ULAM'S CONJECTURE AND GRAPH RECONSTRUCTIONS

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1. Introduction. Henri Poincaré once meditated upon how it can happen that some people do not understand mathematics. Perhaps one reason is that most worthwhile questions nowadays are too difficult to explain. Anyone in a position to understand the problem can usually follow the solution. In this connection, Ulam's conjecture is a pleasant example of a mathematical rarity—a nontrivial, easily stated problem that anyone can take a crack at. At least in its graph-theoretic formulation, Ulam's conjecture is simpler to explain than, say, baseball or Monopoly. Also, reminiscent of seventeenth and eighteenth century mathematics, the clever amateur probably stands as good a chance as anyone of settling it.

In this paper, we discuss Ulam's conjecture as it relates to graph theory, together with some results and questions in the area of graph reconstructions to which it naturally leads. Theorems and proofs will be stated very informally, and, with the exception of the existence theorem of Section 5, none are new.

2. Ulam's version. In his famous problem book [**10**, p. 29], Ulam asked the following question:

Suppose A and B are sets with n elements each ($n \geqq 3$). A metric p is given on A with the property that $p(x, y)$ is either 1 or 2 whenever x and y are in A and $x \neq y$. A similar metric is given on B. Now suppose that the $n-1$ element subsets of A and B can be labelled, $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$, in such a way that each A_i is isometric to B_i. Does this force A to be isometric to B?

Ulam's conjecture is that it does.

This certainly seems a plausible guess, and this author has even been told that it is obvious, although not by anyone who has attempted to write down a proof. We shall not be concerned directly with the statement just given, but shall instead turn immediately to a graph theoretic version. The advantages to be gained are concreteness and a certain visual appeal which the term isometry lacks by itself. These may be more than just fanciful, as most of the progress in this area seems to have been made by adopting the graph theorist's point of view.

3. A graph theoretic formulation. Surprisingly, the barest essentials of graph theory are sufficient to restate the problem. We shall briefly review these below. The interested reader can pursue the subject in [**1, 3, 4, 8, 9**].

A *graph* may be thought of as a finite, nonempty set of points (vertices) and lines (branches) connecting distinct points. It is easy to show that any graph

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can be drawn in 3-space so that branches intersect only at vertices (for a proof, see [3, p. 8]). For obvious reasons, graphs are in practice usually drawn in the plane. Personal taste is allowed, but disregarded, in the drawing of graphs. Thus, the graphs of Figure 1 are considered the same, because a bit of stretching and bending (no tearing) will make them identical. Such graphs are said to be *isomorphic*. More carefully, G and K are isomorphic if there is a 1-1 function f between their vertices such that a branch between x and y is in G exactly when the branch between $f(x)$ and $f(y)$ is in K.

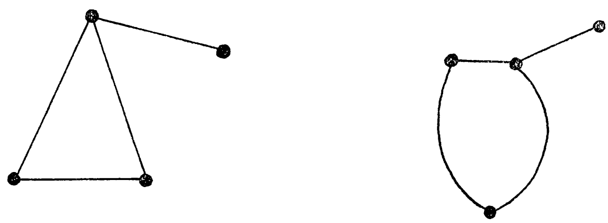

FIG. 1

Finally, if x is a vertex of G, then $G-x$ denotes the graph obtained by deleting x and all branches incident with x from G.

We now have enough to represent Ulam's conjecture graphically. Given the elements $a_1, \cdots, a_n$ of A, mark n points $x_1, \cdots, x_n$ as vertices of a graph G_A. Interpret $p(a_i, a_i)=0$ to mean there is no branch from x_i to x_i. If $i \neq j$, draw a branch between x_i and x_j exactly when $p(a_i, a_j)=1$. Similarly, draw a graph G_B to represent the metric space (B, q). Ulam's conjecture now reads:

If G_A-x_i is isomorphic to G_B-y_i for each i, then G_A is isomorphic to G_B.

A slightly different point of view is provided by the notion of a graph reconstruction. Suppose we start with n graphs, $H_1, \cdots, H_n$, each with $n-1$ vertices. A graph K is called a *reconstruction* of the H_i's if K has n vertices, $t_1, \cdots, t_n$, such that $K-t_i$ is essentially (isomorphic to) H_i. In the above discussion, G_A is a reconstruction of the graphs G_A-x_i, as is G_B of the G_B-y_i. If we regard G_A-x_i as essentially the same graph as G_B-y_i (through the isomorphism), and call it, say, H_i, then we have the following version of the problem.

If K and K' are reconstructions of $H_1, \cdots, H_n$, then K and K' are isomorphic.

Thus, the problem becomes one of essential uniqueness of graph reconstructions. This is Ulam's conjecture in its most concrete form.

4. Some results on graph reconstructions. There are two quite different ways of looking at graph reconstructions. First, given a reconstruction G of $G-x_1, \cdots, G-x_n$, are there any essentially different reconstructions? Of equal interest is a second point of view: given n graphs $H_1, \cdots, H_n$, each with $n-1$ vertices, does a reconstruction exist?

The reader can convince himself that the graphs of Figure 2 have no reconstruction (certain vertices are labelled for later reference). Thus, the existence given in Ulam's conjecture is by no means a trivial assumption. We shall see in Section 5 exactly how nontrivial it is. In this section, we shall discuss some results which have been achieved concerning uniqueness and construction of reconstructions, assuming existence.

The question of uniqueness has been settled for "small" graphs. A theorem due to Kelly [7] says that Ulam's conjecture is true if n is less than 7. This has been extended to $n=7$ by Harary and Palmer [5]. Unfortunately, the arguments used (consideration of all cases) do not suggest a method of proof for arbitrary n.

Some of the best results to date depend upon connectivity conditions. G is said to be *connected* if any two distinct vertices of G can be linked by a path made up of branches of G. In Figure 2, H_1 is connected while H_5 is not. Here H_5 has four *components* (maximal connected subgraphs), and H_4, two components. A *tree* is a connected graph with no circuits (H_1 and H_2 are trees, while H_3, H_4 and H_5 are not).

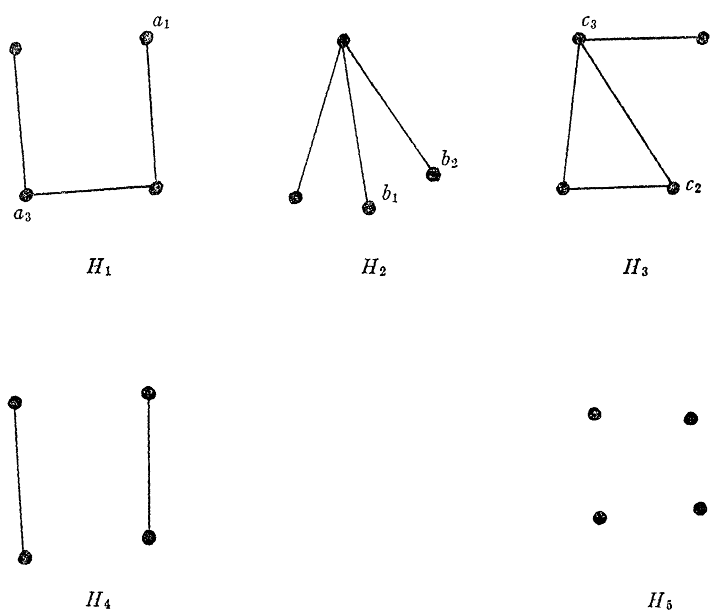

FIG. 2

In the paper referred to above, Kelly was able to show that any tree is uniquely reconstructible from its $n-1$ vertex subgraphs. Thus, Ulam's conjecture is true for trees. The reader might try translating the assumption that G is a tree back into the original language of the problem. Harary and Palmer [6] improved upon Kelly's theorem by showing that any tree can in fact be uniquely reconstructed from just the subgraphs obtained by removing the *endpoints* (vertices with just one incident branch). A more recent improvement, which we shall not discuss here, can be found in [2].

The following result is quite remarkable. Suppose the graphs $G_1, \cdots, G_n$,

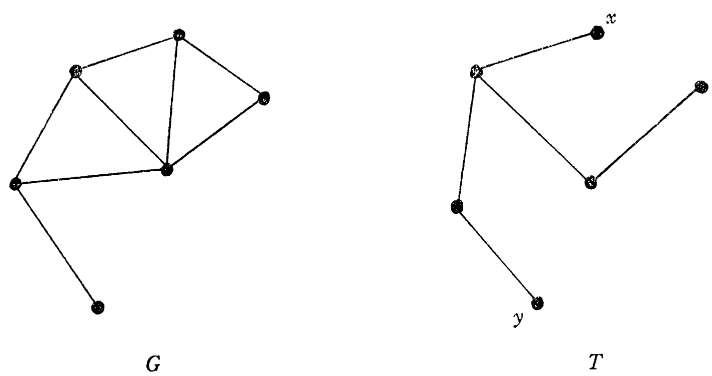

FIG. 3

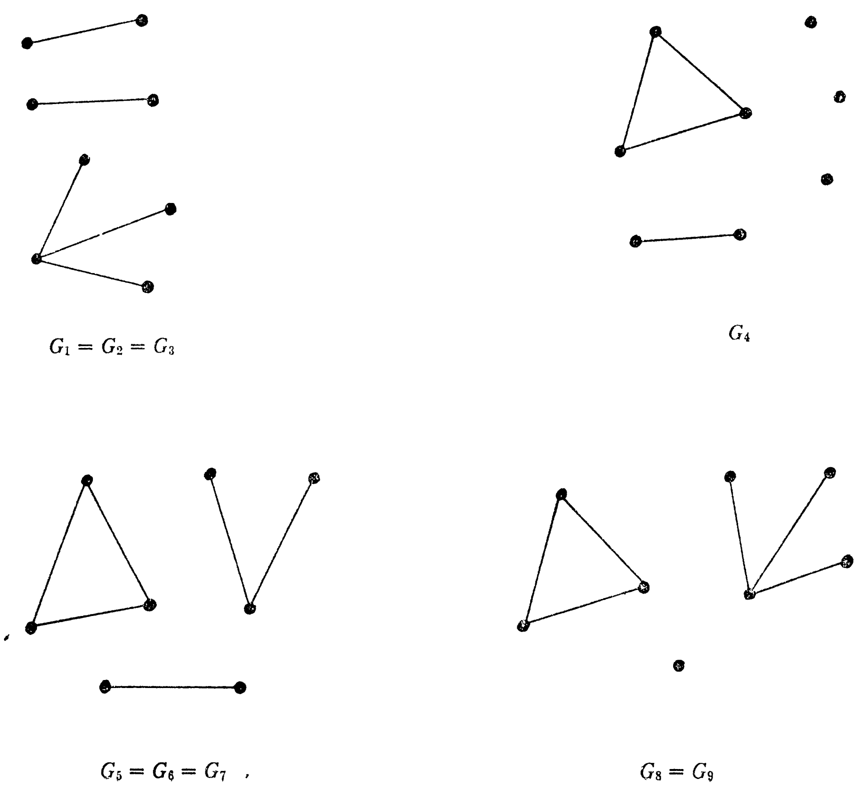

FIG. 4

each with $n-1$ vertices, have a reconstruction G. If at most one G_i is connected, then G is unique.

The hypothesis on the number of connected G_i appears surprising at first. In fact, it is equivalent to saying that any reconstruction which exists must be disconnected. For suppose G is connected. It is easy to see that some set of $n-1$ branches of G forms a tree T (Figure 3). Further, since $n>2$, T has at least two end points, say x and y. Then, at least $G-x$ and $G-y$ are connected. It is straightforward to show that, conversely, if G is disconnected, then all but possibly one G_i are disconnected.

The disconnectedness of any reconstruction can be exploited to actually produce one, component by component. For simplicity, the constructions will be given for particular graphs. The reader might try extending them to the general case.

First consider G_1 through G_9 as shown in Figure 4. Here, all G_i are disconnected. The trick is to observe that the smallest number (greater than 1) of components in any G_i is 3, and to try to produce a suitable G with this number of components. Of those G_i with three components, choose one, say G_9, with a smallest component. For convenience, label its components C_1, C_2 and C_3, with C_3 the smallest (see Figure 5). Keep C_1 and C_2 as candidates for components of

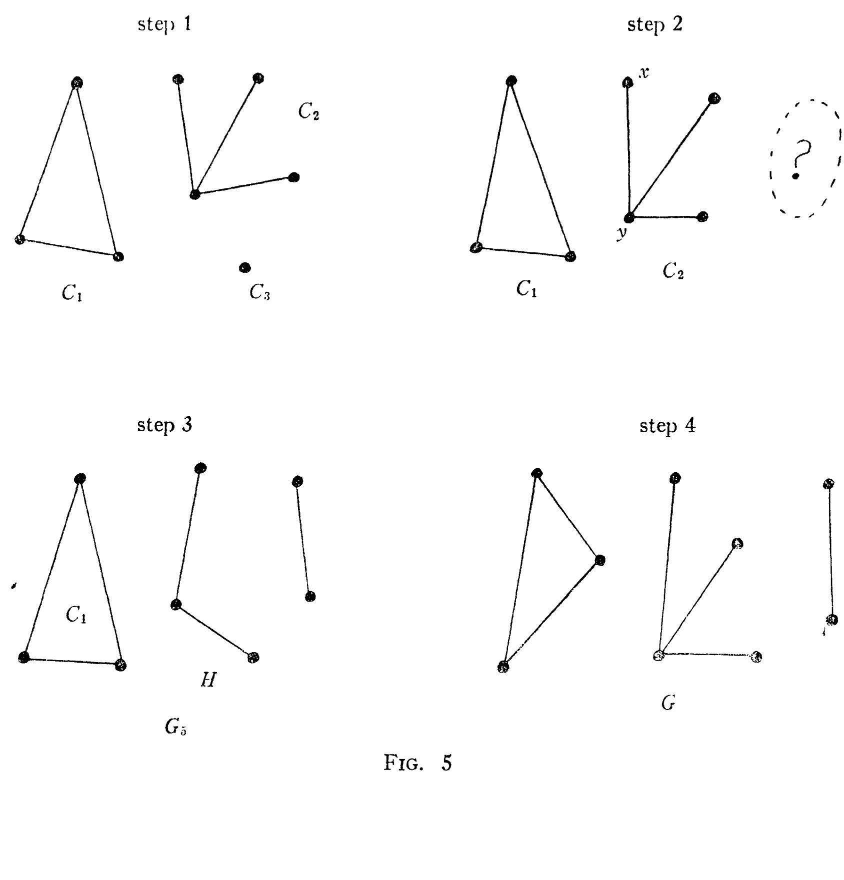

FIG. 5

G. There remains to produce one more. Take C_1 or C_2, say C_2, and remove a vertex which does not disconnect C_2, say x (but not y), forming a graph H. Now identify some G_i having three components, two being C_1 and H. A good choice is G_5. The third component of G_5 can now be taken as the final component of G.

Note that G as constructed has no *isolates* (vertices with no incident branches). This could have been predicted from the given G_i: G will have k isolates exactly when k of the G_i have $k-1$ isolates and each of the remaining $n-k$ has at least (but possibly more than) k isolates. In this event, the construction just given fails. However, when G has isolates, it is particularly easy to reconstruct. For example, in Figure 6, we have $k=3$, and we can simply pick some G_i (say G_1) having two isolates and adjoin another one.

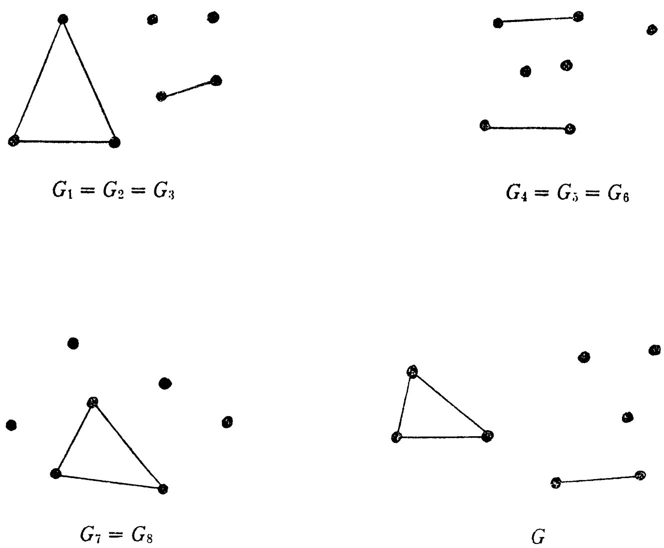

FIG. 6

To sum up, if a reconstruction G of the graphs G_i exists, we can: (1) predict from the G_i when G must be disconnected, and (2) in this event, construct G uniquely.

These results produce another immediately upon taking complements. The *complement* of G, denoted G^c, is formed by erasing all branches of G and drawing in any which were not originally there. It is easy to see that G is a reconstruction of $G_1, \cdots, G_n$ exactly when G^c is a reconstruction of $G_1^c, \cdots, G_n^c$. In Figure 5, note that G^c is connected, so this is an honest extension to some graphs not already treated.

The constructions we have just described are special cases of the general ones discussed by Harary [4], who also has additional details and references on graph reconstructions.

5. A more general reconstruction problem. We have already seen (Figure 2) that n graphs may very well have no reconstruction. To see how bad the situation can be, note that even H_3 and H_5 by themselves have none; that is, there is no five vertex graph G such that $G-x$ is H_3 and $G-y$ is H_5. In this section, we focus attention on the general existence problem: given $G_1, \cdots, G_k$, each with $n-1$ vertices ($k \leqq n$), does there exist an n vertex G with vertices $x_1, \cdots, x_k$ such that $G-x_i$ is essentially G_i? Ideally, we would like necessary and sufficient conditions on the G_i's which will tell us when to expect a reconstruction, and then a method for finding it (or them).

To get some feeling for what is happening, we begin with the simplest nontrivial case, $k=2$. A first impression is that G_1 and G_2 have to be "almost the same" for there to be a reconstruction. Specifically, if $G-x_1 \doteq G_1$ and $G-x_2=G_2$, then

$$(G - x_1) - x_2 = G_1 - x_2$$

and

$$(G - x_2) - x_1 = G_2 - x_1.$$

But it is easy to see that $(G-x_1)-x_2$ and $(G-x_2)-x_1$ are identical. Thus, G_1 and G_2 are the same "to within one vertex." More carefully: if a reconstruction of G_1 and G_2 exists, then there must be vertices x of G_1 and y of G_2 such that G_1-x is isomorphic to G_2-y.

This is a very reasonable necessary condition in that it is both easily stated and applied (try it on H_3 and H_5 of Figure 2). Happily, it is also sufficient. For suppose $G_1-x=G_2-y$. We can produce a suitable G by starting with a copy of G_1-x and drawing in two new vertices and appropriate branches to each. In Figure 2, we have $H_1-a_1=H_3-c_2$, and two possible constructions are indicated in Figure 7, (a) and (b), where $G-u=H_1$ and $G-v=H_3$.

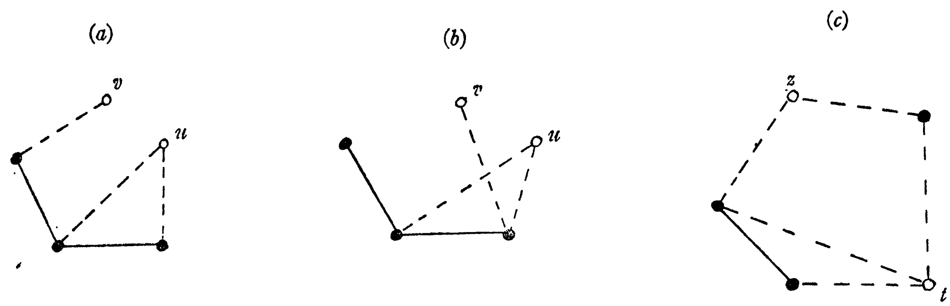

FIG. 7

Incidentally, this shows that a reconstruction is never unique when $k=2$, since we are free to put in or leave out the branch between the two adjoined vertices (u and v of Figure 7 (a) and (b)). It also suggests a method for producing all possible reconstructions of two graphs. Simply list all pairs of vertices a and b such that $G_1-a=G_2-b$, and repeat the constructions for each pair. For example, referring again to Figure 2, note that $H_1-a_3=H_3-c_3$. A reconstruction K begun with a copy of H_1-a_3 is given in Figure 7 (c), where $K-t=H_1$ and $K-z=H_3$.

Having found a nice solution for $k=2$, we are led naturally to attempt a similar theorem for the general case. Half of the above reasoning carries through immediately. If $G-x_i=G_i$, then

$$(G - x_i) - x_j = G_i - x_j = (G - x_j) - x_i$$
$$= G_j - x_i,$$

so that, in pairs, G_i and G_j are identical to within a vertex. This isn't surprising, since any reconstruction of k graphs is a reconstruction of any two of them. The bubble bursts, however, with H_1, H_2 and H_3 of Figure 2. In pairs, H_1-a_1 and H_2-b_1; H_2-b_2 and H_3-c_2; and H_3-c_3 and H_1-a_3, are isomorphic. There is not even any cheating, since no vertex has been used twice. But there is no reconstruction of H_1, H_2 and H_3.

It turns out that, to insure existence, we need not only isomorphisms between G_i-x and G_j-y, but particular kinds of isomorphisms. This is best explained in terms of vertex orderings. If x_i^j is an ordering of the vertices of G_j, for $1\leqq i\leqq n-1$ and $1\leqq j\leqq k$, then we can define maps f_{1+r}^t from $G_t-x_r^t$ to $G_{1+r}-x_t^{1+r}$ by:

$$x_i^t \to x_i^{1+r} \qquad \text{if } 1 \leqq i \leqq t-1 \text{ or } 1+r \leqq i \leqq n-1,$$

and

$$x_i^t \to x_{1+i}^{1+r} \qquad \text{if } t \leqq i < r.$$

Here, values of t are specified by $1\leqq t\leqq \min(3, k-1)$, and, for each t, values of r by $t\leqq r\leqq k-1$. Now, in general, these maps may have no particular properties of interest. In the event that they are all isomorphisms, we call the orderings *compatible*. A general existence theorem can now be stated.

THEOREM. *The graphs $G_1, \cdots, G_k$ have a reconstruction if and only if their vertices can be ordered compatibly.*

Most of the details of the proof are uninteresting, with the exception of the constructive part. When the vertex orderings are compatible, a reconstruction can be produced as follows. Mark n points, say $v_1, \cdots, v_n$, as vertices. Draw a branch between v_i and v_j for $2\leqq i<j\leqq n$ exactly when x_{i-1}^1 and x_{j-1}^1 are connected by a branch in G_1. For $i\geqq 3$, connect v_1 with v_i when x_1^2 and x_{i-1}^2 are connected in G_2. Finally, if $k>2$, connect v_1 and v_2 if x_1^3 and x_2^3 are connected in G_3.

Two observations are in order. First, G is constructed from G_1, G_2 and G_3. The compatibility conditions simply insure that the other G_i are recoverable from G. Second, the conditions are more theoretical than practical, as there is no clear cut way of determining whether or not there are compatible orderings.

It happens that, when $k=2$, the vertices of G_1 and G_2 can always be ordered so that the isomorphism $G_1-x \leftrightarrow G_2-y$ is of the required form. For $k>2$, however, complications mount rapidly. Unfortunately, at least for the present, there does not seem to be any way to avoid this in the general case.

To bring the discussion full cycle, it is easy to verify that the following is an equivalent statement of Ulam's conjecture.

If x_i^j and y_i^j, where $1 \leqq i \leqq n-1$, are compatible vertex orderings of G_j, for $1 \leqq j \leqq n$, then, for each j, the map $x_i^j \rightarrow y_i^j$ is an isomorphism of G_j onto itself.

This formulation clearly lacks the simple appeal of the others. It does serve to pinpoint the real difficulty built into the conjecture, as any proof will in effect show that compatible orderings are essentially unique.

6. Conclusion. Our concluding remarks will be confined to a brief statement of the current extent of ignorance about graph reconstructions.

Virtually nothing is known, for example, about when uniqueness starts happening. For $k=2$, reconstructions are never unique. For $k=3$, uniqueness may occur. For example, three complete graphs of $n-1$ vertices each have exactly one reconstruction, namely the complete graph of n vertices (complete means that every possible branch is drawn in). When else can three graphs uniquely determine a reconstruction? When do k graphs uniquely determine a reconstruction? Is it possible to characterize a reasonable class of n vertex graphs which are uniquely reconstructible from certain subgraphs (as Kelly did for trees and all the $n-1$ vertex subgraphs, and Harary and Palmer did for trees and their maximal subtrees)?

Such questions are interesting in their own right, and even partial solutions would certainly yield some insight into the general problem of existence and uniqueness of graph reconstructions.

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