Cooperative Game Theory

Jennifer Wilson

Department of Natural Sciences and Mathematics
Eugene Lang College The New School for Liberal Arts

August 6, 2008

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Player 3: A

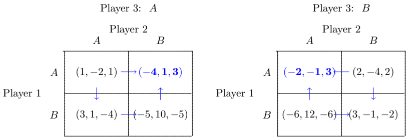
		Player 2	
		A	B
Player 1	A	$(1, -2, 1)$	$(-4, 1, 3)$
	B	$(3, 1, -4)$	$(-5, 10, -5)$

Player 3: B

		Player 2	
		A	B
Player 1	A	$(-2, -1, 3)$	$(2, -4, 2)$
	B	$(-6, 12, -6)$	$(3, -1, -2)$

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example



Player 3: A

Player 2

		A	B
Player 1	A	$(1,-2,1)$	$\mathbf{(-4,1,3)}$
	B	$(3,1,-4)$	$(-5,10,-5)$

Player 3: B

Player 2

		A	B
Player 1	A	$\mathbf{(-2,-1,3)}$	$(2,-4,2)$
	B	$(-6,12,-6)$	$(3,-1,-2)$

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Player 3: A

		Player 2	
		A	B
Player 1	A	$(1,-2,1)$	$(-4,1,3)$
	B	$(3,1,-4)$	$(-5,10,-5)$

Player 3: B

		Player 2	
		A	B
Player 1	A	$(-2,-1,3)$	$(2,-4,2)$
	B	$(-6,12,-6)$	$(3,-1,-2)$

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$(1,-1)$	$(-2,2)$	$(-4,4)$	$(2,-2)$
	B	$(3,-3)$	$(-6,6)$	$(-5,5)$	$(3,-3)$

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$(1,-1)$	$(-2,2)$	$(-4,4)$	$(2,-2)$
	B	$(3,-3)$	$(-6,6)$	$(-5,5)$	$(3,-3)$

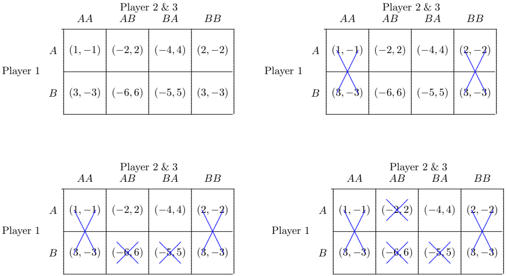
		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	~~$(1,-1)$~~	$(-2,2)$	$(-4,4)$	~~$(2,-2)$~~
	B	~~$(3,-3)$~~	$(-6,6)$	$(-5,5)$	~~$(3,-3)$~~

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	~~$(1,-1)$~~	$(-2,2)$	$(-4,4)$	~~$(2,-2)$~~
	B	~~$(3,-3)$~~	~~$(-6,6)$~~	~~$(-5,5)$~~	~~$(3,-3)$~~

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	~~$(1,-1)$~~	~~$(-2,2)$~~	$(-4,4)$	~~$(2,-2)$~~
	B	~~$(3,-3)$~~	~~$(-6,6)$~~	~~$(-5,5)$~~	~~$(3,-3)$~~

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example



		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$(1,-1)$	$(-2,2)$	$(-4,4)$	$(2,-2)$
	B	$(3,-3)$	$(-6,6)$	$(-5,5)$	$(3,-3)$

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$\cancel{(1,-1)}$	$(-2,2)$	$(-4,4)$	$\cancel{(2,-2)}$
	B	$\cancel{(3,-3)}$	$(-6,6)$	$(-5,5)$	$\cancel{(3,-3)}$

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$\cancel{(1,-1)}$	$(-2,2)$	$(-4,4)$	$\cancel{(2,-2)}$
	B	$\cancel{(3,-3)}$	$\cancel{(-6,6)}$	$\cancel{(-5,5)}$	$\cancel{(3,-3)}$

		Player 2 & 3			
		AA	AB	BA	BB
Player 1	A	$\cancel{(1,-1)}$	$\cancel{(-2,2)}$	$(-4,4)$	$\cancel{(2,-2)}$
	B	$\cancel{(3,-3)}$	$\cancel{(-6,6)}$	$\cancel{(-5,5)}$	$\cancel{(3,-3)}$

So $\nu(1) = -4$ and $\nu(2,3) = 4$.

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

		Player 1 & 3			
		AA	AB	BA	BB
Player 2	A	$(2,-2)$	$(-1,1)$	$(-1,1)$	$(12,-12)$
	B	$(1,-1)$	$(-4,4)$	$(10,-10)$	$(-1,1)$

		Player 1 & 3			
		AA q	AB $1-q$	BA	BB
Player 2	A p	$(2,-2)$	$(-1,1)$	~~$(-1,1)$~~	~~$(12,-12)$~~
	B $1-p$	$(1,-1)$	$(-4,4)$	~~$(10,-10)$~~	~~$(-1,1)$~~

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

		Player 1 & 3			
		AA	AB	BA	BB
Player 2	A	$(2,-2)$	$(-1,1)$	$(-1,1)$	$(12,-12)$
	B	$(1,-1)$	$(-4,4)$	$(10,-10)$	$(-1,1)$

		Player 1 & 3			
		AA q	AB $1-q$	BA	BB
Player 2	A p	$(2,-2)$	$(-1,1)$	~~$(-1,1)$~~	~~$(12,-12)$~~
	B $1-p$	$(1,-1)$	$(-4,4)$	~~$(10,-10)$~~	~~$(-1,1)$~~

Solving, gives $p = \frac{5}{6}$ and $q = \frac{1}{2}$.
So $\nu(2) = -1.5$ and $\nu(1,3) = 1.5$.

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

		Player 1 & 3			
		AA	AB	BA	BB
Player 2	A	$(2,-2)$	$(-1,1)$	$(-1,1)$	$(12,-12)$
	B	$(1,-1)$	$(-4,4)$	$(10,-10)$	$(-1,1)$

		Player 1 & 3			
		AA q	AB $1-q$	BA	BB
Player 2	A p	$(2,-2)$	$(-1,1)$	~~$(-1,1)$~~	~~$(12,-12)$~~
	B $1-p$	$(1,-1)$	$(-4,4)$	~~$(10,-10)$~~	~~$(-1,1)$~~

Solving, gives $p=\frac{5}{6}$ and $q=\frac{1}{2}$.
So $\nu(2)=-1.5$ and $\nu(1,3)=1.5$.
Similarly $\nu(3)=-4.48$ and $\nu(1,2)=4.48$.

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Summary:
$\nu(1) = -4,\ \nu(2) = -1.5,\ \nu(3) = -4.48,$
$\nu(2,3) = 4,\ \nu(1,2) = 4.48$ and $\nu(1,3) = 1.5.$

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Summary:

$\nu(1) = -4,\ \nu(2) = -1.5,\ \nu(3) = -4.48,$
$\nu(2,3) = 4,\ \nu(1,2) = 4.48$ and $\nu(1,3) = 1.5.$

It's advantageous to form a coalition.

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Summary:
$\nu(1) = -4,\ \nu(2) = -1.5,\ \nu(3) = -4.48,$
$\nu(2,3) = 4,\ \nu(1,2) = 4.48$ and $\nu(1,3) = 1.5.$

It's advantageous to form a coalition.

Which coalitions should form?

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Summary:
$\nu(1) = -4,\ \nu(2) = -1.5,\ \nu(3) = -4.48,$
$\nu(2,3) = 4,\ \nu(1,2) = 4.48$ and $\nu(1,3) = 1.5.$

It's advantageous to form a coalition.

Which coalitions should form?
Player 1 versus Players 2 & 3: $(-4, 1, 3)$
Player 2 versus Players 1 & 3: $(-0.58, 1.5, 2.08)$
Player 3 versus Players 1 & 2: $(-0.56, 5.04, -4.48)$

A Three-Person Zero-Sum Game [Rapoport, 1970]

Example

Summary:
$\nu(1) = -4,\ \nu(2) = -1.5,\ \nu(3) = -4.48,$
$\nu(2,3) = 4,\ \nu(1,2) = 4.48$ and $\nu(1,3) = 1.5.$

It's advantageous to form a coalition.

Which coalitions should form?
Player 1 versus Players 2 & 3: $(-4, 1, 3)$
Player 2 versus Players 1 & 3: $(-0.58, 1.5, 2.08)$
Player 3 versus Players 1 & 2: $(-0.56, 5.04, -4.48)$

What should each player receive from the game?

Assumptions

- Players may communicate and form coalitions with one another.
- Players may make sidepayments to each other.

Assumptions

- Players may communicate and form coalitions with one another.
- Players may make sidepayments to each other.
  - Utility is *transferable* between players.

Assumptions

- Players may communicate and form coalitions with one another.
- Players may make sidepayments to each other.
    - Utility is *transferable* between players.
    - Players' utility values are *comparable*.

Assumptions

- Players may communicate and form coalitions with one another.
- Players may make sidepayments to each other.
  - Utility is *transferable* between players.
  - Players' utility values are *comparable*.

We will look at *transferable utitility*(TU) games.

Characteristic Function Form

The characteristic function ν of a game is a function that assigns a value for each coalition (subset) of players.

Characteristic Function Form

The characteristic function ν of a game is a function that assigns a value for each coalition (subset) of players.

If the game is constant-sum then

- $\nu(\emptyset) = 0$

Characteristic Function Form

The characteristic function ν of a game is a function that assigns a value for each coalition (subset) of players.

If the game is constant-sum then

- $\nu(\emptyset) = 0$
- $\nu(N) = K$

Characteristic Function Form

The characteristic function ν of a game is a function that assigns a value for each coalition (subset) of players.

If the game is constant-sum then

- $\nu(\emptyset) = 0$
- $\nu(N) = K$
- $\nu(S) + \nu(N \setminus S) = K$

Cooperative Games

Definition

A cooperative game is a set N of players, with a function $\nu : 2^N \longrightarrow \Re$ such that $\nu(\emptyset) = 0$.

Cooperative Games

Definition

A cooperative game is a set N of players, with a function $\nu : 2^N \longrightarrow \Re$ such that $\nu(\emptyset) = 0$.

Definition

A cooperative game is called *superadditive* if

$$\nu(S \cup T) \geq \nu(S) + \nu(T) \quad \text{for all } S \cap T = \emptyset.$$

Cooperative Games

Definition

A cooperative game is a set N of players, with a function $\nu : 2^N \longrightarrow \Re$ such that $\nu(\emptyset) = 0$.

Definition

A cooperative game is called *superadditive* if

$$\nu(S \cup T) \geq \nu(S) + \nu(T) \quad \text{for all } S \cap T = \emptyset.$$

Every normal form game can be put in characteristic (cooperative) form.

There are cooperative games that do not correspond to any normal form games.

Cooperative Games

In *non-cooperative* game theory, the goals are to

- determine the optimal strategies for each player, and to
- find the equilibria of the game when the players play their optimal strategies.

Cooperative Games

In *non-cooperative* game theory, the goals are to

- determine the optimal strategies for each player, and to
- find the equilibria of the game when the players play their optimal strategies.

In *cooperative* game theory the goals are to

- determine which coalitions should form, and to
- decide how each player should be rewarded.

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns \$10,000 to creditor A, \$20,000 to creditor B and \$30,000 to creditor C. If the company has a total of \$36,000 in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A,B) = 6 \;\; v(A,C) = 16 \;\; v(B,C) = 26 \text{ and } v(A,B,C) = 36$$

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A,B) = 6 \;\; v(A,C) = 16 \;\; v(B,C) = 26 \text{ and } v(A,B,C) = 36$$

What are the minimum requirements for a settlement?

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns \$10,000 to creditor A, \$20,000 to creditor B and \$30,000 to creditor C. If the company has a total of \$36,000 in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$v(A, B) = 6 \;\; v(A, C) = 16 \;\; v(B, C) = 26$ and $v(A, B, C) = 36$

What are the minimum requirements for a settlement?

Let x_1, x_2 and x_3 be the payments offered to each creditor.

The Bankruptcy Game [O'Neill, 1982]

Example

A small company goes bankrupt owing money to three creditors. It owns \$10,000 to creditor A, \$20,000 to creditor B and \$30,000 to creditor C. If the company has a total of \$36,000 in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A, B) = 6 \;\; v(A, C) = 16 \;\; v(B, C) = 26 \;\text{ and }\; v(A, B, C) = 36$$

What are the minimum requirements for a settlement?

Let x_1, x_2 and x_3 be the payments offered to each creditor.

$$x_1 \geq 0 \;\; x_2 \geq 0 \;\; x_3 \geq 6 \;\text{ and }\; x_1 + x_2 + x_3 = 36$$

.

The Bankruptcy Game [O'Neill, 1982]

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \quad v(B) = 0 \quad v(C) = 6$$

The Bankruptcy Game [O'Neill, 1982]

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A, B) = 6 \;\; v(A, C) = 16 \;\; v(B, C) = 26 \text{ and } v(A, B, C) = 36$$

$$x_1 \geq 0 \;\; x_2 \geq 0 \;\; x_3 \geq 6 \text{ and } x_1 + x_2 + x_3 = 36$$

.

The Bankruptcy Game [O'Neill, 1982]

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A,B) = 6 \;\; v(A,C) = 16 \;\; v(B,C) = 26 \;\text{ and }\; v(A,B,C) = 36$$

$$x_1 \geq 0 \;\; x_2 \geq 0 \;\; x_3 \geq 6 \;\text{ and }\; x_1 + x_2 + x_3 = 36$$

.

What additional requirements should there be?

The Bankruptcy Game [O'Neill, 1982]

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A,B) = 6 \;\; v(A,C) = 16 \;\; v(B,C) = 26 \;\text{ and }\; v(A,B,C) = 36$$

$$x_1 \geq 0 \;\; x_2 \geq 0 \;\; x_3 \geq 6 \;\text{ and }\; x_1 + x_2 + x_3 = 36$$

.

What additional requirements should there be?

$$x_1 + x_2 \geq 6 \quad x_1 + x_3 \geq 16 \quad x_2 + x_3 \geq 26$$

The Bankruptcy Game [O'Neill, 1982]

A small company goes bankrupt owing money to three creditors. It owns $\$10,000$ to creditor A, $\$20,000$ to creditor B and $\$30,000$ to creditor C. If the company has a total of $\$36,000$ in assets, how should the money be divided?

$$v(A) = 0 \;\; v(B) = 0 \;\; v(C) = 6$$

$$v(A,B) = 6 \;\; v(A,C) = 16 \;\; v(B,C) = 26 \text{ and } v(A,B,C) = 36$$

$$x_1 \geq 0 \;\; x_2 \geq 0 \;\; x_3 \geq 6 \text{ and } x_1 + x_2 + x_3 = 36$$

.

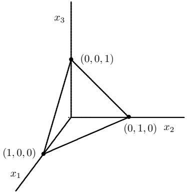
What additional requirements should there be?

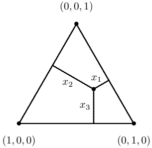
$$x_1 + x_2 \geq 6 \quad x_1 + x_3 \geq 16 \quad x_2 + x_3 \geq 26$$

$$x_3 \leq 30 \quad x_2 \leq 20 \quad x_1 \leq 10$$

Graphing on the Simplex

The 2-dimensional simplex
$\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, \text{ and } x_3 \geq 0.\}$.

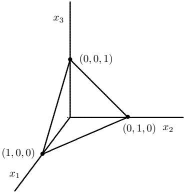


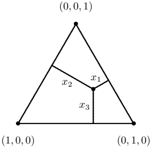


Graphing on the Simplex

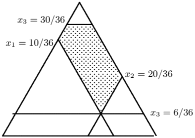
The 2-dimensional simplex

$\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0,$ and $x_3 \geq 0.\}$.





Possible “solutions” of the Bankrupcy Game.



Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i)$$

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

and

$$\sum_i x_i = \nu(N)$$

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

and

$$\sum_i x_i = \nu(N) \quad \text{collective rationality.}$$

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

and

$$\sum_i x_i = \nu(N) \quad \text{collective rationality.}$$

If $\sum_i \nu(i) = \nu(N)$, then $x_i = \nu(i)$.

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

and

$$\sum_i x_i = \nu(N) \quad \text{collective rationality.}$$

If $\sum_i \nu(i) = \nu(N)$, then $x_i = \nu(i)$.
These games are called *inessential*.

Imputations

Definition

Given a cooperative game ν on n players, an *imputation* is a payoff vector

$$x_1, x_2, \ldots, x_n$$

such that

$$x_i \geq \nu(i) \quad \text{individual rationality}$$

and

$$\sum_i x_i = \nu(N) \quad \text{collective rationality.}$$

If $\sum_i \nu(i) = \nu(N)$, then $x_i = \nu(i)$.
These games are called *inessential*.

We will assume ν is essential.

The Core

Definition

The core is the set of imputations such that

$$\sum_{i \in S} x_i \geq v(S) \quad \text{for every } S \subset N.$$

The Glove Market

Example

A market consists of people with left-handed glove and people with right-handed gloves (but not both). The value of a coalition is the number of complete pairs of gloves it has.

The Glove Market

Example

A market consists of people with left-handed glove and people with right-handed gloves (but not both). The value of a coalition is the number of complete pairs of gloves it has.

We will look at the case when A_1 and A_2 have left-handed glove and B_1, B_2 and B_3 have right-handed gloves.

The Glove Market

Example

A market consists of people with left-handed glove and people with right-handed gloves (but not both). The value of a coalition is the number of complete pairs of gloves it has.

We will look at the case when A_1 and A_2 have left-handed glove and B_1, B_2 and B_3 have right-handed gloves.

$$\nu(A_i) = \nu(B_j) = 0 \quad \nu(A_1, A_2) = \nu(B_i, B_j) = 0 \quad \nu(A_i, B_i) = 1$$

The Glove Market

Example

A market consists of people with left-handed glove and people with right-handed gloves (but not both). The value of a coalition is the number of complete pairs of gloves it has.

We will look at the case when A_1 and A_2 have left-handed glove and B_1, B_2 and B_3 have right-handed gloves.

$$\nu(A_i) = \nu(B_j) = 0 \quad \nu(A_1, A_2) = \nu(B_i, B_j) = 0 \quad \nu(A_i, B_i) = 1$$

$$\nu(A_i, B_{i_1}, B_{i_2}) = 1 \quad \ldots$$

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

It is an imputation if

$$x_i \geq 0, \quad y_j \geq 0 \quad y_2 \geq 0 \quad x_1 + x_2 + y_1 + y_2 + y_3 = 2.$$

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

It is an imputation if

$$x_i \geq 0, \quad y_j \geq 0 \quad y_2 \geq 0 \quad x_1 + x_2 + y_1 + y_2 + y_3 = 2.$$

The payoff vector is in the core if

$$x_1 + y_1 \geq 1 \quad x_2 + y_2 \geq 1 \quad \ldots$$

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

It is an imputation if

$$x_i \geq 0, \quad y_j \geq 0 \quad y_2 \geq 0 \quad x_1 + x_2 + y_1 + y_2 + y_3 = 2.$$

The payoff vector is in the core if

$$x_1 + y_1 \geq 1 \quad x_2 + y_2 \geq 1 \quad \ldots$$

But

$$(x_1 + y_1) + (x_2 + y_2) \geq 1 + 1 = 2.$$

Hence

$$(x_1 + y_1) = (x_2 + y_2) = 1 \quad \text{and} \quad y_3 = 0.$$

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

It is an imputation if

$$x_i \geq 0, \quad y_j \geq 0 \quad y_2 \geq 0 \quad x_1 + x_2 + y_1 + y_2 + y_3 = 2.$$

The payoff vector is in the core if

$$x_1 + y_1 \geq 1 \quad x_2 + y_2 \geq 1 \quad \dots$$

But

$$(x_1 + y_1) + (x_2 + y_2) \geq 1 + 1 = 2.$$

Hence

$$(x_1 + y_1) = (x_2 + y_2) = 1 \quad \text{and } y_3 = 0.$$

So $y_1 = y_2 = y_3 = 0$, and $x_1 = x_2 = 1$.

The Glove Market

Let $(x_1, x_2, y_1, y_2, y_3)$ be the payoff vector.

It is an imputation if

$$x_i \geq 0, \quad y_j \geq 0 \quad y_2 \geq 0 \quad x_1 + x_2 + y_1 + y_2 + y_3 = 2.$$

The payoff vector is in the core if

$$x_1 + y_1 \geq 1 \quad x_2 + y_2 \geq 1 \quad \ldots$$

But

$$(x_1 + y_1) + (x_2 + y_2) \geq 1 + 1 = 2.$$

Hence

$$(x_1 + y_1) = (x_2 + y_2) = 1 \quad \text{and } y_3 = 0.$$

So $y_1 = y_2 = y_3 = 0$, and $x_1 = x_2 = 1$.

The core consists of the single point $(1, 1, 0, 0, 0)$.

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 1$$

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 \geq 1 \quad x_1 + x_3 \geq 1 \quad x_2 + x_3 \geq 1.$$

Divide the Dollar
[Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 \geq 1 \quad x_1 + x_3 \geq 1 \quad x_2 + x_3 \geq 1.$$

This is impossible.

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them. The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 \geq 1 \quad x_1 + x_3 \geq 1 \quad x_2 + x_3 \geq 1.$$

This is impossible.

Hence the core is empty.

Divide the Dollar [Von Neumann & Morgenstern, 1947]

Example

Three players are given a dollar to divide amongst them.
The decision is to be made by majority rule.

$$\nu(1) = \nu(2) = \nu(3) = 0.$$
$$\nu(1,2) = \nu(1,3) = \nu(2,3) = \nu(1,2,3) = 1.$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0 \quad x_1 + x_2 + x_3 = 1$$

$$x_1 + x_2 \geq 1 \quad x_1 + x_3 \geq 1 \quad x_2 + x_3 \geq 1.$$

This is impossible.

Hence the core is empty.
All constant-sum games have empty cores.

Divide the Dollar

Problem: If Players 1 and 2 agree to a (0.50, 0.50, 0) split,

Divide the Dollar

Problem: If Players 1 and 2 agree to a (0.50, 0.50, 0) split, Player 3 can offer Player 1 a $(0.60, 0, 0.40)$ split.

Divide the Dollar

Problem: If Players 1 and 2 agree to a (0.50, 0.50, 0) split, Player 3 can offer Player 1 a $(0.60, 0, 0.40)$ split.
Player 2 can retaliate by offering Player 3 a $(0, 0.50, 0.50)$ split.

Divide the Dollar

Problem: If Players 1 and 2 agree to a (0.50, 0.50, 0) split, Player 3 can offer Player 1 a $(0.60, 0, 0.40)$ split.
Player 2 can retaliate by offering Player 3 a $(0, 0.50, 0.50)$ split.
Player 1 can offer....

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

- $y_i > x_i$ for every $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$.

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

- $y_i > x_i$ for every $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$.

Consider the two imputations:

$$\mathbf{x} = (0.50, 0.50, 0) \quad \text{and} \quad \mathbf{y} = (0.60, 0, 0.40).$$

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

- $y_i > x_i$ for every $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$.

Consider the two imputations:

$$\mathbf{x} = (0.50, 0.50, 0) \quad \text{and} \quad \mathbf{y} = (0.60, 0, 0.40).$$

Players 1 and 3 can 'force' player 2 to accept **y**.

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

- $y_i > x_i$ for every $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$.

Consider the two imputations:

$$\mathbf{x} = (0.50, 0.50, 0) \quad \text{and} \quad \mathbf{y} = (0.60, 0, 0.40).$$

Players 1 and 3 can 'force' player 2 to accept **y**.

So $(0.50, 0, 50, 0)$ is dominated by $(0.60, 0, 0.40)$ through $S = \{1, 3\}$

Dominance Relations [Von Neumann and Morgenstern, 1947]

Definition

Imputation **x** is *dominated* by imputation **y** *through* coalition S if

- $y_i > x_i$ for every $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$.

Consider the two imputations:

$$\mathbf{x} = (0.50, 0.50, 0) \quad \text{and} \quad \mathbf{y} = (0.60, 0, 0.40).$$

Players 1 and 3 can 'force' player 2 to accept **y**.

So $(0.50, 0, 50, 0)$ is dominated by $(0.60, 0, 0.40)$ through $S = \{1, 3\}$ which is dominated by $(0, 0.50, 0.50)$ through coalition $S = \{2, 3\}$ which is dominated by ...

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose $\mathbf{x}$ is dominated by $\mathbf{y}$ through the coalition S.

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose **x** is dominated by **y** through the coalition S. Then

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq \nu(S)$$

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose **x** is dominated by **y** through the coalition S. Then

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq \nu(S)$$

hence **x** is not in the core.

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose **x** is dominated by **y** through the coalition S. Then

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq \nu(S)$$

hence **x** is not in the core.

Suppose **x** is an imputation that is not in the core and $\sum_{i \in S} x_i < \nu(S)$.

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose **x** is dominated by **y** through the coalition S. Then

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq \nu(S)$$

hence **x** is not in the core.

Suppose **x** is an imputation that is not in the core and $\sum_{i \in S} x_i < \nu(S)$.
Let

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Dominance Relations

The core consists of all undominated imputations.

Proof.

Suppose **x** is dominated by **y** through the coalition S. Then

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq \nu(S)$$

hence **x** is not in the core.

Suppose **x** is an imputation that is not in the core and $\sum_{i \in S} x_i < \nu(S)$.
Let

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

☐

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \setminus S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$
- $\mathbf{y}$ is an imputation

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$
- **y** is an imputation
  - $y_i \geq \nu(i)$ for all i

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i\in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N\setminus S|}[\nu(N) - (\nu(S) + \sum_{i\notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$
- $\nu(S) \geq \sum_{i\in S} y_i$
- $\mathbf{y}$ is an imputation
  - $y_i \geq \nu(i)$ for all i
  - $\sum_i y_i = \nu(N)$

Dominance Relations

$$y_i = \begin{cases} x_i + \frac{1}{|S|}[\nu(S) - \sum_{i \in S} x_i] & \text{if } i \in S \\ \nu(i) + \frac{1}{|N \backslash S|}[\nu(N) - (\nu(S) + \sum_{i \notin S} \nu(i))] & \text{if } i \notin S. \end{cases}$$

Claim:

- $y_i > x_i$ for all $i \in S$
- $\nu(S) \geq \sum_{i \in S} y_i$
- **y** is an imputation
  - $y_i \geq \nu(i)$ for all i
  - $\sum_i y_i = \nu(N)$

Hence **y** is an imputation that dominates **x**.

Stable Sets

Definition

A *stable set* is a set I of imputations such that

- no imputation in I is dominated by any other imputation in I

Stable Sets

Definition

A *stable set* is a set I of imputations such that

- no imputation in I is dominated by any other imputation in I *internally stable*

Stable Sets

Definition

A *stable set* is a set I of imputations such that

- no imputation in I is dominated by any other imputation in I *internally stable*
- every imputation not in I is dominated by some imputation in I.

Stable Sets

Definition

A *stable set* is a set I of imputations such that

- no imputation in I is dominated by any other imputation in I *internally stable*
- every imputation not in I is dominated by some imputation in I. *externally stable*

Stable Sets and Divide the Dollar

Examples:

$$I_1 = \{(0.50, 0.50, 0), (0.50, 0, 0.50), (0, 0.50, 0.50)\}$$

Stable Sets and Divide the Dollar

Examples:

$$I_1 = \{(0.50, 0.50, 0), (0.50, 0, 0.50), (0, 0.50, 0.50)\}$$

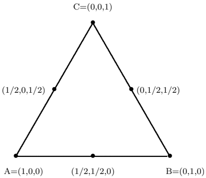
$$I_2 = \{0.7, x, 0.3 - x \mid 0 \leq x \leq 0.7\}$$

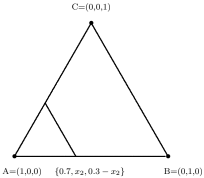
Stable Sets and Divide the Dollar

Examples:

$$I_1 = \{(0.50, 0.50, 0), (0.50, 0, 0.50), (0, 0.50, 0.50)\}$$

$$I_2 = \{0.7, x, 0.3 - x \mid 0 \leq x \leq 0.7\}$$





Other Solution concepts

- Bargaining Set
- Kernel
- Nucleolus
- Shapley Value

Other Solution concepts

- Bargaining Set
- Kernel
- Nucleolus
- Shapley Value

Several solution concepts are based on the idea of the "excess" of an imputation.

Other Solution concepts

- Bargaining Set
- Kernel
- Nucleolus
- Shapley Value

Several solution concepts are based on the idea of the "excess" of an imputation.

$$e(\mathbf{x}, S) = \nu(S) - \sum_{i \in S} x_i$$

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \longrightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$.

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.
  Dummy Property

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \longrightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

  Dummy Property

  (Player i is a dummy if $\nu(S \cup i) = \nu(S)$ for all $S \subset N$.)

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

  Dummy Property

  (Player i is a dummy if $\nu(S \cup i) = \nu(S)$ for all $S \subset N$.)
- $\sum_{i \in N} \phi_i(\nu) = \nu(N)$

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

  Dummy Property

  (Player i is a dummy if $\nu(S \cup i) = \nu(S)$ for all $S \subset N$.)
- $\sum_{i \in N} \phi_i(\nu) = \nu(N)$ *Efficiency*

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \rightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

  Dummy Property

  (Player i is a dummy if $\nu(S \cup i) = \nu(S)$ for all $S \subset N$.)
- $\sum_{i \in N} \phi_i(\nu) = \nu(N)$ *Efficiency*
- $\phi_i(\nu + w) = \phi_i(\nu) + \phi_i(w)$.

The Shapley Value [Shapley, 1953]

Given a cooperative game ν on N players, there exists a unique map $\phi : N \longrightarrow \Re$ which assigns to each player i a value $\phi_i(v)$ satisfying

- If π is a permutation of the players then $\phi_i(\nu_\pi) = \phi_{\pi(i)}(\nu)$. *Symmetry*
- If i is a *dummy* player then $\phi_i(\nu) = 0$.

  Dummy Property

  (Player i is a dummy if $\nu(S \cup i) = \nu(S)$ for all $S \subset N$.)
- $\sum_{i \in N} \phi_i(\nu) = \nu(N)$ *Efficiency*
- $\phi_i(\nu + w) = \phi_i(\nu) + \phi_i(w)$. *Linearity*

Shapley Value

Outline of Proof:

Shapley Value

Outline of Proof:

Let $T \subset N$, let ν_T be the cooperative game

$$\nu_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{if } T \not\subseteq S. \end{cases}$$

Shapley Value

Outline of Proof:

Let $T \subset N$, let ν_T be the cooperative game

$$\nu_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{if } T \nsubseteq S. \end{cases}$$

Let

$$\phi_i(\nu_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T. \end{cases}$$

Shapley Value

Outline of Proof:

Let $T \subset N$, let ν_T be the cooperative game

$$\nu_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{if } T \nsubseteq S. \end{cases}$$

Let

$$\phi_i(\nu_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T. \end{cases}$$

The games ν_T form a basis for the space of cooperative games on N,

Shapley Value

Outline of Proof:

Let $T \subset N$, let ν_T be the cooperative game

$$\nu_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{if } T \not\subseteq S. \end{cases}$$

Let

$$\phi_i(\nu_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T. \end{cases}$$

The games ν_T form a basis for the space of cooperative games on N,

$$\nu = \sum_{T \subset N} c_T \nu_T \quad \text{where } c_T = \sum_{S \subset T} (-1)^{t-s} \nu(S).$$

Shapley Value

Outline of Proof:

Let $T \subset N$, let ν_T be the cooperative game

$$\nu_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{if } T \not\subseteq S. \end{cases}$$

Let

$$\phi_i(\nu_T) = \begin{cases} \frac{1}{|T|} & \text{if } i \in T \\ 0 & \text{if } i \notin T. \end{cases}$$

The games ν_T form a basis for the space of cooperative games on N,

$$\nu = \sum_{T \subset N} c_T \nu_T \quad \text{where } c_T = \sum_{S \subset T} (-1)^{t-s} \nu(S).$$

$$\phi_i(\nu) = \sum_{\substack{S \subset N \\ i \notin S}} \frac{s!(n-1-s)!}{n!} [\nu(S \cup i) - \nu(S)].$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Suppose a river is divided into n segments numbered from upstream to downstream, with one corporation/polluter in each segment. Local environmental agencies dictate the cost c_i in the i^{th} section of the river.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Suppose a river is divided into n segments numbered from upstream to downstream, with one corporation/polluter in each segment. Local environmental agencies dictate the cost c_i in the i^{th} section of the river.

Under the LR (*Local Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} section of the river.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Suppose a river is divided into n segments numbered from upstream to downstream, with one corporation/polluter in each segment. Local environmental agencies dictate the cost c_i in the i^{th} section of the river.

Under the LR (*Local Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} section of the river.

Under the DR (*Downstream Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} through n^{th} sections of the river.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the LR (*Local Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} section of the river.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the LR (*Local Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} section of the river.

For each $S \subset N$, let

$$v^{\mathbf{c}}(S) = \sum_{i \in S} c_i .$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the LR (*Local Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} section of the river.

For each $S \subset N$, let

$$v^{\mathbf{c}}(S) = \sum_{i \in S} c_i.$$

Claim

$$\phi_i(v^{\mathbf{c}}) = c_i.$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(LR)

$$\phi_i(v^{\mathbf{C}}) \quad = \quad \sum_{\substack{S \subset N \\ i \notin S}} \frac{s!(n-1-s)!}{n!}[v(S \cup i) - v(S)]$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(LR)

$$\begin{aligned}\phi_i(v^{\mathbf{C}}) &= \sum_{\substack{S\subset N\\ i\notin S}} \frac{s!(n-1-s)!}{n!}[v(S\cup i)-v(S)]\\ &= \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \sum_{\substack{S\subset N\backslash i\\ |S|=s}} c_i\end{aligned}$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(LR)

$$\begin{aligned}
\phi_i(v^{\mathbf{C}}) &= \sum_{\substack{S\subset N\\ i\notin S}} \frac{s!(n-1-s)!}{n!}[v(S\cup i)-v(S)] \\
&= \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \sum_{\substack{S\subset N\setminus i\\ |S|=s}} c_i \\
&= c_i \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \frac{(n-1)!}{s!(n-1-s)!}
\end{aligned}$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(LR)

$$\begin{aligned}
\phi_i(v^{\mathbf{C}}) &= \sum_{\substack{S \subset N \\ i \notin S}} \frac{s!(n-1-s)!}{n!}[v(S \cup i) - v(S)] \\
&= \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \sum_{\substack{S \subset N \setminus i \\ |S|=s}} c_i \\
&= c_i \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \frac{(n-1)!}{s!(n-1-s)!} \\
&= c_i \sum_{s=0}^{n-1} \frac{1}{n} = c_i
\end{aligned}$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(LR)

$$\begin{aligned}
\phi_i(v^{\mathbf{C}}) &= \sum_{\substack{S \subset N \\ i \notin S}} \frac{s!(n-1-s)!}{n!} [v(S \cup i) - v(S)] \\
&= \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \sum_{\substack{S \subset N \setminus i \\ |S| = s}} c_i \\
&= c_i \sum_{s=0}^{n-1} \frac{s!(n-1-s)!}{n!} \frac{(n-1)!}{s!(n-1-s)!} \\
&= c_i \sum_{s=0}^{n-1} \frac{1}{n} = c_i
\end{aligned}$$

□

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the DR (*Downstream Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} through n^{th} sections of the river.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the DR (*Downstream Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} through n^{th} sections of the river.

For each $S \subset N$, let

$$v^{\mathbf{C}}(S) = \sum_{i=minS}^{n} c_i .$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Under the DR (*Downstream Responsibility*) principle, each player situated along the i^{th} section of the river is responsible for the cleanup of the i^{th} through n^{th} sections of the river.

For each $S \subset N$, let

$$v^{\mathbf{c}}(S) = \sum_{i=minS}^{n} c_i.$$

Claim

$$\phi_i(v^{\mathbf{c}}) = \sum_{j \geq i} \frac{1}{j} c_j = \frac{1}{i} c_i + \frac{1}{i+1} c_{i+1} \cdots + \frac{1}{n} c_n.$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$,

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$, then

$$v^{\mathbf{C}^j}(S) = 0 \text{ if } minS > j \quad \text{and} \quad v^{\mathbf{C}^j}(S) = c_j \text{ if } minS \leq j.$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$, then

$$v^{\mathbf{C}^j}(S) = 0 \text{ if } minS > j \quad \text{and} \quad v^{\mathbf{C}^j}(S) = c_j \text{ if } minS \leq j.$$

So player i is a dummy if $i > j$, and the game is symmetric for all players $i \leq j$.

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$, then

$$v^{\mathbf{C}^j}(S) = 0 \text{ if } minS > j \quad \text{and} \quad v^{\mathbf{C}^j}(S) = c_j \text{ if } minS \leq j.$$

So player i is a dummy if $i > j$, and the game is symmetric for all players $i \leq j$.

Hence

$$\phi_i(v^{\mathbf{C}^j}) = 0 \text{ if } i > j \quad \text{and} \quad \phi_i(v^{\mathbf{C}^j}) = \frac{1}{j} c_j \text{ if } i \leq j.$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$, then

$$v^{\mathbf{C}^j}(S) = 0 \text{ if } minS > j \quad \text{and} \quad v^{\mathbf{C}^j}(S) = c_j \text{ if } minS \leq j.$$

So player i is a dummy if $i > j$, and the game is symmetric for all players $i \leq j$.

Hence

$$\phi_i(v^{\mathbf{C}^j}) = 0 \text{ if } i > j \quad \text{and} \quad \phi_i(v^{\mathbf{C}^j}) = \frac{1}{j} c_j \text{ if } i \leq j.$$

Since

$$v^{\mathbf{C}} = \sum_j v^{\mathbf{C}^j}$$

Sharing Costs of River Cleanup [Ni & Wang, 2007]

Proof.

(DR) Let$\mathbf{C}^j = (0, \ldots, 0, c_j, 0, \ldots, 0)$, then

$$v^{\mathbf{C}^j}(S) = 0 \text{ if } minS > j \quad \text{and} \quad v^{\mathbf{C}^j}(S) = c_j \text{ if } minS \leq j.$$

So player i is a dummy if $i > j$, and the game is symmetric for all players $i \leq j$.

Hence

$$\phi_i(v^{\mathbf{C}^j}) = 0 \text{ if } i > j \quad \text{and} \quad \phi_i(v^{\mathbf{C}^j}) = \frac{1}{j} c_j \text{ if } i \leq j.$$

Since

$$v^{\mathbf{C}} = \sum_j v^{\mathbf{C}^j} \quad \phi_i(v^{\mathbf{C}}) = \sum_j \phi_i(v^{\mathbf{C}^j}) = \sum_{j \geq i} \frac{1}{j} c_j.$$

The Shapley Value and Simple Games

Example

A local county board consists of 4 representatives of different townships with voting weight equal to $3, 1, 2$ and 5 respectively. A motion is passed if it receives at least 6 votes.

The Shapley Value and Simple Games

Example

A local county board consists of 4 representatives of different townships with voting weight equal to $3, 1, 2$ and 5 respectively. A motion is passed if it receives at least 6 votes.

Denote the representatives a, b, c and d.

The Shapley Value and Simple Games

Example

A local county board consists of 4 representatives of different townships with voting weight equal to $3, 1, 2$ and 5 respectively. A motion is passed if it receives at least 6 votes.

Denote the representatives a, b, c and d.

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

The Shapley Value and Simple Games

Example

A local county board consists of 4 representatives of different townships with voting weight equal to $3, 1, 2$ and 5 respectively. A motion is passed if it receives at least 6 votes.

Denote the representatives a, b, c and d.

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

$$\nu(S) = \begin{cases} 1 & \text{if } \text{ S is winning} \\ 0 & \text{else} \end{cases}$$

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The Shapley Value and Simple Games

Winning coalitions: $\{a,d\}$, $\{b,d\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$,$\{b,c,d\}$ and $\{a,b,c,d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The only non-zero terms are those in which $S \cup a$ is winning and S is not winning. (Player a is "critical.")

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The only non-zero terms are those in which $S \cup a$ is winning and S is not winning. (Player a is "critical.") $S = \{d\}$ or $S = \{b, c\}$,

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The only non-zero terms are those in which $S \cup a$ is winning and S is not winning. (Player a is "critical.") $S = \{d\}$ or $S = \{b, c\}$,

$$\phi_a(\nu) = \frac{1!2!}{24!}[1-0] + \frac{2!1!}{24!}[1-0] = \frac{1}{6}$$

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The only non-zero terms are those in which $S \cup a$ is winning and S is not winning. (Player a is "critical.") $S = \{d\}$ or $S = \{b, c\}$,

$$\phi_a(\nu) = \frac{1!2!}{24!}[1-0] + \frac{2!1!}{24!}[1-0] = \frac{1}{6}$$

Similarly,

$$\phi_b(\nu) = \frac{1}{6} \quad \phi_c(\nu) = \frac{1}{6} \quad \phi_d(\nu) = \frac{1}{3}.$$

The Shapley Value and Simple Games

Winning coalitions: $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$,$\{b, c, d\}$ and $\{a, b, c, d\}$.

Applying the Shapley value to this game for Player a,

$$\phi_a(\nu) = \sum_{\substack{S \subset N \\ a \notin S}} \frac{s!(n-1-s)!}{n!}[\nu(S \cup a) - \nu(S)].$$

The only non-zero terms are those in which $S \cup a$ is winning and S is not winning. (Player a is "critical.") $S = \{d\}$ or $S = \{b, c\}$,

$$\phi_a(\nu) = \frac{1!2!}{24!}[1-0] + \frac{2!1!}{24!}[1-0] = \frac{1}{6}$$

Similarly,

$$\phi_b(\nu) = \frac{1}{6} \quad \phi_c(\nu) = \frac{1}{6} \quad \phi_d(\nu) = \frac{1}{3}.$$

This is known as the *Shapley-Shubik Power Index*.

The United Nations Security Council

Example

United Nations Security Council consists of 5 permanent members and 10 non-permanent members. Resolutions are passed if they are supported by all 5 permanent members and at least 4 of the non-permanent members.

The United Nations Security Council

Example

United Nations Security Council consists of 5 permanent members and 10 non-permanent members. Resolutions are passed if they are supported by all 5 permanent members and at least 4 of the non-permanent members.

Let P be the set of permanent members and T be the set of non-permanent members.

$$\nu(S) = \begin{cases} 1 & \text{if } \mid S \cap P \mid = 5 \text{ and } \mid S \cap T \mid \geq 4 \\ 0 & \text{else} \end{cases}$$

The United Nations Security Council

Example

United Nations Security Council consists of 5 permanent members and 10 non-permanent members. Resolutions are passed if they are supported by all 5 permanent members and at least 4 of the non-permanent members.

Let P be the set of permanent members and T be the set of non-permanent members.

$$\nu(S) = \begin{cases} 1 & \text{if } \mid S \cap P \mid = 5 \text{ and } \mid S \cap T \mid \geq 4 \\ 0 & \text{else} \end{cases}$$

After much combinatorics,

$$\phi_P(\nu) = 0.1974 \quad \phi_T(\nu) = 0.0022.$$

The Shapley-Shubik Power Index

Proposition (Straffin, 1977)

The Shapley-Shubik value is equal to the probability that a player's vote will make a difference, given that each player has an equal probability of voting 'yes' or 'no'.

The Shapley-Shubik Power Index

Proposition (Straffin, 1977)

The Shapley-Shubik value is equal to the probability that a player's vote will make a difference, given that each player has an equal probability of voting 'yes' or 'no'.

The probability that coalition S forms is equal to

$$P(S) = \int_0^1 p^s (1-p)^{n-s} dp = \frac{s!(n-s)!}{(n+1)!}$$

The Shapley-Shubik Power Index

Proposition (Straffin, 1977)

The Shapley-Shubik value is equal to the probability that a player's vote will make a difference, given that each player has an equal probability of voting 'yes' or 'no'.

The probability that coalition S forms is equal to

$$P(S) = \int_0^1 p^s(1-p)^{n-s}dp = \frac{s!(n-s)!}{(n+1)!}$$

Hence player i will make a difference with probability

$$\begin{aligned}\sum P(S) + P(S \cup i) &= \sum \left[\frac{s!(n-s)!}{(n+1)!} + \frac{(s+1)!(n-1-s)!}{(n+1)!}\right] \\ &= \sum \frac{s!(n-1-s)!}{(n+1)!}[(n-s)+(s+1)] \\ &= \sum \frac{s!(n-1-s)!}{n!}\end{aligned}$$

Extensions of Cooperative Game Theory

- Multichoice Games [Hsiao & Raghaven, 1993]
- Ternary Voting Games [Felsenthal & Machover, 1997]
- Fuzzy Games
- (j, k) Games [Zwicker & Freixas, 2003]
- Games over Lattices [Grabisch & Lange, 2007]

Models in Cooperative Game Theory, Branzei, Dimitrov & TIjs, 2005

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \rightarrow \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \rightarrow \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Comments:

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \rightarrow \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Comments:

- given $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in M^n$, write $v(x_1, x_2, \ldots, x_n)$

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \to \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Comments:

- given $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in M^n$, write $v(x_1, x_2, \ldots, x_n)$
- each $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ corresponds to a partition of the players $S_0^{\mathbf{x}}, S_1^{\mathbf{x}}, \ldots, S_m^{\mathbf{x}}$, where $S_j^{\mathbf{x}} = \{i \in N : x_i = j\}$ is the set of players acting at level j.

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \to \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Comments:

- given $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in M^n$, write $v(x_1, x_2, \ldots, x_n)$
- each $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ corresponds to a partition of the players $S_0^{\mathbf{x}}, S_1^{\mathbf{x}}, \ldots, S_m^{\mathbf{x}}$, where $S_j^{\mathbf{x}} = \{i \in N : x_i = j\}$ is the set of players acting at level j.
- $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for all i

Multichoice Games

Definition (Hsiao & Raghaven, 1993)

A multichoice game is a game in which n players $N = \{1, 2, \ldots, n\}$ select among $m + 1$ levels of participation $M = \{0, 1, \ldots, m\}$, and $v : M^n \to \mathbb{R}$ is the characteristic or value function satisfying $v(0, 0, \ldots, 0) = 0$.

Comments:

- given $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in M^n$, write $v(x_1, x_2, \ldots, x_n)$
- each $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ corresponds to a partition of the players $S_0^{\mathbf{x}}, S_1^{\mathbf{x}}, \ldots, S_m^{\mathbf{x}}$, where $S_j^{\mathbf{x}} = \{i \in N : x_i = j\}$ is the set of players acting at level j.
- $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for all i
- assume v *monotonic* $v(\mathbf{x}) \geq v(\mathbf{y})$ if $\mathbf{x} \geq \mathbf{y}$

Examples

Example

United Nations Security Council resolutions on non-procedural issues are approved if nine members support it and all permanent members 'concur.' ($n = 15$ and $m = 2$)

Examples

Example

United Nations Security Council resolutions on non-procedural issues are approved if nine members support it and all permanent members 'concur.' ($n = 15$ and $m = 2$)

Example (Hsiao & Raghaven, 1993)

A mathematics department with 50 faculty including 10 distinguished professors must vote to promote a junior colleague. The colleague will be promoted if

- At least 40 faculty marginally or strongly support the candidate and at least 2 distinguished professors attend the meeting
- At least 25 faculty strongly support the candidate including at least 1 distinguished professor

($n = 50$ and $m = 3$)

Examples

It's Not Just 'Ayes' and 'Nays': Obama's Votes in Illinois Echo

By RAYMOND HERNANDEZ and CHRISTOPHER DREW
Published: December 20, 2007

The New York Times December 20, 2007

Seth Perlman/Associated Press

Extensions of the Shapley Value

- using weights [Hsiao & Raghavan, 1993]
- using maximal chains [Nouweland & Tijs, 1995]
- using roll-calls [Freixas, 2005, Felshenthal & Machover, 1997]
- using axiomatic characterizations [Peters & Zenk, 2005, Grabisch & Lange, 2007]
- using multilinear extensions [Jones & Wilson, preprint]

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

- based on *weights* $0 = w_0 \leq w_1 \leq \cdots \leq w_m$

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

- based on *weights* $0 = w_0 \leq w_1 \leq \cdots \leq w_m$
- defined on unanimity games $v_{\mathbf{y}}$ where $v_{\mathbf{y}}(\mathbf{x}) = 1$ iff $\mathbf{x} \geq \mathbf{y}$

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

- based on *weights* $0 = w_0 \leq w_1 \leq \cdots \leq w_m$
- defined on unanimity games $v_{\mathbf{y}}$ where $v_{\mathbf{y}}(\mathbf{x}) = 1$ iff $\mathbf{x} \geq \mathbf{y}$

$$\tau_{i,j}(v_{\mathbf{y}}) = \begin{cases} \frac{w_{y_i}}{\sum w_{y_k}} & \text{if} j \geq y_i \\ 0 & \text{if} j < y_i \end{cases}$$

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

- based on *weights* $0 = w_0 \leq w_1 \leq \cdots \leq w_m$
- defined on unanimity games $v_{\mathbf{y}}$ where $v_{\mathbf{y}}(\mathbf{x}) = 1$ iff $\mathbf{x} \geq \mathbf{y}$

$$\tau_{i,j}(v_{\mathbf{y}}) = \begin{cases} \frac{w_{y_i}}{\sum w_{y_k}} & \text{if} j \geq y_i \\ 0 & \text{if} j < y_i \end{cases}$$

- extend linearly

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

- based on *weights* $0 = w_0 \leq w_1 \leq \cdots \leq w_m$
- defined on unanimity games $v_{\mathbf{y}}$ where $v_{\mathbf{y}}(\mathbf{x}) = 1$ iff $\mathbf{x} \geq \mathbf{y}$

$$\tau_{i,j}(v_{\mathbf{y}}) = \begin{cases} \frac{w_{y_i}}{\sum w_{y_k}} & \text{if} j \geq y_i \\ 0 & \text{if} j < y_i \end{cases}$$

- extend linearly

$$\tau_{i,j}(v) = \sum_{k \leq j} \sum_{\mathbf{x}_{n-1} \in M^{n-1}} \sum_{T \subset B_i(\mathbf{x}_{-i})} (-1)^t \frac{w_k}{||(\mathbf{x}_{-i}, k)||_w + \sum_{r \in T}[w(x_r + 1} \times [v(\mathbf{x}_{-i}, k) - v(\mathbf{x}_{-i}, k-1)]$$

where $B_i(\mathbf{x}_{-i}) = \{i' \in N \setminus \{i\} | i' \neq m\}$ and $t = |T|$.

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

The extension $\tau_{i,j}$ uniquely satisfies

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

The extension $\tau_{i,j}$ uniquely satisfies

- $\tau_{i,j}(v_{\mathbf{y}})$ is proportional to w_{y_i}

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

The extension $\tau_{i,j}$ uniquely satisfies

- $\tau_{i,j}(v_{\mathbf{y}})$ is proportional to w_{y_i}
- $\sum_i \tau_{i,m}(v) = v(m, m \ldots, m)$

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

The extension $\tau_{i,j}$ uniquely satisfies

- $\tau_{i,j}(v_{\mathbf{y}})$ is proportional to w_{y_i}
- $\sum_i \tau_{i,m}(v) = v(m, m \ldots, m)$
- $\tau_{i,j}(v + w) = \tau_{i,j}(v) + \tau_{i,j}(w)$

Extensions of the Shapley Value

Shapley Value [Hsiao & Raghavan, 1993]

The extension $\tau_{i,j}$ uniquely satisfies

- $\tau_{i,j}(v_{\mathbf{y}})$ is proportional to w_{y_i}
- $\sum_i \tau_{i,m}(v) = v(m, m \ldots, m)$
- $\tau_{i,j}(v + w) = \tau_{i,j}(v) + \tau_{i,j}(w)$
- If $v(\mathbf{x}) = 0$ for all $\mathbf{x} \not\geq \mathbf{y}$, then $\tau_{i,j}(v) = 0$ for all $j < y_i$

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

Based on projection of games with levels of approval $M = \{0, 1, \ldots, m\}$ to $\{0, m\}$

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

Based on projection of games with levels of approval $M = \{0, 1, \ldots, m\}$ to $\{0, m\}$

$$\rho_{i,m}(v) = \sum_{\mathbf{x}_{-i} \in \{0,m\}^{n-1}} \frac{(n - s_m - 1)! s_m!}{n!} \times [v(\mathbf{x}_{-i}, m) - v(\mathbf{x}_{-i}, 0)]$$

where $s_m = \mid S_m \mid$.

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

The extension $\rho_{i,m}$ uniquely satisfies

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

The extension $\rho_{i,m}$ uniquely satisfies

- linearity
- dummy property
- monotonicity
- symmetry
- efficiency

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

The extension $\rho_{i,m}$ uniquely satisfies

- linearity
- dummy property
- monotonicity
- symmetry
- efficiency
- invariance

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

The extension $\rho_{i,m}$ uniquely satisfies

- linearity
- dummy property
- monotonicity
- symmetry
- efficiency
- invariance

  If $v_1(\mathbf{x}_{-i}, j) = v_2(\mathbf{x}_{-i}, j-1)$ for $j \geq 1$ and $v_1(\mathbf{x}_{-i}, 0) = v_2(\mathbf{x}_{-i}, 0)$, then

Extensions of the Shapley Value

Shapley value over lattices [Grabisch & Lange, 2007]

The extension $\rho_{i,m}$ uniquely satisfies

- linearity
- dummy property
- monotonicity
- symmetry
- efficiency
- invariance

  If $v_1(\mathbf{x}_{-i}, j) = v_2(\mathbf{x}_{-i}, j-1)$ for $j \geq 1$ and $v_1(\mathbf{x}_{-i}, 0) = v_2(\mathbf{x}_{-i}, 0)$, then

  $\rho_{i,j}(v_1) = \rho_{i,j-1}(v_2)$ for $j \geq 1$

Extensions of the Shapley Value

Shapley-Shubik power index for simple multichoice games [Jones & Wilson]

Extensions of the Shapley Value

Shapley-Shubik power index for simple multichoice games [Jones & Wilson]

$$\phi_{i,j}(v) = \sum_{\mathbf{x}_{-i} \in M^{n-1}} \frac{s_0!s_1!\ldots s_m!m!}{(n+m-1)!}[v(\mathbf{x}_{-i}, j) - v(\mathbf{x}_{-i}, 0)].$$

where $s_j = \mid S_j \mid$.

Extensions of the Shapley Value

Shapley-Shubik power index for simple multichoice games [Jones & Wilson]

$$\phi_{i,j}(v) = \sum_{\mathbf{x}_{-i} \in M^{n-1}} \frac{s_0!s_1!\ldots s_m!m!}{(n+m-1)!}[v(\mathbf{x}_{-i}, j) - v(\mathbf{x}_{-i}, 0)].$$

where $s_j = | S_j |$.

$\phi_{i,m}$ is the probability that player i will make a difference given that each player has equal probability of participating at each level.

Comparison of Shapley Values

Example

United Nations Security Council resolutions on non-procedural issues are approved if nine members support it and all permanent members 'concur.'

Comparison of Shapley Values

Example

United Nations Security Council resolutions on non-procedural issues are approved if nine members support it and all permanent members 'concur.'

- $\tau_{i,j}$ values: 0.1963 permanent members and 0.0016 temporary members
- $\phi_{i,j}$ values: 0.1329 permanent members and 0.0213 temporary members

Advantages to Teaching (Cooperative) Game Theory in Undergraduate Courses

Advantages to Teaching (Cooperative) Game Theory in Undergraduate Courses

- Introduces students to mathematical applications in the social sciences

Advantages to Teaching (Cooperative) Game Theory in Undergraduate Courses

- Introduces students to mathematical applications in the social sciences
- Exposes them to mathematical formalism and structure of proofs in a context where no background knowledge is required

Advantages to Teaching (Cooperative) Game Theory in Undergraduate Courses

- Introduces students to mathematical applications in the social sciences
- Exposes them to mathematical formalism and structure of proofs in a context where no background knowledge is required
- Integrates formal, algebraic and geometric techniques

Advantages to Teaching (Cooperative) Game Theory in Undergraduate Courses

- Introduces students to mathematical applications in the social sciences
- Exposes them to mathematical formalism and structure of proofs in a context where no background knowledge is required
- Integrates formal, algebraic and geometric techniques
- They like it

References

- Branzei, R. et. al., "A simple algorithm for the mucleolus of airport surplus games", 2004.
- Branzei, R. et. al., *Models in Cooperative Game Theory*, Springer, Berlin, 2005.
- Felsenthal, D. and Machover, M. "Ternary Voting Games", Int. J. of Game Theory (26), 335-351, 1997.
- Felsenthal, D. and Machover, M., *The Measurement of Voting Power*, Edward Elgar, Cheltenham, 1998.
- Hsiao, C. and Raghaven, T., "Shapley Value for Multichoice Cooperative Games, I", Games and Econ. Beh. (5),240-256, 1993.
- Littlechild S.C. and Owen, G., A, "A simple expression for the Shapley value in a special case", Management Science (20), 370-2, 1974.

References

- Littlechild S.C. and Owen, G. "A further note on the nucleolus of the airport game", International Journal of Game Theory, (5), 91-5, 1977.
- Luce, R.D. and Raiffa, H., *Games and Decisions*, Wiley and Sons, NY, 1957.
- Ni, D. and Wang, Y., "Sharing a polluted river", Games and Economic Behavior (60), 176-186, 2007.
- Owen,G., *Game Theory*, Academic Press, New York, 1982.
- Rapoport, A., *N-Person Game Theory*, Dover, 2001.
- Straffin, P. "Homogeneity, Independence, and Power Indices", Public Choice (30), 107-119, 1977.
- Straffin, P., Game Theory and Strategy, Mathematical Association of America, 2004.