An Infinite Product for the Golden Ratio

We give a Wallis-type infinite product for the golden ratio \( \phi \), in which the number 25, the number 9, and twice the triangular numbers \( T_k = k(k+1)/2 \) play a crucial role.

**Theorem 1.**

\[
\phi = \frac{1 + \sqrt{5}}{2} = \frac{1}{2} \prod_{k=0}^{\infty} \frac{100k(k+1) + 5^2}{100k(k+1) + 3^2} = \frac{1}{2} \cdot \frac{25}{9} \cdot \frac{225}{209} \cdot \frac{625}{609} \cdot \frac{1225}{1209} \cdot \frac{2025}{2009} \cdots
\]

**Proof.** Substituting Euler’s infinite product for the gamma function [1, p. 357]

\[
\Gamma(x) = \frac{x^n}{\prod_{k=1}^{\infty} \left(1 + 1/k\right)^x}, \quad x \neq 0, -1, -2, \ldots
\]

into the reflection formula [1, p. 364] \( \Gamma(x)\Gamma(1-x) = \pi \csc \pi x, x \notin \mathbb{Z} \), with the choice \( x = 1/n, n > 1 \), we obtain

\[
\frac{\pi/n}{\sin \pi/n} = \frac{n}{n-1} \cdot \frac{n \cdot 2n}{(n+1)(2n-1)} \cdot \frac{2n \cdot 3n}{(2n+1)(3n-1)} \cdot \frac{3n \cdot 4n}{(3n+1)(4n-1)} \cdots
\]

Since \( \sin \pi/10 = (\sqrt{5} - 1)/4 \), then \( \csc \pi/10 = 2\phi \). Thus, the division of (11) with \( n = 10 \) by (11) with \( n = 2 \) gives, after straightforward manipulations, the desired conclusion. \( \blacksquare \)

**REFERENCES**


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