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An Infinite Product for the Golden Ratio

We give a Wallis-type infinite product for the golden ratio ϕ, in which the number 25, the number 9, and twice the triangular numbers $T_k = k(k+1)/2$ play a crucial role.

Theorem 1.

$$\begin{aligned}\phi &= \frac{1+\sqrt{5}}{2} \\ &= \frac{1}{2}\prod_{k=0}^{\infty}\frac{100k(k+1)+5^2}{100k(k+1)+3^2} \\ &= \frac{1}{2}\cdot\frac{25}{9}\cdot\frac{225}{209}\cdot\frac{625}{609}\cdot\frac{1225}{1209}\cdot\frac{2025}{2009}\cdots.\end{aligned}$$

Proof. Substituting Euler's infinite product for the gamma function [**1**, p. 357]

$$\Gamma(x) = \frac{1}{x}\prod_{k=1}^{\infty}\frac{(1+1/k)^x}{1+x/k}, \qquad x \neq 0, -1, -2, \ldots$$

into the *reflection formula* [**1**, p. 364] $\Gamma(x)\Gamma(1-x) = \pi \csc \pi x$, $x \notin \mathbb{Z}$, with the choice $x = 1/n$, $n > 1$, we obtain

$$\begin{aligned}\frac{\pi/n}{\sin \pi/n} &= \frac{n}{n-1}\cdot\frac{n\cdot 2n}{(n+1)(2n-1)} \\ &\quad \cdot\frac{2n\cdot 3n}{(2n+1)(3n-1)}\cdot\frac{3n\cdot 4n}{(3n+1)(4n-1)}\cdots. \qquad (11)\end{aligned}$$

Since $\sin \pi/10 = (\sqrt{5}-1)/4$, then $\csc \pi/10 = 2\phi$. Thus, the division of (11) with $n = 10$ by (11) with $n = 2$ gives, after straightforward manipulations, the desired conclusion. ■

REFERENCES

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