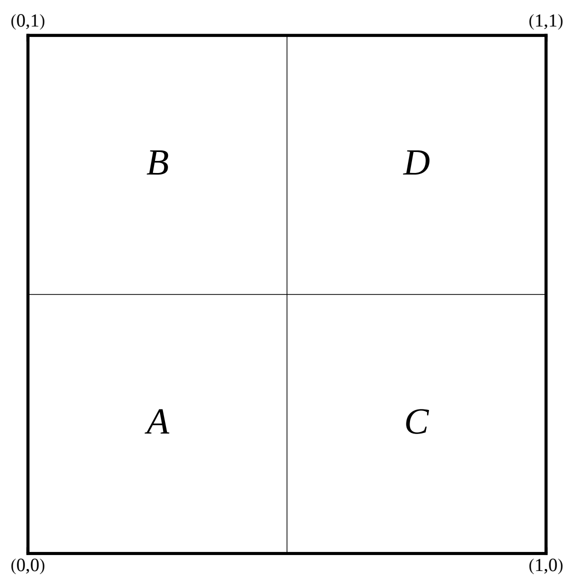

Figure 1. The subsets of $\overline{S} \setminus \{(0,0) \cup (1,1)\}$.

It should be apparent from the definition that $\widetilde{J}_\gamma(a,b)$ satisfies James conditions (a) and (b). Condition (e) follows from the inequalities

$$0 \le \gamma x \le x$$

and

$$x \le \gamma x + 1 - \gamma \le 1$$

for $0 \le x \le 1$, along with the fact that γx and $\gamma x + 1 - \gamma$ are both strictly increasing functions of x. We will now verify the two remaining conditions:

(c) If a point (a,b) belongs either to the subset A or the subset D, then (b,a) belongs to the same subset. In either case,

$$\widetilde{J}_\gamma(b,a) = J(b,a) = 1 - J(a,b) = 1 - \widetilde{J}_\gamma(a,b).$$

If (a,b) belongs to B, then (b,a) belongs to C, from which it follows that

$$\begin{aligned}\widetilde{J}_\gamma(b,a) = \gamma J(b,a) + 1 - \gamma =& \gamma\big(1 - J(a,b)\big) + 1 - \gamma \\ =& 1 - \gamma J(a,b) = 1 - \widetilde{J}_\gamma(a,b).\end{aligned}$$

An analogous argument pertains to the case where (a,b) belongs to C.

(d) If a point (a,b) belongs either to B or to C, then $(1-b,1-a)$ belongs to the same subset. If (a,b) belongs to A or D, then $(1-b,1-a)$ belongs to D or A respectively. In all these cases, it follows that $\widetilde{J}_\gamma(1-b,1-a) = \widetilde{J}_\gamma(a,b)$, since $J(a,b)$ possesses the same property.

In other words, every function defined in this manner for $0 < \gamma < 1$ must be a Jamesian function. Note that $\widetilde{J}_\gamma(a,b)$ is discontinuous at every point of the form $(a, \frac{1}{2})$ for $0 < a < 1$ or $(\frac{1}{2}, b)$ for $0 < b < 1$, which is the purpose of this construction.

It is also instructive to consider the situation where $\gamma = 0$:

$$\widetilde{J}_0(a,b) = \begin{cases} J(a,b), & (a,b) \in A = \big([0,\frac{1}{2}] \times [0,\frac{1}{2}]\big) \setminus \{(0,0)\} \\ 0, & (a,b) \in B = [0,\frac{1}{2}) \times (\frac{1}{2}, 1] \\ 1, & (a,b) \in C = (\frac{1}{2}, 1] \times [0, \frac{1}{2}) \\ J(a,b), & (a,b) \in D = \big([\frac{1}{2},1] \times [\frac{1}{2},1]\big) \setminus \{(1,1)\} \end{cases}.$$

Because of its behavior on the subsets B and C, this function does not satisfy James condition (e). One can readily verify, however, that it does satisfy all six of the proto-James conditions. In other words, this function provides a counterexample to the James conjecture, as stated in [3], that is not actually a Jamesian function.

The following proposition, whose proof we leave to the reader, is similar in spirit to the preceding construction.

Proposition 1. *Suppose that $J_1(a,b), J_2(a,b), \ldots, J_n(a,b)$ are Jamesian functions and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive constants with $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$. The linear combination*

$$\alpha_1 J_1(a,b) + \alpha_2 J_2(a,b) + \ldots + \alpha_n J_n(a,b)$$

is also a Jamesian function.

3. A Partial Fractions Expansion

Theorem 9 in [3] shows that

$$J(a,b) = g^{-1}\big(g(a) - g(b)\big)$$

is a Jamesian function whenever $g\colon (0,1) \to \mathbb{R}$ is a continuous, strictly increasing function with $g(1-a) = -g(a)$ and $\lim_{a \to 0^+} g(a) = -\infty$. In the particular case where

$$g_n(a) = \int_{\frac{1}{2}}^{a} \frac{1}{(t(1-t))^n} dt \tag{3.1}$$

for some real number $n > 1$, we refer to the resulting function

$$H_n(a,b) = g_n^{-1}\big(g_n(a) - g_n(b)\big)$$

as a *hyper-James function.*

The following partial fraction decomposition pertains to the integral that defines the function g_n.

Lemma 2. *For positive integers m and n, we have*

$$\frac{1}{t^m(1-t)^n} = \sum_{k=0}^{m-1} \binom{n+k-1}{k} \frac{1}{t^{m-k}} + \sum_{k=0}^{n-1} \binom{m+k-1}{k} \frac{1}{(1-t)^{n-k}}. \tag{3.2}$$

Proof. For $n = 1$ and an arbitrary positive integer m we can write

$$\begin{aligned}\frac{1}{1-t} + \frac{1}{t} + \frac{1}{t^2} + \cdots + \frac{1}{t^m} &= \frac{1}{1-t} + \frac{1}{t}\left(\frac{1-\frac{1}{t^m}}{1-\frac{1}{t}}\right) \\ &= \frac{1}{1-t} - \frac{1}{1-t}\left(1 - \frac{1}{t^m}\right) \\ &= \frac{1}{t^m(1-t)},\end{aligned}$$

so (3.2) holds in this case. Equation (3.2) is unchanged if we switch m and n while changing t to $1-t$, so it also holds for $m = 1$ and an arbitrary positive integer n.

Now assume (3.2) holds for all values of m and n with the fixed sum $m+n+1$, and consider

$$\frac{1}{t^{m+1}(1-t)^{n+1}} = \frac{1-t+t}{t^{m+1}(1-t)^{n+1}} = \frac{1}{t^{m+1}(1-t)^n} + \frac{1}{t^m(1-t)^{n+1}}$$

for positive integers m and n. By induction, we have

$$\begin{aligned}\frac{1}{t^{m+1}(1-t)^{n+1}} &= \sum_{k=0}^{m} \binom{n+k-1}{k} \frac{1}{t^{m+1-k}} + \sum_{k=0}^{n-1} \binom{m+k}{k} \frac{1}{(1-t)^{n-k}} \\ &\quad + \sum_{k=0}^{m-1} \binom{n+k}{k} \frac{1}{t^{m-k}} + \sum_{k=0}^{n} \binom{m+k-1}{k} \frac{1}{(1-t)^{n+1-k}}.\end{aligned}$$

Changing k to $k-1$ in the second and third sums then gives

$$\begin{aligned}\frac{1}{t^{m+1}(1-t)^{n+1}} &= \sum_{k=0}^{m} \binom{n+k-1}{k} \frac{1}{t^{m+1-k}} + \sum_{k=1}^{n} \binom{m+k-1}{k-1} \frac{1}{(1-t)^{n+1-k}} \\ &\quad + \sum_{k=1}^{m} \binom{n+k-1}{k-1} \frac{1}{t^{m+1-k}} + \sum_{k=0}^{n} \binom{m+k-1}{k} \frac{1}{(1-t)^{n+1-k}} \\ &= \sum_{k=0}^{m} \binom{n+k}{k} \frac{1}{t^{m+1-k}} + \sum_{k=0}^{n} \binom{m+k}{k} \frac{1}{(1-t)^{n+1-k}},\end{aligned}$$

as desired. □

We have found Lemma 2 in several different forms in the literature. Changing t to $\frac{p}{p+q}$, one obtains

$$\frac{1}{p^m q^n} = \sum_{k=0}^{m-1} \binom{n+k-1}{k} \frac{1}{p^{m-k}(p+q)^{n+k}} + \sum_{k=0}^{n-1} \binom{m+k-1}{k} \frac{1}{q^{n-k}(p+q)^{m+k}}. \tag{3.3}$$

This result appears in Eisenstein's great paper [1], which was beautifully recapitulated and extended by Weil [5]. Weil suggests proving (3.3) by starting with

$$\frac{1}{pq} = \frac{1}{p(p+q)} + \frac{1}{q(p+q)},$$

the case where $m = n = 1$, and taking $m-1$ derivatives with respect to p and $n-1$ derivatives with respect to q. Euler gives the less attractive form

$$\frac{1}{x^m(x+a)^n} = \sum_{k=0}^{m-1} \binom{n+k-1}{k} \frac{(-1)^k}{x^{m-k}a^{n+k}} + (-1)^m \sum_{k=0}^{n-1} \binom{m+k-1}{k} \frac{1}{(x+a)^{n-k}a^{m+k}}$$

in [2]. This equation can be obtained by taking $p = -x$ and $q = x + a$ in (3.3).

Applying Lemma 2, we see that

$$g_n(a) = \binom{2n-2}{n-1} \log\left(\frac{a}{1-a}\right) + \sum_{k=0}^{n-2} \binom{n+k-1}{k} \frac{1}{n-k-1} \left(\frac{1}{(1-a)^{n-k-1}} - \frac{1}{a^{n-k-1}}\right)$$

for any integer $n \geq 1$, where the sum is empty if $n = 1$. While this formula is certainly worth having, we still cannot obtain an explicit representation for $H_n(a, b)$ without a similar formula for g_n^{-1}.

It is also possible to evaluate (3.1) with n replaced by $m + \frac{1}{2}$ for a positive integer m. Substituting $t = \sin^2\theta$, we have

$$\int_{\frac{1}{2}}^{a} \frac{dt}{(t(1-t))^{m+\frac{1}{2}}} = 2\int_{\frac{\pi}{4}}^{\sin^{-1}\sqrt{a}} \frac{d\theta}{\sin^{2m}\theta\cos^{2m}\theta}.$$

One can break this expression apart using Lemma 2, with $t = \sin^2\theta$, but it is better to write

$$\begin{aligned} 2\int_{\frac{\pi}{4}}^{\sin^{-1}\sqrt{a}} \frac{d\theta}{\sin^{2m}\theta\cos^{2m}\theta} &= 2^{2m}\int_{\frac{\pi}{4}}^{\sin^{-1}\sqrt{a}} \frac{2\,d\theta}{\sin^{2m}(2\theta)} \\ &= 2^{2m}\int_{\frac{\pi}{2}}^{2\sin^{-1}\sqrt{a}} \left(\cot^2\phi + 1\right)^{m-1} \csc^2\phi\, d\phi. \end{aligned}$$

Substituting $u = \cot\phi$ gives us

$$\begin{aligned} g_{m+\frac{1}{2}}(a) =& 2^{2m}\int_{\frac{1-2a}{2\sqrt{a(1-a)}}}^{0} (u^2+1)^{m-1}\, du \\ =& \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{2^{2m-2k-1}}{2k+1} \left(\frac{2a-1}{\sqrt{a(1-a)}}\right)^{2k+1}. \end{aligned}$$

In particular, taking $m = 1$ we have

$$g_{\frac{3}{2}}(a) = \frac{2(2a-1)}{\sqrt{a(1-a)}}.$$

In this case, since

$$g_{\frac{3}{2}}^{-1}(s) = \frac{s + \sqrt{s^2 + 16}}{2\sqrt{s^2 + 16}},$$

we see that

$$H_{\frac{3}{2}}(a, b) = \frac{1}{2} + \frac{v'\sqrt{u} - u'\sqrt{v}}{2\sqrt{u + v - 4uv - 2u'v'\sqrt{uv}}}$$

for $u = a(1 - a)$, $v = b(1 - b)$, $u' = 1 - 2a$, and $v' = 1 - 2b$, as stated in [3].

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