The Itô integral problem

Definition
Let $W$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A process $\Delta(s, \omega)$, a function of $s \geq 0$ and $\omega \in \Omega$, is adapted if the dependence of $\Delta(s, \omega)$ on $\omega$ is as a function of the initial path fragment $W(u, \omega)$, $0 \leq u \leq s$. In particular, $\Delta(s)$ is independent of $W(t) - W(s)$ whenever $0 \leq s \leq t$.

We want to make sense of
\[
\int_0^t \Delta(s) \, dW(s), \quad 0 \leq t \leq T.
\]

Remark
If $g(s)$ is a differentiable function, then we can define
\[
\int_0^t \Delta(s) \, dg(s) = \int_0^t \Delta(u) g'(s) \, ds.
\]

This won’t work for Brownian motion, however, because the paths of Brownian motion are not differentiable.
Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$, i.e.,

$$0 = t_0 \leq t_1 \leq \cdots \leq t_n = T.$$ 

Assume that $\Delta(s)$ is constant in $s$ on each subinterval $[t_k, t_{k+1})$. We call such a $\Delta$ a simple process.

Example

$$\Delta(s) = W(t_k), \quad t_k \leq s < t_{k+1}$$

Interpretation of Simple Integrand

- Think of $W(s)$ as the price per share of an asset at time $s$.
- Think of $t_0, t_1, \ldots, t_{n-1}$ as the trading dates in the asset.
- Think of $\Delta(t_0), \Delta(t_1), \ldots, \Delta(t_{n-1})$ as the number of shares of the asset acquired at each trading date and held to the next trading date.

Gain from trading.

$$I(t) = \Delta(t_0)[W(t) - W(t_0)] = \Delta(t_0)W(t), \quad 0 \leq t \leq t_1,$$

$$I(t) = \Delta(t_0)[W(t_1) - W(t_0)] + \Delta(t_1)[W(t) - W(t_1)], \quad t_1 \leq t \leq t_2,$$

$$I(t) = \Delta(t_0)[W(t_2) - W(t_0)] + \Delta(t_1)[W(t_2) - W(t_1)] + \Delta(t_2)[W(t) - W(t_2)], \quad t_2 \leq t \leq t_3.$$

The process $I$ is the Itô integral of the simple process $\Delta$, i.e.,

$$I(t) = \int_0^t \Delta(s) \, dW(s), \quad 0 \leq t \leq T.$$ 

Expectation of Itô integral

Theorem

The Itô integral of a simple process has expectation zero.

Proof: By definition

$$I(T) = \sum_{j=0}^{n-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)).$$

Compute expectation term by term. Because $\Delta(t_j)$ is independent of $W(t_{j+1}) - W(t_j)$, we have

$$E[\Delta(t_j)(W(t_{j+1}) - W(t_j))] = E[\Delta(t_j)] \cdot E[W(t_{j+1}) - W(t_j)]$$

$$= E[\Delta(t_j)] \cdot 0$$

$$= 0.$$
Exercise (5.1)
Suppose \( Y(t), 0 \leq t \leq T, \) is a stochastic process (a function of \( t \) and \( \omega \)) such that if \( 0 \leq s \leq t, \) then the increment \( Y(t) - Y(s) \) is independent of the path of \( Y \) up to time \( s \) and has expectation zero. Let \( \{\Delta(s)\}_{0 \leq s \leq T} \) be a simple process adapted to \( Y, \) i.e., there is a partition \( \Pi = \{t_0, t_1, \ldots, t_n\} \) of \( [0, T] \) such that \( \Delta(s) \) is constant in \( s \) in each subinterval \([t_j, t_{j+1})\), and for each \( s \in [0, T], \) the random variable \( \Delta(s) \) depends on \( \omega \) only through the path of \( Y \) up to time \( s \), and hence \( \Delta(s) \) is independent of \( Y(t) - Y(s) \) for all \( t \in [s, T] \). Define the Itô integral

\[
I(T) = \sum_{j=0}^{n-1} \Delta(t_j)(Y(t_{j+1}) - Y(t_j)).
\]

(i) Show that \( \mathbb{E}\{I(T)\} = 0. \)

(ii) A simple arbitrage is a simple process \( \Delta \) such that \( I(T) \geq 0 \) almost surely and \( \mathbb{P}\{I(T) > 0\} > 0. \) Show that there is no simple arbitrage under the assumptions of this exercise.

Proof of (QV)
For \( s \in [t_j, t_{j+1}), \) we have \( \Delta(s) = \Delta(t_j) \) and

\[
I(s) = I(t_j) + \Delta(t_j)\left[W(s) - W(t_j)\right] = \left[I(t_j) - \Delta(t_j)W(t_j)\right] + \Delta(t_j)W(s).
\]

On this subinterval, quadratic variation of \( I \) comes from the quadratic variation of \( W, \) which is scaled by \( \Delta(t_j). \) Therefore

\[
[I, I](t_{j+1}) - [I, I](t_j) = \Delta^2(t_j)(W(t_{j+1}) - W(t_j))^2 - \Delta^2(t_j)(t_{j+1} - t_j)
\]

\[
= \int_{t_j}^{t_{j+1}} \Delta^2(s) \, ds.
\]

Summing over subintervals, we obtain

\[
[I, I](T) = \sum_{j=0}^{n-1} \left([I, I](t_{j+1}) - [I, I](t_j)\right) = \int_0^T \Delta^2(s) \, ds.
\]

Quadratic Variation of Itô Integral

**Theorem**
The simple Itô integral

\[
I(t) = \int_0^t \Delta(u) \, dW(u)
\]

has quadratic variation

\[
[I, I](T) = \int_0^T \Delta^2(u) \, du \quad (QV)
\]

and variance

\[
\mathbb{E}[I^2(T)] = \mathbb{E}\int_0^T \Delta^2(u) \, du. \quad (VAR)
\]

**Remark**
Both sides of \((QV)\) are random, but the expressions in \((VAR)\) are not. \((VAR)\) is called Itô’s Isometry.

Proof of (VAR)

\[
I(T) = \sum_{j=0}^{n-1} \Delta(t_j)(W(t_{j+1}) - W(t_j)).
\]

Squaring and taking expectations, we obtain

\[
\mathbb{E}[I^2(T)] = \sum_{k=j}^{n-1} \mathbb{E}\left[\Delta^2(t_j)(W(t_{j+1}) - W(t_j))^2\right] + 2 \sum_{j<k} \mathbb{E}\left[\Delta(t_j)\Delta(t_k)(W(t_{j+1}) - W(t_j))(W(t_{k+1}) - W(t_k))\right].
\]

We use independence to simplify the pure square terms:

\[
\mathbb{E}[\Delta^2(t_j)(W(t_{j+1}) - W(t_j))^2] = \mathbb{E}[\Delta^2(t_j)] \cdot \mathbb{E}[W(t_{j+1}) - W(t_j)]^2
\]

\[
= \mathbb{E}[\Delta^2(t_j)] \cdot (t_{j+1} - t_j) = \int_{t_j}^{t_{j+1}} \mathbb{E}\Delta^2(s) \, ds.
\]

The sum of the pure square terms is \( \mathbb{E}\int_0^T \Delta^2(s) \, ds. \)
Proof of (VAR) (continued)

It remains to show that the cross-terms have zero expectation. For $j < k$, the increment $W(t_{k+1}) - W(t_k)$ is independent of
$\Delta(t_j)\Delta(t_k)(W(t_{j+1}) - W(t_j))$, and hence

$$
\text{E}
\left[
\Delta(t_j)\Delta(t_k)(W(t_{j+1}) - W(t_j))(W(t_{k+1}) - W(t_k))
\right]
$$

$$
= \text{E}
\left[
\Delta(t_j)\Delta(t_k)(W(t_{j+1}) - W(t_j))
\right] \cdot \text{E}(W(t_{k+1}) - W(t_k))
$$

$$
= \text{E}
\left[
\Delta(t_j)\Delta(t_k)(W(t_{j+1}) - W(t_j))
\right] \cdot 0
$$

$$
= 0.
$$

Outline of construction for general integrands

- Given $\Delta(s), 0 \leq s \leq T$, satisfying
  $$
  \text{E} \int_0^T \Delta^2(s) \, ds < \infty,
  $$
  construct an approximating sequence of simple processes $\Delta_n(s), 0 \leq s \leq T$, such that
  $$
  \lim_{n \to \infty} \text{E} \int_0^T (\Delta(s) - \Delta_n(s))^2 \, ds = 0.
  $$

- Set $I_n(T) = \int_0^T \Delta_n(s) \, dW(s)$. Itô’s isometry implies that
  $$
  \text{E} [(I_n(T) - I_m(T))^2] = \text{E} \int_0^T (\Delta_n(s) - \Delta_m(s))^2 \, ds.
  $$

- Because the sequence $\{\Delta_n\}_{n=1}^\infty$ converges in
  $L_2(\Omega \times [0, T], F \otimes \text{Borel}([0, T]), \mathbb{P} \times \text{Lebesgue})$, it is Cauchy in this space. Therefore, $\{I_n(T)\}_{n=1}^\infty$ is Cauchy in $L_2(\Omega, F, \mathbb{P})$.

Outline of construction (continued)

- $L_2(\Omega, F, \mathbb{P})$ is complete, and so the sequence $\{I_n(T)\}_{n=1}^\infty$ has a limit $I(T)$ in this space.

- We define
  $$
  \int_0^T \Delta(s) \, dW(s) = I(T) = \lim_{n \to \infty} I_n(T).
  $$

  This limit does not depend on the approximating sequence $\{\Delta_n\}_{n=1}^\infty$.

- By choosing approximating sequences that converge rapidly, we can in fact make the convergence of $I_n(T)$ to $I(T)$ be almost sure (almost everywhere with respect to $\mathbb{P}$) rather than in $L_2$.

- With additional work, one can choose the approximating sequence so that the paths of $I_n(t), 0 \leq t \leq T$, converge uniformly in $t \in [0, T]$ almost surely. This guarantees that there is a limit $I(t), 0 \leq t \leq T$, that is a continuous function of $t \in [0, T]$ for $\mathbb{P}$-almost every $\omega$.  

5 Stochastic Calculus

5.3 Construction for General Integrands
Theorem

Under the assumption $E[\int_0^T \Delta^2(s) \, ds] < \infty$, the Itô integral

$$I(t) = \int_0^t \Delta(s) \, dW(s), \quad 0 \leq t \leq T,$$

is defined and continuous in $t \in [0, T]$. We have

$$E[I(t)] = 0, \quad 0 \leq t \leq T.$$

The quadratic variation of the Itô integral is

$$[I, I](t) = \int_0^t \Delta^2(s) \, ds, \quad 0 \leq t \leq T,$$

and the Itô integral satisfies Itô's Isometry

$$\text{Var}[I(t)] = E[I^2(t)] = E \left[ \int_0^t \Delta^2(s) \, ds \right], \quad 0 \leq t \leq T.$$

\[
\int_0^T W(s) \, dW(s)
\]

Divide $[0, T]$ into $n$ equal subintervals. Define

$$\Delta_n(s) = W \left( \frac{jT}{n} \right) \text{ for } \frac{jT}{n} \leq s < \frac{(j+1)T}{n}.\]

By definition,

$$\int_0^T W(s) \, dW(s) = \lim_{n \to \infty} \sum_{j=0}^{n-1} W \left( \frac{jT}{n} \right) \left[ W \left( \frac{(j+1)T}{n} \right) - W \left( \frac{jT}{n} \right) \right].$$

To simplify notation, we denote $W_j = W \left( \frac{jT}{n} \right)$. Then $W_0 = W(0) = 0$, $W_n = W(T)$, and

$$\int_0^T W(s) \, dW(s) = \lim_{n \to \infty} \sum_{j=0}^{n-1} W_j (W_{j+1} - W_j).$$
\[ \frac{1}{2} \sum_{j=0}^{n-1} (W_{j+1} - W_j)^2 = \frac{1}{2} \sum_{j=0}^{n-1} W_{j+1}^2 + \frac{1}{2} \sum_{j=0}^{n-1} W_j^2 - \sum_{j=0}^{n-1} W_j W_{j+1} + \frac{1}{2} \sum_{j=0}^{n-1} W_j^2 \]

\[ = \frac{1}{2} \sum_{k=1}^{n} W_k^2 - \sum_{j=0}^{n-1} W_j W_{j+1} + \frac{1}{2} \sum_{j=0}^{n-1} W_j^2 \]

\[ = \frac{1}{2} W_n^2 + \frac{1}{2} \sum_{k=0}^{n-1} W_k^2 - \sum_{j=0}^{n-1} W_j W_{j+1} \]

\[ = \frac{1}{2} W_n^2 + \sum_{j=0}^{n-1} W_j (W_j - W_{j+1}). \]

\[ \sum_{j=0}^{n-1} W_j (W_{j+1} - W_j) = \frac{1}{2} W_n^2 - \frac{1}{2} \sum_{j=0}^{n-1} (W_{j+1} - W_j)^2. \]

From the previous page, we have

\[ \sum_{j=0}^{n-1} W_j (W_{j+1} - W_j) = \frac{1}{2} W_n^2 - \frac{1}{2} \sum_{j=0}^{n-1} (W_{j+1} - W_j)^2. \]

Letting \( n \to \infty \), we get

\[ \int_0^T W(s) \, dW(s) = \frac{1}{2} W^2(T) - \frac{1}{2} [W, W](T) = \frac{1}{2} W^2(T) - \frac{1}{2} T. \]

Remark

If \( g \) is a differentiable function with \( g(0) = 0 \), then

\[ \int_0^T g(s) \, dg(s) = \int_0^T g(s) g'(s) \, ds = \frac{1}{2} g^2(s) \bigg|_0^T = \frac{1}{2} g^2(T). \]

The extra term \( \frac{1}{2} T \) in \( \int_0^T W(s) \, dW(s) \) comes from the nonzero quadratic variation of Brownian motion.

Exercise (5.2)

Show that

\[ \lim_{n \to \infty} \sum_{j=0}^{n-1} W \left( \frac{j + 1}{n} \right) \left[ W \left( \frac{j + 1}{n} \right) - W \left( \frac{j}{n} \right) \right] = \frac{1}{2} W^2(T) + \frac{1}{2} T. \]

5 Stochastic Calculus

5.5 Itô’s Formula for One Process
Along the path of a Brownian motion, we want to “differentiate” \( f(W(t)) \), where \( f(x) \) is a differentiable function. If the path of the Brownian motion \( W(t) \) could be differentiated with respect to \( t \), then the ordinary chain rule would give

\[
\frac{d}{dt} f(W(t)) = f'(W(t)) W'(t),
\]

which could be written in differential notation as

\[
df(W(t)) = f'(W(t)) dW(t).
\]

Because \( W \) has nonzero quadratic variation, the correct formula has an extra term, namely,

\[
df(W(t)) = f'(W(t)) dW(t) + \frac{1}{2} f''(W(t)) \, dt.
\]

This is Itô’s formula in differential form.

**Application of Itô’s Formula**

Consider \( f(x) = \frac{1}{2} x^2 \), so that

\[
f'(x) = x, \quad f''(x) = 1.
\]

Itô’s formula in integral form

\[
f(W(T)) - f(W(0)) = \int_0^T f'(W(s)) \, dW(s) + \frac{1}{2} \int_0^T f''(W(s)) \, ds
\]

becomes

\[
\frac{1}{2} W^2(T) = \int_0^T W(s) \, dW(s) + \frac{1}{2} \int_0^T W(s) \, ds + \frac{1}{2} T,
\]

or equivalently,

\[
\int_0^T W(u) \, dW(u) = \frac{1}{2} W^2(T) - \frac{1}{2} T.
\]

**Remark**

The mathematically meaningful form of Itô’s formula is Its form in integral form:

\[
f(W(T)) - f(W(0)) = \int_0^T f'(W(s)) \, dW(s) + \frac{1}{2} \int_0^T f''(W(s)) \, ds.
\]

This is because we have definitions for both integrals appearing on the right-hand side. The first,

\[
\int_0^T f'(W(s)) \, dW(s)
\]

is an Itô integral. The second

\[
\int_0^T f''(W(s)) \, ds
\]

is a Riemann integral with respect to time, computed path by path.

**Derivation of Itô’s Formula**

Consider \( f(x) = \frac{1}{2} x^2 \), so that

\[
f'(x) = x, \quad f''(x) = 1.
\]

Let \( x_{j+1} \) and \( x_j \) be numbers. Taylor’s formula implies

\[
f(x_{j+1}) - f(x_j) = (x_{j+1} - x_j)f'(x_j) + \frac{1}{2}(x_{j+1} - x_j)^2 f''(x_j).
\]

In this case, Taylor’s formula to second order is exact because \( f \) is a quadratic function.

In the general case, the above equation is only approximate, and the error is of the order of \((x_{k+1} - x_k)^3\). The total error will have limit zero in the last step of the following argument (see Exercise 4.6(iii) of Lecture 4).
Fix $T > 0$ and let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, T]$.

\[
f(W(T)) - f(W(0)) = \sum_{j=0}^{n-1} \left[ f(W(t_{j+1})) - f(W(t_j)) \right]
\]

\[
= \sum_{j=0}^{n-1} \left[ W(t_{j+1}) - W(t_j) \right] f'(W(t_j)) + \frac{1}{2} \sum_{j=0}^{n-1} \left[ W(t_{j+1}) - W(t_j) \right]^2 f''(W(t_j))
\]

\[
= \sum_{j=0}^{n-1} W(t_j) \left[ W(t_{j+1}) - W(t_j) \right] + \frac{1}{2} \sum_{j=0}^{n-1} \left[ W(t_{j+1}) - W(t_j) \right]^2.
\]

From the previous page, we have

\[
f(W(T)) - f(W(0))
\]

\[
= \sum_{j=0}^{n-1} W(t_j) \left[ W(t_{j+1}) - W(t_j) \right] + \frac{1}{2} \sum_{k=0}^{n-1} \left[ W(t_{j+1}) - W(t_j) \right]^2.
\]

We let $||\Pi|| \rightarrow 0$, to obtain

\[
f(W(T)) - f(W(0))
\]

\[
= \int_0^T W(s) \, dW(s) + \frac{1}{2} \left[ W, W \right](T)
\]

\[
= \int_0^T f'(W(s)) \, dW(s) + \frac{1}{2} \int_0^T f''(W(s)) \, ds.
\]

This is Itô’s formula in integral form for the special case

\[
f(x) = \frac{1}{2} x^2.
\]

**Exercise (5.3)**

Let $u \in \mathbb{R}$ be constant and define $f(x) = e^{ux}$. Use Itô’s formula applied to $f(W(t))$ to obtain the moment-generating function formula

\[
\mathbb{E} e^{uW(T)} = e^{u^2 T}.
\]

(Compare with Exercise 4.4 of Lecture 4.)

**Exercise (5.4)**

Let $f(x) = x^4$. Use Itô’s formula applied to $f(W(t))$ to obtain the fourth-moment formula

\[
\mathbb{E} W^4(T) = 3 T^2.
\]

(Compare with Exercise 4.5 of Lecture 4.)
Solution to Exercise 5.1

(i) As in the proof of the theorem preceding the exercise, we use independence to compute $\mathbb{E}I(T)$ term by term:

$$E[\Delta(t_j)(Y(t_{j+1}) - Y(t_j))] = E[I(t_j) \cdot E[Y(t_{j+1}) - Y(t_j)]] = E[\Delta(t_j) \cdot 0] = 0.$$  

(ii) If $I(T) \geq 0$ almost surely and $P\{I(T) > 0\} > 0$, then $E[I(T)] > 0$. This contradicts part (i).

Solution to Exercise 5.2

Let $t_j = \frac{IT}{n}$. The quadratic variation result for Brownian motion is

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 = T.$$  

The Example shows that

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} W(t_j)(W(t_{j+1}) - W(t_j)) = \frac{1}{2} W^2(T) - \frac{1}{2} T.$$  

Adding these two equations, we obtain

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} W(t_{j+1})(W(t_{j+1}) - W(t_j)) = \frac{1}{2} W^2(T) + \frac{1}{2} T.$$  

This is the desired result.

Solution to Exercise 5.3

We have $f'(x) = uf(x)$ and $f''(x) = u^2f(x)$. Therefore, Itô’s formula becomes

$$e^{uW(T)} = e^{uW(0)} + u \int_0^T e^{uW(t)} dW(t) + \frac{1}{2} u^2 \int_0^T e^{uW(t)} dt.$$  

Taking expectations and using the fact that the expectation of the Itô integral is zero, we obtain

$$\mathbb{E}e^{uW(T)} = 1 + \frac{1}{2} u^2 \int_0^T \mathbb{E}e^{uW(t)} dt.$$  

We differentiate both sides with respect to $T$ to obtain

$$\frac{d}{dT} \mathbb{E}e^{uW(T)} = \frac{1}{2} u^2 \mathbb{E}e^{uW(T)}.$$  

The unique solution to this ordinary differential equation satisfying $\mathbb{E}e^{uW(0)} = 1$ is

$$\mathbb{E}e^{uW(T)} = e^{\frac{1}{2} u^2 T}.$$
Exercise 5.4

With \( f(x) = x^4 \), we have \( f'(x) = 4x^3 \) and \( f''(x) = 12x^2 \).

Therefore, Itô’s formula becomes

\[
W^4(T) = 4 \int_0^T W^3(t) \, dW(t) + 6 \int_0^T W^2(t) \, dt.
\]

Taking expectations of both sides and using the fact that the Itô integral has expectation zero, we obtain

\[
\mathbb{E}W^4(T) = 6 \int_0^T \mathbb{E}W^2(t) \, dt
\]

\[
= 6 \int_0^T t \, dt
\]

\[
= 3T^2.
\]