# Appendix to Modeling a Diving Board

Michael A. Karls
Department of Mathematical Sciences
Ball State University
Muncie, Indiana 47306–0490
mkarls@bsu.edu

Brenda M. Skoczelas
Department of Physics and Astronomy
Ball State University
Muncie, Indiana 47306–0490
bmskoczelas@bsu.edu

## Solving Our Model

The standard way to solve problems such as the following model for the vertical displacement $y(x,t)$ of a cantilever beam with left end fixed ($x = 0$) and right end free ($x = L$),

$$\frac{\partial^2 y}{\partial t^2} = -c^2 \frac{\partial^4 y}{\partial x^4} - k\frac{\partial y}{\partial t} - g; \;\; 0 < x < L, \;\; t > 0, \tag{1}$$

$$y(0,t) = 0, \;\; t > 0, \tag{2}$$

$$\frac{\partial y}{\partial x}(0,t) = 0, \;\; t > 0, \tag{3}$$

$$\frac{\partial^2 y}{\partial x^2}(L,t) = 0, \;\; t > 0, \tag{4}$$

$$\frac{\partial^3 y}{\partial x^3}(L,t) = 0, \;\; t > 0, \tag{5}$$

$$y(x,0) = f(x), \; 0 < x < L, \tag{6}$$

$$\frac{\partial y}{\partial t}(x,0) = g(x), \; 0 < x < L, \tag{7}$$

is to reduce the partial differential problem to simpler ordinary differential equations. In this case, we look at two related problems – the *steady–state problem* and the *transient problem.*

After the beam has been set in motion and a large amount of time has passed, we'd expect the displacement $y$ to be independent of time, i.e. as $t \to \infty$, $y(x,t) \to v(x)$. With this assumption in (1)–(7), we find that $v(x)$ must satisfy the ordinary differential equation

$$0 = -c^2 \frac{d^4 v}{dx^4} - g; \quad 0 < x < L, \tag{8}$$

with boundary data

$$v(0) = 0,\ v'(0) = 0,\ v''(L) = 0,\ v'''(L) = 0. \tag{9}$$

We call (8), (9) the *steady–state problem* and its solution

$$v(x) = -\frac{g\,L^2\,x^2}{4\,c^2} + \frac{g\,L\,x^3}{6\,c^2} - \frac{g\,x^4}{24\,c^2}, \tag{10}$$

the *steady–state solution.*

Next, we consider the function $w(x,t) := y(x,t) - v(x)$. Notice that as $t \to \infty$, $w(x,t) \to 0$. For this reason, we call $w(x,t)$ the *transient solution.* Substituting $y(x,t) = w(x,t) + v(x)$ into (1)–(7) and using (8) and (9), we obtain the *transient problem*

$$\frac{\partial^2 w}{\partial t^2} = -c^2 \frac{\partial^4 w}{\partial x^4} - k\frac{\partial w}{\partial t}; \quad 0 < x < L, \quad t > 0. \tag{11}$$

$$w(0,t) = 0, \quad t > 0, \tag{12}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad t > 0, \tag{13}$$

$$\frac{\partial^2 w}{\partial x^2}(L,t) = 0, \quad t > 0, \tag{14}$$

$$\frac{\partial^3 w}{\partial x^3}(L,t) = 0, \quad t > 0, \tag{15}$$

$$w(x,0) = f(x) - v(x),\ 0 < x < L, \tag{16}$$

$$\frac{\partial w}{\partial t}(x,0) = g(x),\ 0 < x < L. \tag{17}$$

To solve (11)–(17) we use the standard technique of *separation of variables* [1], [2], [4], [5], in which solutions to (11) of the form $w(x,t) = X(x)T(t)$ are assumed. With this assumption, (11)–(15) become

$$X(x)T''(t) = -c^2X''''(x)T(t) - kX(x)T'(t);\ \ 0 < x < L,\ \ t > 0, \tag{18}$$

$$X(0)T(t) = 0,\ \ t > 0, \tag{19}$$

$$X'(0)T(t) = 0,\ \ t > 0, \tag{20}$$

$$X''(L)T(t) = 0,\ \ t > 0, \tag{21}$$

$$X'''(L)T(t) = 0,\ \ t > 0. \tag{22}$$

After dividing (18) by $X(x)T(t)$, we see that non–zero $X(x)$ and $T(t)$ must satisfy

$$\frac{T''(t) + kT'(t)}{-c^2T(t)} = \frac{X''''(x)}{X(x)} = \lambda;\ \ 0 < x < L,\ \ t > 0, \tag{23}$$

for some constant $\lambda$. Also note that for boundary conditions (19)–(22) to be satisfied, either $T(t) \equiv 0$ for $t > 0$, or

$$X(0) = X'(0) = X''(L) = X'''(L) = 0 \tag{24}$$

must hold. To avoid the trivial solution $w(x,t) \equiv 0$, we require (24).

Rearranging (23), we are led to the ordinary differential equations

$$X^{(4)}(x) - \lambda X(x) = 0,\ \ 0 < x < L \tag{25}$$

and

$$T''(t) + kT'(t) + \lambda c^2T(t) = 0,\ \ t > 0. \tag{26}$$

If $X(x)$ is a solution of (25) that satisfies (24) and $T(x)$ is a solution of (26), then it will follow that $w(x,t) = X(x)T(t)$ is a solution of (11)–(15). Since the only choice for $\lambda$ that leads to a non–trivial solution of (25) with boundary data (24) is $\lambda > 0$, writing $\lambda = \alpha^4$ for convenience and imposing boundary conditions (24), we find for each integer $n \geq 1$, (25) has a solution of the form

$$X_n(x) = \cos\alpha_n x - \cosh\alpha_n x - \frac{\cosh\alpha_n L + \cos\alpha_n L}{\sinh\alpha_n L + \sin\alpha_n L}(\sin\alpha_n x - \sinh\alpha_n x), \quad (27)$$

where $\alpha_n$ satisfies

$$\cos(\alpha L) + \mathrm{sech}(\alpha \mathrm{L}) = 0. \quad (28)$$

Note that for large $\alpha L$, $\mathrm{sech}(\alpha\mathrm{L})$ is very close to zero, so for large positive integers $n$, $\alpha_n$ satisfies

$$\alpha_n \approx \frac{(2n+1)\pi}{2L}. \quad (29)$$

This is useful for finding the $\alpha_n$ values numerically. (The $n = 0$ case leads to a trivial transient solution and since the sine and hyperbolic sine functions are odd, no "new" solutions are given for integers $n < 0$.) We call solutions $X_n(x)$ of (25) *eigenfunctions* with corresponding *eigenvalues* $\alpha_n^2$.

Following [4], we assume $k$ is small and find that for each positive integer $n$, corresponding to $X_n(x)$ given by (27) is a solution to (26) of the form

$$T_n(t) = \exp\left(\frac{-kt}{2}\right)[A_n\cos(\mu_n t) + B_n\sin(\mu_n t)], \quad (30)$$

with

$$\mu_n = \frac{1}{2}\sqrt{4\alpha_n^4 c^2 - k^2}, \quad (31)$$

for arbitrary constants $A_n$ and $B_n$, provided $0 \leq k < 2c\alpha_n^2$.

Thus, for each positive integer $n$, we have a solution to (11)–(15) of the form

$$w_n(x,t) = X_n(x,t)T_n(x,t), \quad (32)$$

with $X_n(x)$ and $T_n(t)$ given by (27) and (30), respectively. Since each solution of form (32) solves the *homogeneous* problem (11)–(15) and by the *Principle of Superposition* [4], any finite linear combination of these solutions will also solve (11)–(15), let's suppose a solution to (11)–(15) of the form

$$w(x,t) = \sum_{n=1}^{\infty} w_n(x,t) \quad (33)$$

exists and determine the coefficients $A_n$ and $B_n$ needed for (33) to satisfy initial conditions (16) and (17). Substituting (33) into (16) and (17) we find that formally,

$$f(x) - v(x) = w(x,0) = \sum_{n=1}^{\infty} A_n X_n(x), \tag{34}$$

and

$$g(x) = \frac{\partial w}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left( -\frac{A_n k}{2} + B_n \mu_n \right) X_n(x). \tag{35}$$

Since the the set of eigenfunctions $\{X_n(x)|n = 1, 2, ...\}$, is an *orthonormal* set that satisfies:

$$\int_0^L X_n(x) X_m(x) dx \begin{cases} = 0 \text{ if } n \neq m, \\ \neq 0 \text{ if } n = m, \end{cases} \tag{36}$$

the coefficients $A_n$ and $B_n$ can be found by formally multiplying equations (34) and (35) by $X_m(x)$ for *fixed* $m$ and integrating from 0 to $L$, to get

$$A_n = \frac{\int_0^L (f(x) - v(x)) X_n(x) dx,}{\int_0^L X_n(x)^2 dx} \tag{37}$$

and

$$-\frac{A_n k}{2} + B_n \mu_n = \frac{\int_0^L g(x) X_n(x) dx}{\int_0^L X_n(x)^2 dx}, \tag{38}$$

for integers $n \geq 1$. Note that since problem (25) with boundary data (24) is a special case of a *Stürm-Liouville problem*, it follows that with these choices of $A_n$ and $B_n$, the *generalized Fourier series* (34) and (35) converge on $(0, L)$ to $f(x)$ and $g(x)$, respectively, [1].

## Another Way to Find $c$

The following method, which is suggested as a homework exercise in [8], can be used to experimentally find the value of the coefficient $c$ in the beam equation (1). The way to do this is to hang a weight on the free end of the beam and measure the resulting steady–state deflection $d$. The weight causes a vertical shearing force $F$ at the free end and the beam's displacement can be found by solving the steady–state equation (8) with boundary data

$$v(0) = 0, \; v'(0) = 0, \; v(L) = d, \; v'''(L) = \frac{F}{mc^2}, \tag{39}$$

where $m$ is the mass per unit length of the beam.

The last boundary condition follows from our definition of $c$ and the relationships between shearing force $V(x)$, bending moment $M(x)$ at any point $x$ along the beam,

$$c^2 = \frac{EI}{m}, \tag{40}$$

$$V(x) = M'(x), \tag{41}$$

and

$$M(x) = EIv''(x), \tag{42}$$

where $EI$ is the beam's flexural rigidity and $m$ is the beam's mass per unit length [3], [6], [7]. Assuming that the flexural rigidity is constant and differentiating (42), it follows from (40) and (41) that

$$v'''(x) = \frac{V(x)}{EI} = \frac{V(x)}{mc^2}.$$

Solving (8), (39), we find

$$v(x) = \frac{x^2\left(24c^2dm - 4FL^3 - 3gL^4m\right)}{24c^2L^2m} + \frac{x^3(F + gLm)}{6c^2m} - \frac{gx^4}{24c^2}, \tag{43}$$

and if we know the deflection at the midpoint of the beam,

$$v\left(\frac{L}{2}\right) = e, \tag{44}$$

then it follows from (43) and (44) that

$$c = \sqrt{\frac{8FL^3 + 5gL^4m}{96m(d - 4e)}}. \tag{45}$$

Using a two meter stick, clamped to a table with a C-clamp to make a cantilever beam of length $L = 1.6$ m, we place (flat disk) masses of 0.1 kg, 0.05 kg, and 0.15 kg on top of the free end and measure the displacements $d$ and $e$ at the free end and midpoint, respectively to get estimates for $c$ from (45). Measurements are made with a meter stick from the floor to the top of the beam in equilibrium. With $F$ given by the mass in kilograms times the acceleration due to gravity $g = 9.8$ m/sec$^2$ and the beam's mass per unit length of 0.1507 kg/m, we find find values for $c$ as given in TABLE 1.

| mass (kg) | d (m) | e (m) | c (m$^2$/sec) |
|---|---|---|---|
| 0.1 | -0.099 | -0.035 | 11.65 |
| 0.1 | -0.099 | -0.033 | 12.99 |
| 0.05 | -0.05 | -0.018 | 14.23 |
| 0.15 | -0.15 | -0.049 | 12.05 |

Table 1: Estimates for $c$ from different masses at free end of beam

The reason for two measurements with 0.1 kg is that two different people made these measurements. Thus, it is clear that we can get a rough estimate for $c$, but this estimate is sensitive to small changes in measured displacements and the amount of mass placed at the end of the beam.

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