ARTICLES

Rick's Tricky Six Puzzle: S_5 Sits Specially in S_6

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LOCI: Interactive Supplements to "Rick's Tricky Six Puzzle"

Many of you will be familiar with the Fifteen Puzzle (FIGURE 1, left). Singmaster [**16**, §5A, pp. 77–84] gives nearly a hundred references to it. It is often associated with the name of Sam Loyd, but Sam continues to be a controversial figure [**9**, Chapter 2, pp. 18–30; **17**]. In the unlikely event that you've never seen the Fifteen Puzzle, you can read about it in the review quoted in the next section.

Sliding block puzzles may be represented by graphs in which the vertices represent possible positions of the blocks and the edges represent the permissible moves of a block from one position to another. For example, the Fifteen Puzzle may be thought of as being played on the sixteen vertices of the graph in FIGURE 1. In this graph, don't think of the numbers as labels for the vertices, but as labeled blocks that can be slid from a vertex to an empty vertex. For example, in the figure, either block 12 or block 15 may be slid onto the vertex where □ indicates that there is no block.

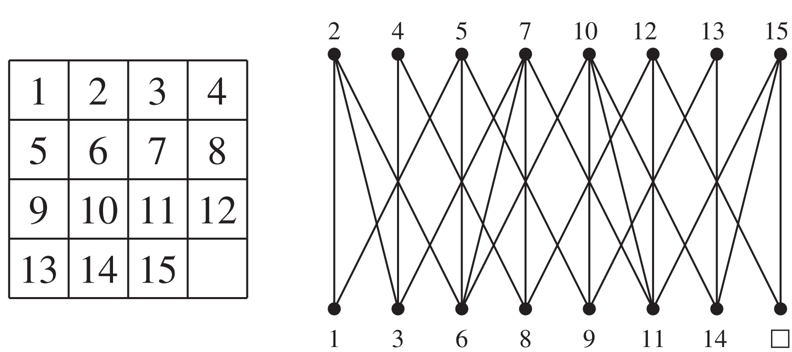

Figure 1 The Fifteen Puzzle and its bipartite graph

The notoriety of the puzzle derives from the impossibility of being able to swap the positions of 14 and 15 in the bottom row, while keeping all the other numbers fixed. This parity property was noted as early as 1879 [**18**, Chapter 1; **19**].

How many people know Rick Wilson's general theorem on sliding block puzzles? We retain Rick's first name to avoid confusion with the well known theorem of Sir John Wilson, first proved by Lagrange, that if p is a prime then $(p-1)!+1$ is divisible by p.

The set of attainable positions in a sliding block puzzle of n pieces sliding on the edges of a graph with $n + 1$ vertices form a group. Rick Wilson's theorem [**25**] states that, apart from simple polygons, and the graph that is the subject of this article, the group of permutations of attainable positions is either S_n, the full symmetric group, if the graph contains an odd circuit, or A_n, the alternating group of even permutations, if the graph contains only even circuits. In the latter case the graph is bipartite, the vertices separate into two sets and there are no edges between members of the same set—the Fifteen Puzzle is the classical example.

We mention that Rick Wilson's theorem applies only to nonseparable graphs, that is, graphs that are 2-connected, or without cut-points, so that there are always at least two paths between any pair of vertices that have no intermediate vertex in common.

What is the exception?

Math Reviews **48** #10882 offers a review by Derek Smith of Wilson's paper [**25**], quoted here with permission from the AMS.

> The 15-puzzle consists of fifteen small movable square tiles numbered $1, 2, \ldots, 15$ and one empty square, arranged in a 4×4 array. One is permitted to interchange the empty square with a tile next to it as often as desired. The challenge is to move by a sequence of such interchanges from one position of the tiles to another specified position. The author generalizes this problem to an arbitrary simple graph and proves that for a finite simple nonseparable graph, with one exception, any position can be reached from any other position unless the graph is bipartite. In the bipartite case, the set of positions splits into two sets, with no position in one set reachable from a position of the other set.

This might be misconstrued to read as though the exception is the set of bipartite graphs. In fact the exception is shown in FIGURE 2. It is a graph on 7 points with 8 edges. It contains two 5-circuits and a 6-circuit, so that we might expect to be able to obtain all $6! = 720$ permutations of the six counters, labeled with the symbols **0**, **1**, **2**, **3**, **4**, ∞. Why do we use ∞ instead of **5**? Our labels represent the field $\mathbb{F}_5$ with ∞ adjoined; this will make the connection with the automorphism group of the puzzle clearer.

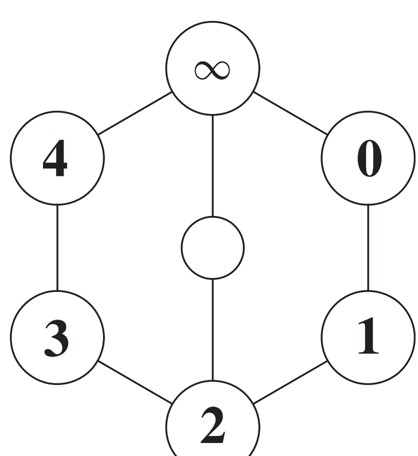

Figure 2 Rick's Tricky Six Puzzle

A little experimentation reveals that there are many arrangements that cannot be attained. The 6! possible arrangements separate into six equivalence classes, with 5! positions in each class. We shall see that

$$\infty\mathbf{01234},\quad \infty\mathbf{01243},\quad \infty\mathbf{01324},\quad \infty\mathbf{01342},\quad \infty\mathbf{01423},\quad \infty\mathbf{01432}$$

are representatives, one from each equivalence class. Note that we always read a position clockwise, starting from twelve o'clock. It is not possible to get from any one of these six positions to any other by sliding the disks along the eight edges of the graph.

Not much of a puzzle?

John Conway tells us that he once made a copy of the Tricky Six Puzzle, and we made one that Art Benjamin helped us demonstrate at the 2006 MathFest, but we doubt if it will ever catch on commercially. However, it does have considerable mathematical interest. We shall see that it is related to the projective plane of order 4, to the Hoffman-Singleton graph, to the Steiner system $S(5, 6, 12)$, to a binary (12, 132, 4) code, to the ternary Golay code $\mathcal{C}_{12}$, and to shuffling a deck of cards [**15, 6**]. It is also related to the invariant theory of six points, to "mystic pentagons" and the two-colorings of the three-subsets of a six-element set [**10**], and to the tetracode, the Minimog, and the Rubicon [**5**, pp. 320–330], and to many other things that we don't have room for here.

Many mathematicians are interested in word play, so we asked our favorite anagrammatist, Andrew Bremner, to supply a set of six letters that had many anagrams. He suggested A, C, E, N, R, T. Among the 720 possibilities we found the following twenty words, names and acronyms.

TABLE 1: Six equivalence classes of anagrams

RECANT	ARCNET	CARTEN	CENTRA	CARNET	TANCER
	CANTER	CRANET	CRETAN	CANTRE	TRANCE
	CERANT		NECTAR	CREANT	
	ENCART		TARNEC	NETCAR	
	TERCAN		TRACEN	TANREC	

If you encode these anagrams with R = ∞, E = 0, C = 1, A = 2, N = 3, and T = 4, you will find that it's possible to get from one word to any other in the same column of TABLE 1, but not to any word in a different column. For example, from RECANT, you can't CANTER to any of the other words. We list below four things you CAN do (have we always found the shortest sequence of moves?). If you want to follow along, and to avoid what Conway calls the "alias-alibi problem" (is it the counter? or the position it's in?), then you should label six counters or slips of paper with the symbols ∞, 0, 1, 2, 3, 4 and the letters R, E, C, A, N, T and slide them about on an improvised board. When we write a permutation (ABC . . . Z) this means that A ends up where B started, B ends up where C started, and so on, cyclically, with Z arriving where A started. By the usual convention, when we string together several such permutations it is the one on the right that acts first: they don't act in the order in which you would normally read them. Compare the out-shuffle with the in-shuffle in the second example below.

1. Cut the deck: swap the first three symbols ∞, 0, 1, with the last three, 2, 3, 4 respectively. The moves $210\infty4310\infty4310\infty432$ take RECANT into ANTREC.

This is the permutation $(\infty 2)(03)(14)$. [In anticipation of the next section we will also write this as $x \to (x+2)/(3x+4)$ mod 5. Such a mapping is called a *Möbius transformation*.]

2. Perform an *out-shuffle*, or an *in-shuffle*: cut the deck RECANT into REC and ANT and interleave letters alternately from each half. In an out-shuffle the top card remains on top: RAENCT = (0132) $[x \to 2x+1]$. This can be achieved by the moves $234\infty 23102\infty 413$. An in-shuffle results in ARNETC = $(\infty 02)(431)$ $[x \to 2/(2x+1)]$ and results from the moves $\infty 012\infty 012\infty 3412\infty 30$. Note that shuffling one way then unshuffling the other performs a cut: $(\infty 20)(134)(0132) = (\infty 2)(03)(14)$. On the other hand, unshuffling then shuffling swaps alternate cards: $(0132)(\infty 20)(134) = (\infty 0)(12)(34)$ $[x \to 2/x]$.

These manipulations of cards don't generate the whole group of the puzzle; they only yield 4! of the 5! possible states, those in which the pairs of cards $\infty 4$, 03, 12, that are equidistant from the centre of the deck, remain so. It doesn't take much experimentation to discover sequences of moves that break up these pairs and generate the whole group:

3. Cycle the first four symbols. The moves $210\infty 2$ followed by $10\infty 21$ and $0\infty 210$ and $\infty 210\infty$ take RECANT → ARECNT → CARENT → ECARNT and back into RECANT. These are the transformations $(\infty 012)$ $[x \to 1/(2x+1)]$, $(\infty 012)^2 = (\infty 1)(02)$ $[x \to (2x+1)/(2x+3)]$, $(\infty 012)^3 = (\infty 210)$ $[x \to (2x+3)/x]$, and $(\infty 012)^4 =$ the identity $[x \to x]$.
4. Fix the first symbol and cycle the other five. The moves $\infty 432104\infty$ send RECANT to RTECAN, $\infty 01234$ to $\infty 40123$, the permutation (01234) $[x \to x+1]$. In fact, combined with the out-shuffle (0132) $[x \to 2x+1]$, this cycle allows us to apply any invertible linear polynomial mod 5 to the finite symbols 0, 1, 2, 3, 4, yielding positions such as (0412) $[x \to 3x+4]$, and its inverse (0214) $[x \to 2x+2]$. These are illustrated in the first of the six diagrams of FIGURE 4 below as all ways to travel round the pentagon or the pentagram.

What is the group of the Tricky Six Puzzle?

As you may have guessed from the brackets in the last section, it is the group $PGL(2, \mathbb{F}_5)$ of Möbius transformations over the field $\mathbb{F}_5$.

$$x \to \frac{px+q}{rx+s} \qquad ps - qr \neq 0$$

This $\mathbb{F}_5$ is the first of several *finite fields* we will encounter. In fact for each prime power q there is a unique field with q elements, which we will denote by $\mathbb{F}_q$. So working in $\mathbb{F}_5$ means working modulo 5—but only because 5 is prime.

There are $5^2 - 1 = 24$ possible nonzero vectors (p, q) for the top row of the matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and then $5^2 - 5 = 20$ vectors (r, s) that are independent of the first row, as possibilities for the second row; a total of $24 \times 20 = 480$ nonsingular matrices. But the matrices M, $2M$, $3M$, $4M$, for example

$$\begin{pmatrix} 1 & 0 \\ 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix},$$

all give the same transformation, $(0)(3)(\infty 4)(1\,2)$, taking $\infty 01234$ into 40213∞, or RECANT into TEACNR, so that the number of different transformations is only $480/4 = 120$.

To the surprise of at least one of the authors, this group is isomorphic to S_5, the group of permutations of five objects. We will show that the isomorphism establishing this extends naturally to an automorphism of S_6, under which the group of the puzzle maps to an S_5 subgroup of S_6 given by fixing a point. It's in this context that the isomorphism is most illuminatingly presented.

Two different group actions

An *inner automorphism* of a group is one given by conjugation, that is, each element $x \mapsto a^{-1}xa$ for some fixed element a. The automorphisms of a group themselves form a group, of which the inner automorphisms form a normal subgroup [**2**, pp. 140–141]. The *outer automorphisms* are those automorphisms this doesn't account for: by one definition any non-inner automorphism is outer; by another the outer automorphism group is the quotient of the automorphism group by the inner automorphism group. The symmetric group S_6 is the only finite symmetric group that supports a (nontrivial) outer automorphism [**11**; **14**, Theorem 7.3].

Suppose an abstract group acts on a finite set T (that is, each element of the group permutes T, and permuting by two group elements in succession is the same as permuting by their product). If we were to relabel the elements of T by a permutation a, then an element that acts via the permutation x after the relabelling would have acted by $a^{-1}xa$ before it. Now suppose our abstract group was the symmetric group S_T all along. Then a is in S_T, so $x \mapsto a^{-1}xa$ is an inner automorphism of S_T.

So the existence of an outer automorphism of S_6 means that it can act on sets of size 6 in a fundamentally different way than the obvious one. We'll realize the outer automorphism by constructing such an action, following Sylvester [**20, 21, 22, 23, 24**].

Consider the complete graph on the six points A, B, C, D, E, F. Sylvester calls the six points *monads*, and its $\binom{6}{2} = 15$ edges *duads*. These duads form $15 = 5 \times 3$ matchings, or triads of independent edges, that Sylvester called *synthemes*, and graph theorists know as one-factors. Note that there are 5 choices for A's partner and 3 ways to pair the remaining four.

The graph supports six partitions, or *synthematic totals*, into five synthemes, shown in TABLE 2 and labeled with their associated Tricky Six blocks, ∞, **0, 1, 2, 3, 4**.

TABLE 2: The six totals: the edge-colorings of K_6 with five colors

color	∞	**0**	**1**	**2**	**3**	**4**
r	AB CF DE	AB DE CF	AB FD CE	AB DC FE	AB FE DC	AB EC DF
o	AC DB EF	AC FD EB	AC EF DB	AC BE DF	AC ED BF	AC BF ED
y	AD EC FB	AD CB FE	AD BE FC	AD FB EC	AD CF EB	AD FE CB
i	AE FD BC	AE BF DC	AE DC BF	AE CF BD	AE BC FD	AE DB FC
v	AF BE CD	AF EC BD	AF CB ED	AF ED CB	AF DB CE	AF CD BE

The complete graph K_6 underlying this construction shouldn't be confused with FIGURE 2, the graph of the puzzle itself. As an example, the coloring associated with the label **2**, with AB DC FE colored red, AC BE DF colored orange, etc., is illustrated in FIGURE 3.

If we fix the monad A and operate on the six totals with the $5! = 120$ permutations of the other five monads, we generate the set of possible arrangements of the Tricky Six symbols.

Consider the action of our inner automorphism on conjugacy classes. Within a symmetric group such as S_6 conjugacy classes are just *cycle shapes*, which we write as

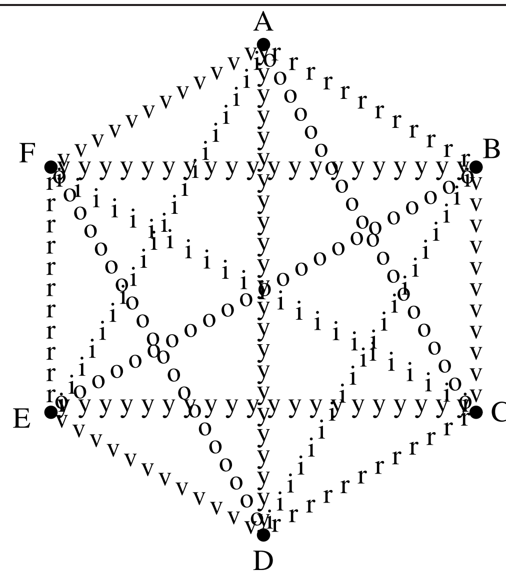

Figure 3 The edge-coloring **2** of K_6, the complete graph on six points

partitions of 6. The cycle shapes on the totals attainable in the puzzle are those that arise from permutations of the monads which fix A, and these have a fixed point in their cycle shape.

For example, if we fix A and three other vertices, we obtain $\binom{5}{2} = 10$ odd permutations of order 2. These are involutions; each is its own inverse. They appear as the first ten entries in TABLE 3:

TABLE 3: Swapping two vertices of K_6

(DE) $(\infty 0)(12)(34)$ 0210	(EF) $(\infty 1)(23)(40)$ 1114	(FB) $(\infty 2)(34)(01)$ 4121	(BC) $(\infty 3)(40)(12)$ 1124	(CD) $(\infty 4)(01)(23)$ 4111
(CF) $(\infty 0)(13)(24)$ 0310	(DB) $(\infty 1)(24)(30)$ 1214	(EC) $(\infty 2)(30)(41)$ 4321	(FD) $(\infty 3)(41)(02)$ 1324	(BE) $(\infty 4)(02)(13)$ 4211
(AB) $(\infty 0)(14)(23)$	(AC) $(\infty 1)(20)(34)$	(AD) $(\infty 2)(31)(40)$	(AE) $(\infty 3)(42)(01)$	(AF) $(\infty 4)(03)(12)$

together with the permutations of ∞ 0 1 2 3 4 that they realize, and the entries $pqrs$ of the corresponding Möbius transformation.

For later reference we include as well the five transpositions that move the monad A; these don't realize Möbius transformations.

We thus find that permutations of ABCDEF of shape

$$1^6 \quad 2\cdot 1^4 \quad 2^2 1^2 \quad 2^3 \quad 3\cdot 1^3 \quad 321 \quad 3^2 \quad 4\cdot 1^2 \quad 42 \quad 51 \quad 6$$

map respectively to permutations of $\infty 01234$ of shape

$$1^6 \quad 2^3 \quad 2^2 1^2 \quad 2\cdot 1^4 \quad 3^2 \quad 6 \quad 3\cdot 1^3 \quad 4\cdot 1^2 \quad 42 \quad 51 \quad 321.$$

When A is fixed, respectively

$$1 \quad 10 \quad 15 \quad 0 \quad 20 \quad 20 \quad 0 \quad 30 \quad 0 \quad 24 \quad 0$$

of these are attainable. For example, at the entry $4 \cdot 1^2$ we fix A, and one other letter (5 ways) and cycle the remaining four ($4!/4 = 6$ ways), contributing $5 \times 6 = 30$ to the total of 120. As another example, if we fix A and two other vertices and cycle the rest, $3 \cdot 1^3$, we obtain $\binom{5}{3} \times 2 = 20$ even permutations of order 3. They are displayed in TABLE 4.

TABLE 4: Cycling three of five vertices of K_6

(FBC)	(BCD)	(CDE)	(DEF)	(EFB)
$(\infty41)(032)$	$(\infty02)(143)$	$(\infty13)(204)$	$(\infty24)(310)$	$(\infty30)(421)$
1341	0113	1312	2311	3110
(FCB)	(BDC)	(CED)	(DFE)	(EBF)
$(\infty14)(023)$	$(\infty20)(134)$	$(\infty31)(240)$	$(\infty42)(301)$	$(\infty03)(412)$
1211	1341	1123	3121	0112
(DEB)	(EFC)	(FBD)	(BCE)	(CDF)
$(\infty32)(014)$	$(\infty43)(120)$	$(\infty04)(231)$	$(\infty10)(342)$	$(\infty21)(403)$
1121	2131	0411	1410	1132
(DBE)	(ECF)	(FDB)	(BEC)	(CFD)
$(\infty23)(041)$	$(\infty34)(102)$	$(\infty40)(213)$	$(\infty01)(324)$	$(\infty12)(430)$
1431	3211	1140	0141	1213

It will be found that any of the $6 \times 5 \times 4 = 120$ possible arrangements of the first three, or indeed of any three, symbols in a Tricky Six position is attainable, the order of the remaining three then being determined.

All 120 positions are conveniently displayed as the set of six diagrams of FIGURE 4. The first symbol is in the middle of the appropriate diagram. The next two symbols determine a directed edge of a pentagon or pentagram. The final three symbols are then found by continuing to cycle round the pentagon or pentagram in the sense defined by the edge. For example, the position $241xyz$ is found in the diagram having 2 in the middle, where the edge 41 defines the counterclockwise pentagram $41\infty30$, so that $xyz = \infty30$.

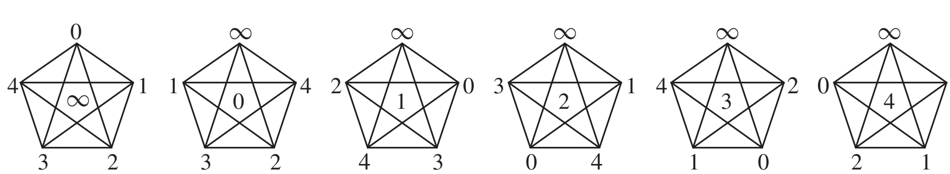

Figure 4 All 120 Tricky Six positions at a glance

The six diagrams of FIGURE 4 are also conveniently viewed as the six pentagonal pyramids that may be sliced from the icosahedron of FIGURE 5, whose opposite vertices are identified. Each pyramid comprises four cycles. For example,

$$\infty(01234)^1 = \infty(01234), \quad \infty(01234)^2 = \infty(02413),$$
$$\infty(01234)^3 = \infty(03142), \quad \infty(01234)^4 = \infty(04321),$$

where the superscripts denote powers, that is the lengths of the steps round the pentagon.

As you can see from TABLE 2, there is a unique synthematic total that is invariant under any five-cycle (JKLMN) on the monads A, B, C, D, E, F. Conway, who introduced

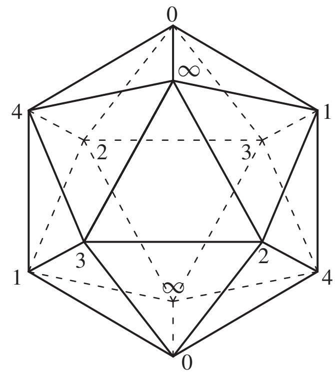

Figure 5 Another good way to see them all

us to Sylvester's notation, denotes it by I(JKLMN). The total I(JKLMN) contains the syntheme IJ KN LM and its images under powers of (JKLMN). FIGURE 6 shows an example.

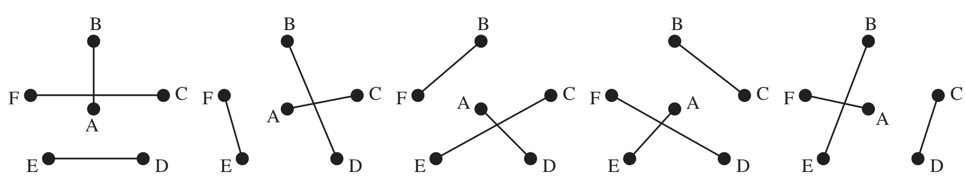

Figure 6 A(BCDEF), the unique synthematic total, also known as ∞, invariant under (BCDEF)

The identities

$$\mathrm{I(JKLMN)} = \mathrm{I(JKLMN)}^{\text{power}} = \mathrm{J}(\overleftrightarrow{\mathrm{I}}\ \overleftrightarrow{\mathrm{LK}}\ \overleftrightarrow{\mathrm{NM}})$$

let us bring any of the 6 monads into the initial position, and write the remainder as any of 5 presentations of any of 4 powers of the five-cycle left over, giving $6 \times 5 \times 4 = 120$ names for each total.

For instance A(BCDEF) = A(BCDEF)$^?$ for any exponent ? not divisible by 5, and its other names are B(ADCFE)$^?$ = C(AEDBF)$^?$ = D(AFECB)$^?$ = E(ABFDC)$^?$ = F(ACBED)$^?$. Each group of names can be thought of as associated with a pentagram labeled with letters, with the first letter in the centre, like those in FIGURE 4. Such pentagrams are fixed by one of the six subgroups of S_6 of order 20 that fixes ∞ = A(BCDEF).

Indeed, observe that there is a duality of our construction exchanging monads with totals and duads with synthemes, realizable as $\infty 01234 \leftrightarrow$ ABCDEF. Under this exchange the names of the total ∞ become just the attainable Tricky Six permutations. Our situation can be schematized as in FIGURE 7, the symmetry of which makes the duality obvious.

We saw that the first three symbols determine the whole position, and how to read it from FIGURE 4. In fact *any* three symbols determine the position. For example, to find which of ∞, 0, 2 should be assigned to x in $x31y4z$, look for the edge 31 in the ∞, 0 and 2 diagrams of FIGURE 4. It respectively defines the pentagram $(\infty)31420$, the pentagram $(0)31\infty42$, and the pentagon $(2)310\infty4$, of which the second has 4 in

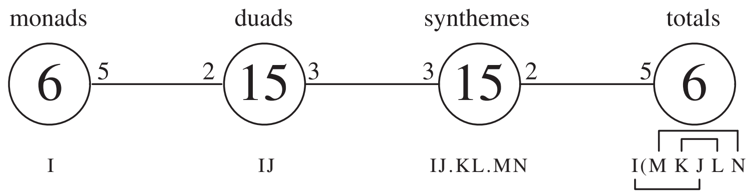

Figure 7 Schematic view of Sylvester's construction

the required position, 031∞42. For another example we may complete $x3y1z4$, by looking in the same three diagrams for the edge 43 (why 43? Think of x as fixed, and notice that 4 and 3 are adjacent in the remaining cycle $3y1z43$). This determines the pentagons (∞)43210, (0)43∞21, (2)4310∞, of which the first has 1 in the required position, ∞32104.

It is through this automorphism that Rick's Tricky Six puzzle is related to the other objects named at the start of the "Not much of a puzzle" section.

Here's a first brief example. Implicit in the way we've written TABLE 2 is another set of six objects paired with the totals, the *mystic pentagons* which begin the interesting paper [**10**]. The ten duads that don't contain A form two sets of five: the second and third columns of each total. Each monad appears twice in each column. If we forget the synthemes and remember only the column divisions, we get a mystic pentagon, that is, a partition of the edges of complete graph on vertices BCDEF into two five-cycles. There are in fact only six mystic pentagons, and we get each of them once (FIGURE 8). Therefore the permutations of the mystic pentagons which can be attained by permuting BCDEF exactly form the Tricky Six group.

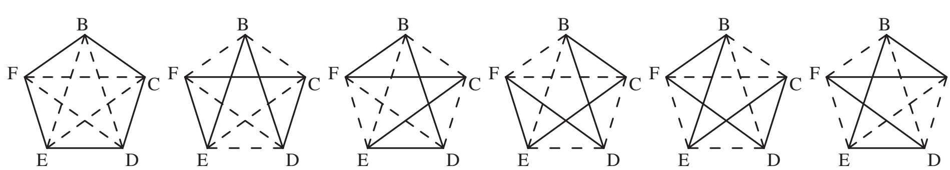

Figure 8 The six mystic pentagons

The remainder of this paper is devoted to a more leisurely examination of several other examples.

The projective plane of order 4

The projective plane of order four, PG(2, $\mathbb{F}_4$), is often defined by means of a *cyclic difference set*, for example {3, 6, 12, 7, 14} modulo 21, whose five members generate the $\binom{5}{2}$ differences $\pm1, \pm2, \ldots, \pm10$. Note that the first three elements generate the multiples of 3, and the last two generate the multiples of 7. Think of the difference set as a complete pentagon which cycles round a complete regular 21-gon as in FIGURE 9. Among its 10 edges there is exactly one of every possible length, so that every pair of the 21 points belongs to just one pentagon. Dually, any two pentagons have just one vertex in common.

Call the pentagon {3, 6, 12, 7, 14} the *line* **0**. Subtract 3, 4, 9, 11 modulo 21 to give the respective lines

3: {0, ♠, 9, 4, 11}, **4**: {20, 2, 8, ♡, 10},
9: {15, 18, ◇, 19, 5}, **11**: {13, 16, 1, 17, ♣}.

These four lines each pass through the point 3 which is denoted differently in each of them, by ♠, ♡, ◇, and ♣ in turn, and is circled in FIGURE 9. The other four points on each of these lines are represented in the figure by the corresponding suit symbols. They exactly cover the 16 points which are not on line **0**.

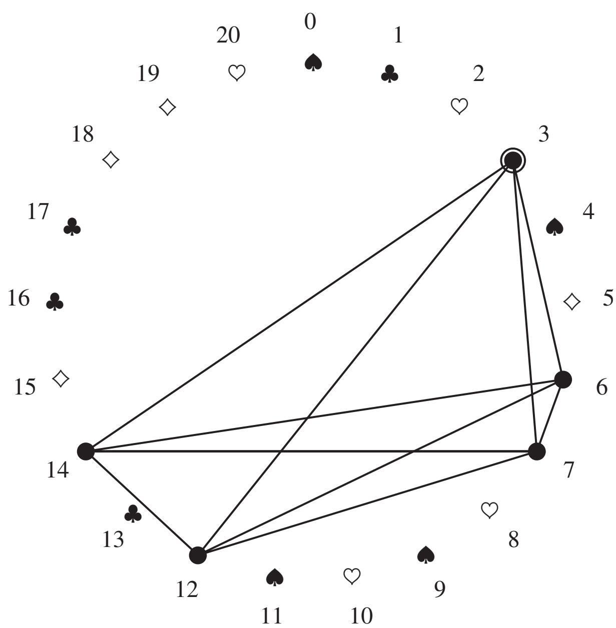

Figure 9 A difference set generates the projective plane of order 4

In general, we give the *line* $\{3-n, 6-n, 12-n, 7-n, 14-n\}$ modulo 21 the name **n**, $\mathbf{0} \le \mathbf{n} \le \mathbf{20}$, as in TABLE 5, which displays a configuration of 21 points and 21 lines with 5 points on each line, 5 lines through each point, every pair of lines intersecting in a point and every pair of points determining a line. Bold numbers refer to lines, ordinary numbers to points (or vice versa, since the configuration is self-dual). The line **i** passes through the point j if and only if the point i lies on the line **j**.

TABLE 5: Incidences in the projective plane of order 4

lines	**0**	**20**	**19**	**18**	**17**	**16**	**15**	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
points	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2
points	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
points	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11
points	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6
points	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Twenty-one is not a prime power, so the numbers 0, 1, ... , 20 do not form a field. However, they do form an additive cyclic group, and the twelve numbers which are not multiples of 3 or 7 form a multiplicative group, of which the powers of 2 are a subgroup.

Let's find two different actions of S_6 in this projective plane. As the two sets of size six let us take the points 1 2 4 8 16 11 (the powers of two, $2^0, 2^1, 2^2, 2^3, 2^4, 2^5$, mod 21) and the lines **0 18 15 9 14 7** (zero and the negatives of the original difference set).

We can begin to rewrite TABLE 2 for the projective plane by replacing the labels A B C D E F of the vertices of K_6 with the respective point numbers 1 2 4 8 16 11. We also relabel the totals, ∞ **0 1 2 3 4** with the respective line numbers **18 15 14 0 7 9**. Then, with TABLE 5 as our guide, we label the edges AB, CF, DE which join the points 1 & 2, 4 & 11, 8 & 16, with the line-numbers **5 3 19** and similarly for all the fifteen synthemes. The lines **5 3 19** concur in the point 9 and each syntheme corresponds to a point. The labels of these fifteen points are just those numbers that are not powers of two, and TABLE 2 turns into TABLE 6. You can check that this is the same configuration, with the same labelling, as before.

TABLE 6: An assignment of numbers to TABLE 2

18				**15**				**14**				**0**				**7**				**9**			
5	**3**	**19**	9	**5**	**19**	**3**	9	**5**	**16**	**8**	19	**5**	**20**	**17**	7	**5**	**17**	**20**	7	**5**	**8**	**16**	19
2	**4**	**17**	10	**2**	**16**	**12**	12	**2**	**17**	**4**	10	**2**	**12**	**16**	12	**2**	**19**	**1**	5	**2**	**1**	**19**	5
6	**8**	**1**	6	**6**	**10**	**17**	18	**6**	**12**	**3**	0	**6**	**1**	**8**	6	**6**	**3**	**12**	0	**6**	**17**	**10**	18
11	**16**	**10**	17	**11**	**1**	**20**	13	**11**	**20**	**1**	13	**11**	**3**	**4**	3	**11**	**10**	**16**	17	**11**	**4**	**3**	3
13	**12**	**20**	15	**13**	**8**	**4**	20	**13**	**10**	**19**	14	**13**	**19**	**10**	14	**13**	**4**	**8**	20	**13**	**20**	**12**	15

The points 1, 2, 4, 8, 16, of which no three are collinear, form a *conic*, that is, the solution set of a homogeneous quadratic over the field of order four. The *tangents* to the conic are the lines that meet the conic in just one point (indicated by a hat):

$$\mathbf{13}\ \{11\ 14\ 20\ 15\ \hat{1}\},\quad \mathbf{16}\ \{\hat{8}\ 11\ 17\ 12\ 19\},\quad \mathbf{1}\ \{\hat{2}\ 5\ 11\ 6\ 13\},$$

$$\mathbf{17}\ \{7\ 10\ \widehat{16}\ 11\ 18\},\quad \mathbf{3}\ \{0\ 3\ 9\ \hat{4}\ 11\}.$$

These are the five lines through the point 11. This point combines with the conic to form a *hyperconic*, six points no three of which are collinear. These six points are the monads, and determine $\binom{6}{2} = 15$ lines, the duads, which meet in threes at the other fifteen points; these correspond to the synthemes. The remaining six lines (**0**, **7**, **14**, **9**, **18**, **15**) that don't meet the hyperconic correspond to the totals; no three of them concur and they form a set of lines dual to the set of six points.

We repeat FIGURE 7 as FIGURE 10, annotating the nodes further to make clear the interpretation of the figure as the projective plane.

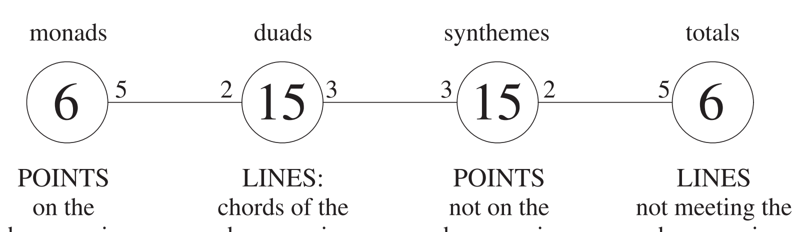

Figure 10 Schematic view of the projective plane of order 4

Our two nonisomorphic S_6-actions show up here as the action that permutes the points of any six-point hyperconic, like 1 2 4 8 16 11, and the action induced on the lines not meeting it, in this case **0 7 14 15 18 9**. Our numbering makes it easy to check that doubling all the vertex labels modulo 21 is an automorphism that fixes the hyperconic under which line labels are also doubled, so the cycle (1 2 4 8 16 11) induces the permutation (**0**)(**7 14**)(**9 18 15**) of the six lines. If we swap 1 and 2 and fix the other four points, (1 2)(4)(8)(16)(11), this induces (**0 7**)(**15 18**)(**9 14**) on the lines and these two automorphisms are enough to generate the whole group.

We can't draw the plane with straight lines, so, in FIGURE 11, although the twenty-one points 0, 1, 2, ... , 20 are clear, the lines are less so. The line **9** is the incircle of the pentagon and the lines **0, 14, 15, 18, 7** look like petals. The lines **3, 16, 17, 13, 1** are the diameters through the point 11. The lines **4, 8, 6, 12, 2** are pentagram edges, that need to be bent round to pass through the respective points 3, 19, 18, 15, 5; and the lines **11, 5, 10, 20, 19** are pentagon edges, both ends of which should be bent round to pass through the respective pairs of points 17&13, 7&9, 14&17, 13&7, 9&14.

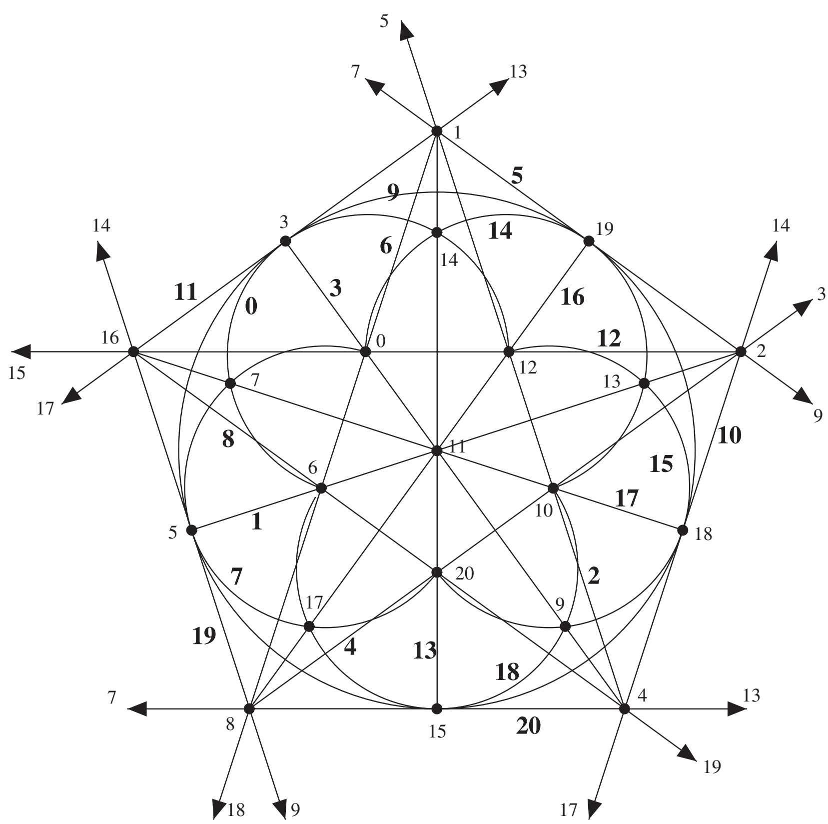

Figure 11 The projective plane of order 4

The points 3, 6, 12, 7, 14 of line **0** thus lie on the respective lines **3, 6, 12, 7, 14** and, of course, lie just one on each of the remaining fifteen lines. The other four points on such a line comprise two pairs that form triples with the line number, each member of a triple being the number of the line containing the other two points. For example, line **18** contains the point 6 and the four points 9, 15, 10, 17 whose joins to the point 18 are the respective lines **15, 9, 17, 10** which form the triples {**18**, **15**, **9**}, and {**18**, **17**, **10**}. There are ten such triples and they exhibit the ten differences $1 \le d \le 10$ exactly three times each. For example, the difference 5 occurs in the triples {**8**, **13**, **4**}, {**11**, **16**, **7**}, and {**15**, **20**, **13**}. These ten triples correspond to the sets of edges of pairs of opposite faces of an icosahedron, half of which is shown in FIGURE 12.

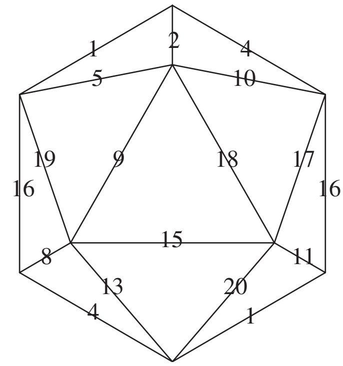

Figure 12 Ten triples form half an icosahedron

Buy one; get several free!

We noticed that the difference set {3, 6, 12, 7, 14} comprised two difference sets: {7, 14} generates the multiples of 7 and {3, 6, 12} generates the multiples of 3. So the projective plane of order four contains the not very exciting projective plane of order one: the triangle {0, 7, 14} and 1119 other copies of it, and the much more interesting projective plane of order two, the so-called Fano configuration (although it was known more than 40 years earlier to the Rev. T. P. Kirkman [**12**]). Besides the obvious example, whose point-numbers are congruent to 0 modulo 3, which is self-dual in the sense that it has the same line-numbers, and is shown in FIGURE 13, there are 359 others: including the dual pair whose point- and line-numbers are respectively congruent to 1 and 2 (or to 2 and 1) modulo 3. The figure also shows a dual pair whose point-numbers differ by 3 from the line-numbers.

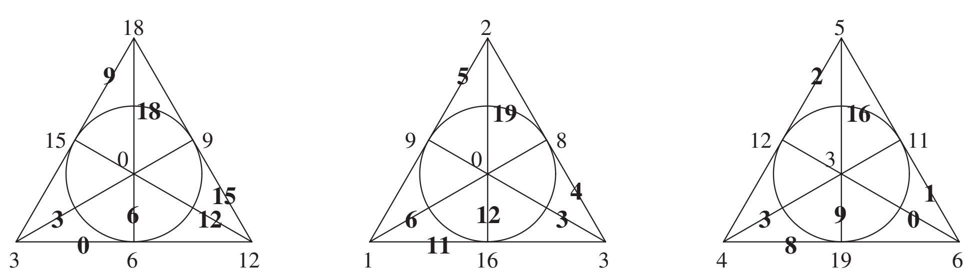

Figure 13 Kirkman-Fano configurations

More surprising is the fact [**1**] that if we throw away a hyperconic we are left with fifteen points which form a projective geometry of order two in three dimensions! For example, throw away 1, 2, 4, 8, 16, 11. The remaining points are those of the line {0, 5, 7, 17, 20}, and its double {0, 10, 14, 13, 19}, together with the multiples of 3. FIGURE 14 shows this geometry as a tetrahedron, as Polster would draw it [**13**]. Its fifteen points are the vertices, 5 7 17 20, the midpoints of the edges (multiples of 3), the centroids of the faces, 10 14 13 19, and the centroid, 0. Fifteen of the thirty-five lines, those which meet the hyperconic, are inherited from the plane: they are the twelve medians of the faces and the three joins of midpoints of opposite edges. The other twenty are the vertex sets of triangles formed by three of the six lines **0**, **7**, **14**, **9**, **18**, **15** which avoid the hyperconic. They appear as the six edges of the tetrahedron, the four joins of the vertices to the centroids of the opposite faces, and ten lines which cannot be drawn in Euclidean space: the four incircles of the faces and six similar curves circumscribing the "medial triangles":

$$\{3, 14, 19\} \quad \{6, 10, 14\} \quad \{9, 10, 13\} \quad \{12, 13, 14\} \quad \{15, 10, 19\} \quad \{18, 13, 19\}$$

formed by the triples of lines

$$\mathbf{14, 9, 0} \qquad \mathbf{14, 0, 18} \qquad \mathbf{14, 15, 18} \qquad \mathbf{14, 0, 15} \qquad \mathbf{14, 9, 18} \qquad \mathbf{14, 9, 15}.$$

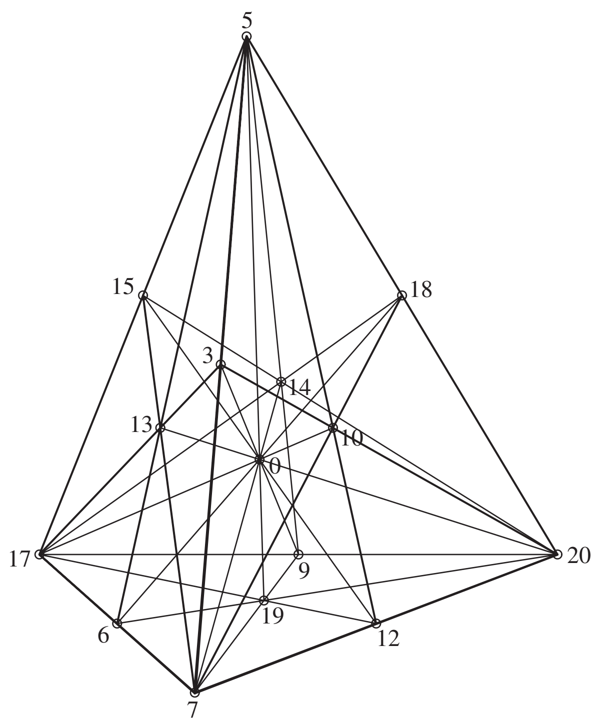

Figure 14 The projective geometry PG$(3, \mathbb{F}_2)$

A different and quite revealing labelling of the $15 = 2^4 - 1 = 4 + 6 + 4 + 1$ points is to assign 1, 2, 4, 8 to the vertices, sums of pairs of these to the midpoints of the edges, sums of three to the centroids of the faces, and the sum of all four, 15, to the centroid.

old numbers	5	7	3	17	15	6	13	20	18	12	10	9	14	19	0
new numbers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

The thirty-five lines are then those triples whose nim-sums (XOR, binary addition without carry) are zero: the ten "noneuclidean" lines correspond to those nim-sums which are not ordinary sums, for example, $3 \oplus 5 = 6$ and $5 \oplus 11 = 14$. The $15 = 2^4 - 1 = 4 + 6 + 4 + 1$ planes of the geometry are Kirkman-Fano configurations: the four faces of the tetrahedron, the six "medial planes" joining the midpoint of an edge to the opposite edge, the four "cones" joining a vertex to the incircle of the opposite face, and the "sphere" of midpoints of edges together with its centre, 15.

Remarkably, the thirty-five lines can be partitioned, in 240 different ways, into seven sets of five lines, with no two of the five intersecting, each set exactly covering the fifteen points. That is, the thirty-five lines can be arranged as rows in a *Kirkman*

(15, 3, 1)-*design*; they provide solutions to the famous Kirkman schoolgirls problem, with which readers of the previous issue of this MAGAZINE will already be familar [**4**]. An example is shown in TABLE 7.

TABLE 7: The thirty-five lines of PG(3,$\mathbb{F}_2$) form a Kirkman (15,3,1)-design

Sun	Mon	Tue	Wed	Thu	Fri	Sat
1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
5 8 14	3 9 10	3 8 11	2 4 6	2 5 7	3 4 7	3 5 6
4 11 15	2 12 14	2 13 15	3 12 15	3 13 14	2 9 11	2 8 10
7 9 14	7 8 15	5 9 12	5 11 14	4 8 12	5 10 15	4 9 13
6 10 12	6 11 13	4 10 14	7 10 13	6 9 15	6 8 14	7 11 12

The fifteen Kirkman-Fano planes each appear as seven triples, one from each day of the week. For example, the "cone" 1 6 7 10 11 12 13 is represented by 6 10 12, 6 11 13, 1 6 7, 7 10 13, 1 10 11, 1 12 13, and 7 11 12.

A somewhat surprising connection between PG(3, $\mathbb{F}_2$) and the Lehmers' method of factoring integers by means of quadratic forms is made in [**7**, §§26 & 27].

The Hoffman-Singleton graph

A *Moore graph* of type v, k is a regular graph of valence v and diameter k with the maximum possible number of vertices, namely $\big(v(v-1)^k - 2\big)/(v-2)$. This formula doesn't make sense if $v = 2$, but it tends to the limit $2k + 1$ as v approaches 2, and this is the number of vertices in the valence 2 case. Hoffman & Singleton [**8**] showed that for diameter 2 there are at most four such. Their valences are 2 (the pentagon), 3 (the Petersen graph), 7 (the Hoffman-Singleton graph) and possibly 57 (though the existence of this last remains an unsolved problem). The Hoffman-Singleton graph has 50 vertices and 175 edges, and like every Moore graph of diameter 2 its shortest cycles are pentagons so that its *girth* is 5. Its automorphism group has order $252000 = 2^5 3^2 5^3 7$. It is *arc-transitive*, that is it has an automorphism sending a particular edge to any of its 175 edges with either of 2 orientations. The stabilizer of an oriented edge thus has order $252000/(175 \cdot 2) = 720$, and indeed is isomorphic to S_6, as reflected in the following construction of the graph from our versatile TABLE 2.

To draw the Hoffman-Singleton graph, start with an edge joining vertices which we label $\star$ and G. Label the six other vertices adjacent to $\star$ with the letters A B C D E F and the other six adjacent to G with the symbols ∞ 0 1 2 3 4 as in FIGURE 15. The other 36 vertices are $\{\text{X}n\}$, where X runs through the letters A B C D E F and n runs through the symbols ∞ 0 1 2 3 4, and there are the implied adjacencies, for example vertex C2 is adjacent to vertices C and 2. It remains to insert the other $175 - (1 + 12 + 36 + 36) = 90$ edges. Again, they correspond to our edge-colorings of K_6.

Recall the fifteen swaps of TABLE 3. They each provide six adjacencies, for example

$$(\text{CE}) \qquad (\infty 2)(30)(41)$$

provides the six adjacencies

$$\text{C}\infty\text{—E2} \quad \text{C2—E}\infty \quad \text{C3—E0} \quad \text{C0—E3} \quad \text{C4—E1} \quad \text{C1—E4}$$

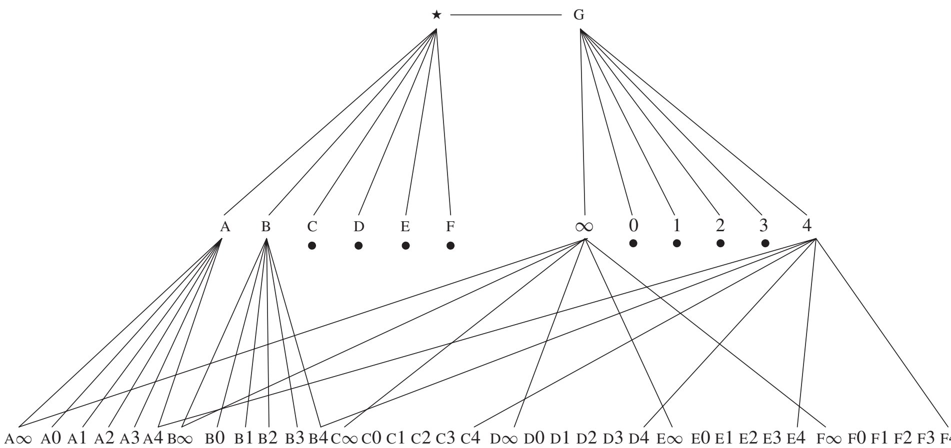

Figure 15 How to construct the Hoffman-Singleton graph

We can also succinctly describe the 6! automorphisms of the graph fixing the edge ⋆—G: they permute the vertices A B C D E F arbitrarily and the vertices ∞ 0 1 2 3 4 as dictated by construction.

Other constructions for the Hoffman-Singleton graph are given in [**3**, §13.1]. Conway showed us his perspective, which begins with a distinguished vertex rather than an edge. We'll choose ⋆ in FIGURE 15 as this vertex. Its neighbors are the six monads ABCDEF and G, and the other neighbors of G are the totals. This suggests that to place all seven neighbors of ⋆ on an equal footing we should recognize G as a seventh monad and interpret the other neighbors of an original monad I as the totals on the set of the six other monads, so that what we before called Xn is reinterpreted as the total n with X replaced by G. Therefore the vertices adjacent to a numbered total n on ABCDEF are just the totals Xn on ABCDEFG that differ from it only by a single-letter substitution. In fact this turns out to be true of any pair of totals, determining all remaining edges of the graph. The resulting picture of the Hoffman-Singleton graph is FIGURE 16.

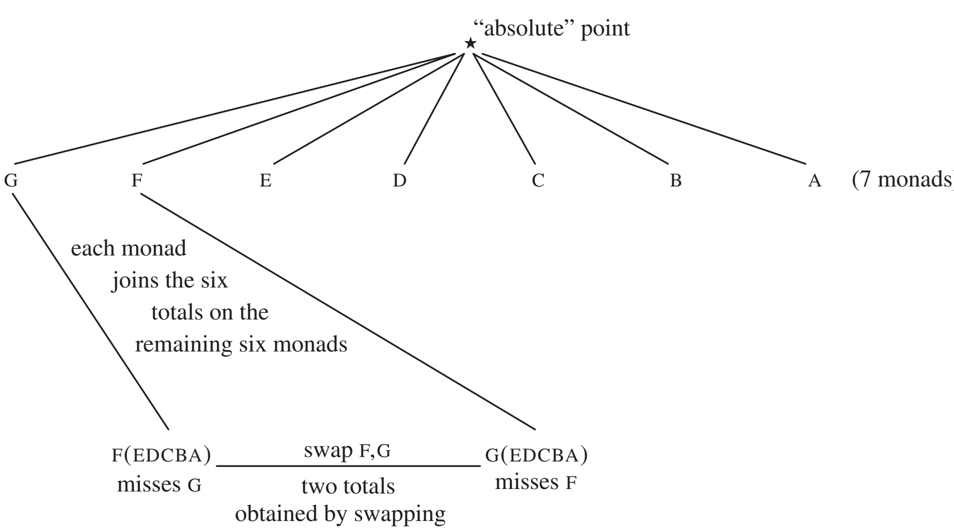

Figure 16 Conway's description of the Hoffman-Singleton graph

The Steiner system $S(5, 6, 12)$

The Steiner system $S(5, 6, 12)$ is a set of *blocks* of 6 elements, *hexads*, chosen from a set of 12 so that each *pentad*, or choice of 5 elements from the 12, occurs exactly once in a block. Hence the number of blocks is $\binom{12}{5}/\binom{6}{5} = 132$.

We use A B C D E F ∞ 0 1 2 3 4 for our 12 elements: in fact ABCDEF and ∞01234 will be two of the blocks. We get $15 \times 6 = 90$ blocks that contain four letters and two numbers, or two letters and four numbers, from the fifteen swaps of TABLE 3.

For example, the swap

$$(\text{FB})\ (\infty 2)(34)(01)$$

yields the six blocks

A2CDE∞, A4CDE3, A1CDE0, F01B34, ∞012FB, ∞FB234

where the pairs of numbers ∞2, 34, 01 have been substituted for the pair of letters FB in ABCDEF and, conversely, the letters FB have been substituted for the pairs of numbers in ∞01234.

The other 40 blocks have three letters and three numbers and may be generated in pairs from the $\binom{6}{3} = 20$ three-cycles of TABLE 4, by substitutions exchanging three letters and three digits. That table omits the three-cycles moving the monad A, but all we need here is the partition of the totals into the two three-cycles that these induce, and this partition is the same one that arises from the cycles on the other three monads. So for instance the cycles (BDE) and (BED) correspond to the permutations (∞32)(014) and (∞23)(041) while (ACF) and (AFC) correspond to (∞23)(014) and (∞32)(041).

For example, the cycle (BEF) associated with the permutation (∞30)(214) gives rise to the four blocks

A∞CD30 A1CD42 ∞0BF3E BF12E4

How do we know that each pentad occurs exactly once? If a pentad consists of 5 letters, or 5 numbers, then the hexad is ABCDEF or ∞01234. If it consists of 4 letters and a number n the hexad will contain a second number. This is found in TABLE 3 which displays all $\binom{6}{2} = 15$ swaps of two vertices. Select the swap of the two letters which are not in the pentad and take the number paired with n. For example, given the pentad ACEF3, look at the entry (DB) (∞1)(24)(30) where 3 is paired with 0, so that the pentad belongs to the unique hexad A3C0EF. If the pentad contains 4 numbers and a letter, for example, ∞024B, find the entries of TABLE 3 that contain the missing numbers 13, namely (AD), (CF), (BE). Here B is paired with E, so the hexad is ∞0B2E4. If the pentad contains 3 letters and 2 numbers, or 3 numbers and 2 letters, we use TABLE 4. For example, for BCF23 we find (FBC) (∞41)(203) so that the hexad is completed with 0. But if the pentad were BCF24, with 2 and 4 in different triples, the hexad must be completed with a letter. In TABLE 3 the pair (24) occurs in the swaps (AE), (BD) and (FC), so the missing letter is D: 2BCD4F.

If the pentad were BCF02, then (02) occurs in (AC), (DF), (EB) with B C F in three different pairs: the pentad requires a number; TABLE 4 gives (FBC) (∞41)(203); the missing number is 3.

A (12, 132, 4) binary code and the ternary Golay code $\mathcal{C}_{12}$

In a binary code, the letters of the codewords are zeroes and ones. The number of letters in a codeword is its length and the number of ones is its weight. The 132 hexads of the Steiner system $S(5, 6, 12)$ form a basis for a binary code with words of length 12 and weight 6.

The blocks of the Steiner system indicate which letters of the 12-letter codewords are occupied by six ones or by six zeroes. In anticipation of the construction of the ternary Golay code $\mathcal{C}_{12}$ we will put the letters in the order

$$\text{A}\ \ 0\ 1\ 2\ 3\ 4\ \infty\ \text{B C D E F}$$

and, for ease of reading, we will leave space round the 1st and 7th letters.

For example, our initial blocks ABCDEF and ∞01234 correspond to the codewords 1 00000 0 11111 and 0 11111 1 00000; the blocks

A2CDE∞, A4CDE3, A1CDE0,

and their complements

F01B34, ∞012FB, ∞FB234

correspond respectively to the codewords

1 00100 1 01110, 1 00011 0 01110, 1 11000 0 01110,

and their complements

0 11011 0 10001, 0 11100 1 10001, 0 00111 1 10001,

while the blocks

∞B30EF, 1B24EF, AD12C4, ∞0AC3D

correspond to

0 10010 1 10011, 0 01101 0 10011, 1 01101 0 01100, 1 10010 1 01100.

Each codeword differs from every other in at least four places, that is, the *Hamming distance* between any two words is at least 4.

Suppose that you received a codeword 0 01101 1 10101. This contains seven ones, so there is an error. Assume that the zeroes are correct. They correspond to the pentad A03CE. TABLE 4 has the (complementary to A C E) entry (FBD) (∞04)(231); 0 and 3 are in different triples, so the missing element is a letter. In TABLE 3 the pair (30) occurs in (DB), (EC), and (AF), so that the missing letter is F and the final 1 in the erroneous codeword should have been 0, making it 0 01101 1 10100.

We can pass from this binary code to a ternary code, which we now present in outline.

To incorporate the words of our binary code into a ternary code we will leave the zeroes as they are and endow the ones with signs. With a correct choice of signs the resulting 132 words of length 12 can be made to generate by addition a *linear code* of dimension 6, that is a 6-dimensional subspace of the ambient vector space $\mathbb{F}_3^{12}$ over the finite field $\mathbb{F}_3 = \{-1, 0, +1\}$. Our code will thus contain $3^6 = 729$ codewords.

Aside from the zero word 0 00000 0 00000, the words will come in pairs of opposite sign. In fact, we will obtain no nonzero codewords with more zeroes than the signed manifestations, two apiece, of our 132 words from the binary code. So the minimal distance of our code will increase to 6. The resulting code is known as the *ternary Golay code* and denoted as $\mathcal{C}_{12}$.

From [**5**, p.85] we learn that $\mathcal{C}_{12}$ may be obtained by appending a zero-sum check digit to $\mathcal{C}_{11}$, the quadratic residue code of length 11 over $\mathbb{F}_3$; that a generator matrix is

$$\begin{array}{cccccccccccc}
\text{A} & 0 & 1 & 2 & 3 & 4 & \infty & \text{B} & \text{C} & \text{D} & \text{E} & \text{F}
\end{array}$$
$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 0
\end{bmatrix}$$

that it has weight enumerator

$$x^{12} + 264x^6y^6 + 440x^3y^9 + 24y^{12}$$

that is it contains 1, 264, 440, and 24 words with respectively 0, 6, 9, and 12 nonzero letters, and that its automorphism group is $2.M_{12}$, that is it has the Mathieu group M_{12} as a normal subgroup with quotient (cyclic of order) 2.

But by now we've roved far enough from the Tricky Six puzzle, so we pursue codes no further and turn to the

Conclusion

Our favorite for an actual puzzle changes C into W and T into D, turning RECANT into REWAND. Manoeuvre #1 of the "Not much of a puzzle" section then gives the figure on the cover of this MAGAZINE, which should be read clockwise, starting from twelve o'clock. The solution: move the letters R E W A R E W A R E and read clockwise from noon again.

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LOCI: Interactive Supplements to "Rick's Tricky Six Puzzle"