The Man Who Found God's Number

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> Like other species, humans can assimilate information through the rapid processing that specialized pattern recognition allows, but unlike other species we also seek, shape, and share information in an open-ended way. Since pattern makes data swiftly intelligible, we actively pursue patterns . . .
>
> —Brian Boyd [**1**, p. 14]

In the early 1980s, Tomas Rokicki, a high school junior in Texas, began losing his sense of hearing. Something was spreading through his ear canal, slowly killing the hair cells that hearing depends on. What made it more frustrating was that no one knew why it was happening.

At the same time, people all over the world were fascinated by the Rubik's Cube. There were Rubik's Cube solving contests, some even televised. Someone, possibly a friend or a thoughtful relative, gave Tom one. The curious obsession the world had with the Rubik's Cube soon spread to Tom as well.

During that era, hearing aids were very primitive and no one knew how to restore hearing to those who were totally deaf. Hearing aids were so-called "in the ear" transistor devices. By the time Tom got his first one, while he was a student at Texas A&M, they were small ear-buds that fit inside the ear, but not reaching into the ear canal [**15**]. The device was only useful if you had at least some hearing ability. Although Tom's hearing was getting worse, he did still have some ability and he relied on these crude devices to get through school.

The Rubik's Cube speedsolving contest winners relied on knowing various solving strategies and having very fast finger movement. These contestants never won because they happened upon the most efficient solution, i.e., the smallest number of moves needed to solve the cube. Indeed, the smallest number of moves needed to solve the cube in its most scrambled state, a quantity sometimes called "God's number," was unknown. In 1981, cubologist extraordinaire David Singmaster guessed it might be 20 [**7**], but it would be another 30 years before anyone knew the answer.

We now know, thanks to Tom Rokicki's recent work, the value of God's number. This was a very difficult problem to solve. By the time he solved this, Tom was completely deaf. Also, we now know, due to decades of work by medical researchers, how

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to restore some hearing to the totally deaf. Digitizing human hearing is a very difficult problem to solve. Thanks to two cochlear implant surgeries, Tom's hearing has been restored.

This is a tale of these two problems.

The Rubik's Cube

The Rubik's Cube was unleashed on the world in the late 1970s. It became a sensation world wide and remains a popular mechanical puzzle for people of all ages.

For the sake of orientation, here is a short description. Image a cube sitting on a table. Call its six faces up, down, left, right, front, and back, and color each face a different color. Now cut up the cube into 27 "cubies" of equal size using slices parallel to the faces. This gives rise to 8 corner cubies, 12 edge cubies, and 6 center cubies (we never see the 27th cubie inside the cube). In 1974, Ernő Rubik, a Hungarian professor of architecture and design at the Budapest College of Applied Arts, did just this and also figured out an ingenious internal mechanism for connecting the 26 external cubies in a way that allows them to be rotated around the center. This mechanism allows each face to be rotated by 90° or 180° or 270°, clockwise or counterclockwise. A move of the Rubik's cube may be regarded as a permutation of the 12 edges, a permutation of the 8 corners, with possible edge flips and corner twists. If you imagine the front face of the cube as always facing you and that you are holding the cube with your right hand on the right face, the total number of different positions that the cube could be scrambled is

$$8! \cdot 12! \cdot 2^{10} \cdot 3^{7} = 43{,}252{,}003{,}274{,}489{,}856{,}000 \approx 4.3 \cdot 10^{19},$$

some 43 billion billion positions [**4**, § 11.2.2]. If we made a Rubik's Cube, side length three inches, for each position and lined them up, they would wrap around the Earth's equator 80 billion times. Imagine having to search through this! Here are some other comparisons with things commonly considered large.

- A straw of hay is about 1 gram. For a 2000 pound haystack, that works out to about $1{,}000{,}000 = 10^6$ straws in a haystack. In other words, finding a needle in a haystack is more than 10^{13} times easier.
- If you stacked sheets of paper on top of each other until the total thickness reached the moon, the number of sheets would be over a million times smaller than this number of cube positions.
- The largest publicly known database in commerce or government is at the World Data Centre for Climate, with about 6 petabytes, or 6×10^{15} bytes [**3**]. (It is conceivable that the National Security Agency or Google has a much larger database, but that information is not publicly known.) The "Rubik's Cube database" is over 7,000 times larger.

Clearly, this number of cube positions is a very big number. This is what Tom was searching through. The following and all subsequent quotations are from private correspondence with Rokicki [**12**].

> I do not recall who gave me the cube, but I do remember quite clearly trying to solve it, and being surprised at how difficult it was. For months I tried to solve it, on and off, and just could not restore it; I could get two layers, but the last

layer was just too difficult. Finally, I started keeping a notebook on the cube, and looked for some sequences (algorithms, I now know they are called) that would affect only a small portion of the cube, leaving the rest alone. At one point I had all the algorithms I needed except the one that permuted three corners. With a bit more work I finally figured out those pesky corners, and had a full solution. It took me about six months from originally getting the cube until this point! I did top corners first, then bottom corners, then top and bottom edges, and finally middle edges.

Since the late 1970s, mathematicians and computer scientists have been wondering how many moves are needed to solve the Rubik's Cube in the worst case scenario. There are two popular ways to measure the number of moves you made while manipulating the cube. One is called the *quarter-turn metric*. In this measurement, each quarter turn of any face of the cube, whether clockwise or counterclockwise, is called one move. The other is the *face-turn metric*. In this measurement, each turn of any face of the cube, whether by 90° or 180° or 270° in either direction, counts as one move. Both of these measurements are ways of thinking of how far a given position is from the solved position.

Let us assume we have access to a computer that can tell us, for any given position of the Rubik's cube, the *minimum* number of of quarter-turn or face-turn moves required to restore the position of the cube to the solved state. We call this number the position's *distance* (in the respective metric). How "far" from the solved position of the cube can you get? The answer of course depends on which metric you use. Following David Singmaster [**7**], the exact answer is known as *God's number* in the face-turn metric (respectively, in the quarter-turn metric, if you prefer that measurement).

The determination of God's number was the Mount Everest of mathematical questions connected with the Rubik's Cube. With over 4.3×10^{19} possible positions, one might think that God's number could be very large indeed.

Hearing loss

Sound is basically a vibration of the molecules of the air created by small differences in air pressure. Different sound frequencies create different pressure differences. Deep inside the ear, in the cochlea, is a surface called the basilar membrane of the inner ear. In very rough terms, you can think of this surface as like your skin with hair on it. When it experiences a sound, the hair on it will vibrate. Remarkably, different frequencies will cause hair in different locations on this surface to vibrate. It is as though a high-pitched frequency would vibrate the hair on your legs and not your arms, but a low-pitched frequency would vibrate the hair on your arms and not your legs. This "frequency-to-place" mapping is the way the ear sorts out different frequencies, enabling our brain to process that sound. In a normal ear, high-frequency sounds are registered by the hair cells located at the shallower end of the cochlea, and low frequency sounds are registered by the hair cells further inside. When these hair cells vibrate, they create a tiny electrical impulse that the nerve cells in the basilar membrane pass to the brain. The brain interprets this activity to determine which sound frequency is being heard.

If you lack hair cells, then this process completely breaks down and you cannot distinguish frequencies. Words become noise.

As Michael Chorost explains in his book *Rebuilt* [**2**], some teenagers in the 1980s developed hearing problems due to the fact that, in the United States, there was a rubella outbreak in the 1960s. One of the consequences of this virus was that some pregnant women exposed to rubella gave birth to children who were more likely to gradually lose cochlear hair cells starting in their teenage years. However, other illnesses can cause hearing loss and the cause of Tom's hearing problems is not known.

Tom graduated from Wolfe City High School in 1981, earned a bachelor's degree in electrical engineering from Texas A&M University in 1985, and a Ph.D. at Stanford University in computer science in 1993. During these years, Tom was becoming deaf.

> I am post-lingually progressively deafened. I started losing my hearing in high school (no cause is known), and in college it became a huge problem. A very good friend's parents noticed how deaf I was and sprung for hearing aids for me (I was working my way through school at the time; no way I could afford them) and they helped a lot but school was still a challenge. Once I started working full-time I could afford more and better hearing aids, but as the aids got better my hearing got worse, until I reached a point that hearing aids weren't really helping much at all.

The mathematics of the cube

In this section, we give a brief technical summary of the group-theoretical aspects of the Rubik's Cube.

Consider the group G of transformations of Rubik's Cube. As before, think of the cube geometrically with faces labelled R (right), L (left), U (up), D (down), F (front), and B (back), each with nine cubies. Note that neither a quarter-turn move nor a face-turn move will move the cubie in the center of a face. If we number the noncenter cubies as in Figure 1, then we see that the Rubik's Cube group G can be regarded as permutations of the numbers $1, \ldots, 48$, i.e., a subgroup of the symmetric group S_{48}.

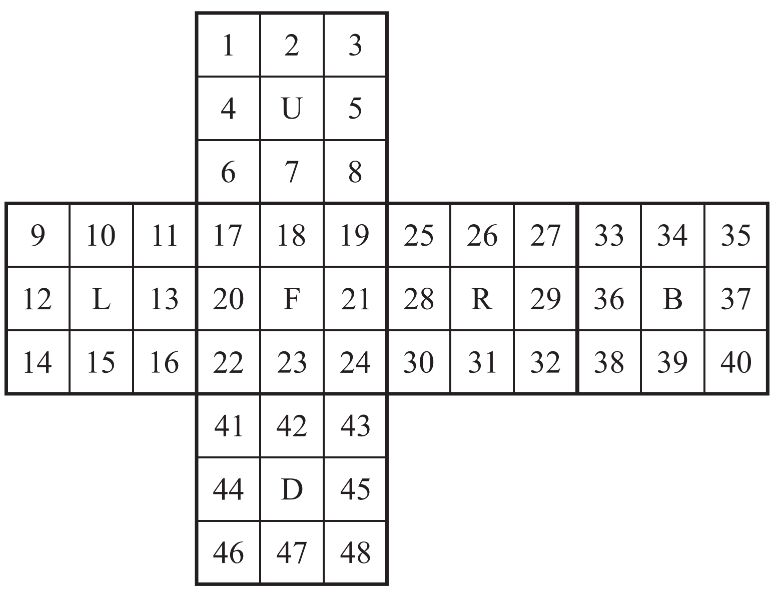

Figure 1. Cubie labeling showing $G < S_{48}$.

For example, if you rotate the "up" face clockwise 90°, then the U corner cubies get permuted by $1 \to 3 \to 8 \to 6 \to 1$, which we abbreviate using disjoint cycle notation as $(1, 3, 8, 6)$. The 1, 3, 6, and 8 cubies are not the only ones permuted by rotating U; the U side cubies are permuted too, as well as some on the R, L, F, and B faces. Taking all of these into account gives

$$U = (1, 3, 8, 6)(2, 5, 7, 4)(9, 33, 25, 17)(10, 34, 26, 18)(11, 35, 27, 19).$$

as the effect of rotating the U face clockwise 90°. Notice that, since U is a product of disjoint 4-cycles, $U^{-1} = U^3$.

Indeed, G is generated by U and the following permutations, corresponding to the quarter turns of the other five faces of the cube.

$$L = (9, 11, 16, 14)(10, 13, 15, 12)(1, 17, 41, 40)(4, 20, 44, 37)(6, 22, 46, 35),$$

$$F = (17, 19, 24, 22)(18, 21, 23, 20)(6, 25, 43, 16)(7, 28, 42, 13)(8, 30, 41, 11),$$

$$R = (25, 27, 32, 30)(26, 29, 31, 28)(3, 38, 43, 19)(5, 36, 45, 21)(8, 33, 48, 24),$$

$$B = (33, 35, 40, 38)(34, 37, 39, 36)(3, 9, 46, 32)(2, 12, 47, 29)(1, 14, 48, 27),$$

$$D = (41, 43, 48, 46)(42, 45, 47, 44)(14, 22, 30, 38)(15, 23, 31, 39)(16, 24, 32, 40).$$

This defines $G = \langle U, L, F, R, B, D\rangle$ as a specific subgroup of S_{48} (where $\langle\cdot\rangle$ denotes "generated by"). The solved position of the cube corresponds to the identity element e of G. More generally, the positions of the Rubik's Cube are in one-to-one correspondence with the elements of G.

The structure of this group, such as expressing it as a semidirect product of well-known groups, has been known for many years. It is not needed here; see [**4**, ch. 11] for details. In the sense of Brian Boyd's quotation, the Rubik's Cube is full of "patterns."

To connect to the search for God's number, we will generate G with a larger, redundant set of permutations. Denote the set of *quarter turns* as

$$S_{QT} = \{U, U^{-1}, L, L^{-1}, F, F^{-1}, R, R^{-1}, B, B^{-1}, D, D^{-1}\},$$

and the set of *face turns*

$$S_{FT} = \{U, U^2, U^{-1}, L, L^2, L^{-1}, F, F^2, F^{-1}, R, R^2, R^{-1}, B, B^2, B^{-1}, D, D^2, D^{-1}\}.$$

Notice that both of these subsets S of G are *symmetric* in the sense that if $s \in S$, then $s^{-1} \in S$ also. Since $\{U, L, F, R, B, D\} \subset S_{QT} \subset S_{FT} \subset G$, it is also true that $G = \langle S_{QT}\rangle = \langle S_{FT}\rangle$. Why generate G with these larger sets? For each $g \in G \setminus \{e\}$, writing $g = s_1 \cdots s_k$ for $s_i \in S$ means that the cube position corresponding to g can be solved in k moves (in the quarter-turn metric for $S = S_{QT}$, face-turn metric for S_{FT}). This representation of g in terms of s_i is called a *word*. If $g = e$, then we say g is the empty word and $k = 0$. Realize that different words of various lengths can correspond to the same g.

For $g \in G$, let $d_{QT}(g)$ denote the smallest possible $k \geq 0$ such that $g = s_1 \cdots s_k$ for $s_i \in S$ holds with $s_i \in S_{QT}$. For example, $d_{QT}(e) = 0$, $d_{QT}(R) = 1$ and $d_{QT}(R^2) = 2$. We call d_{QT} the *quarter-turn metric* on G and define the *face-turn metric* d_{FT} analogously. For example, $d_{FT}(e) = 0$, $d_{FT}(R) = 1$ and $d_{FT}(R^2) = 1$. Clearly, $d_{QT}(g) \geq d_{FT}(g)$, for all $g \in G$.

God's number then is the maximum value for the chosen metric.

A good way to consider G is via graph theory. The *Cayley graph* of G generated by S is the graph $\Gamma = (V, E)$, where

- the set V of vertices (or nodes) of Γ is the set of elements of G,
- the set E of edges of Γ is the subset $E \subset G \times G$ defined by $(g, h) \in E$ if and only if there is an $s \in S$ such that $g = sh$.

The Cayley graph Γ_{QT} associated to the quarter-turn metric has 12 edges incident to each vertex. The meaning of the metric in the graph is the smallest number of edges needed to travel from a vertex g to a vertex h is $d_{QT}(g^{-1}h)$. God's number for the quarter-turn metric is the diameter of Γ_{QT}, the length of the minimal path between two vertices at maximal distance. The situation for the Cayley graph Γ_{FT} associated to the face-turn metric is similar, with each vertex incident to 18 edges.

Recent progress on God's number

As long ago as the early 1980s, it was known that every position the cube could be solved in 52 moves or less. This was a painstaking coset analysis of various subgroups that arise in Thistlewaite's method (see [**4**, § 15.2.2] for a brief discussion).

Michael Reid showed (1995) that a particular position is exactly 20 face turns from the solved state. This *superflip* is the position of the Rubik's Cube where the centers and corners are correct, and all edges are correctly placed but "flipped." See Figure 2(a) on [**10**, p. 247] in this current issue. In other words, $d_{FT}(\text{superflip}) = 20$. The *superflip-fourspot* is the position of the Rubik's Cube similar to the super flip with four of the six centers swapped pairwise, Figure 2(b) on [**10**, p. 247]. Michael Reid showed that the superflip-fourspot is also exactly 20 face turns from solved and also 26 quarter turns from solved: $d_{QT}(\text{superflip-fourspot}) = 26$. These results give lower bounds for two variations of God's number.

$$\max_{g \in G} d_{FT}(g) \geq 20, \quad \max_{g \in G} d_{QT}(g) \geq 26.$$

For years, Tom Rokicki has been working on lowering the upper bound for the face-turn metric of the Rubik's Cube. His main work (finished in March 2008) is described in [**8, 11**], using group theory and prodigious programming skills.

> In December of 2003 I decided to solve the "edges-with-centers" space (the entire space of Rubik's Cube, ignoring all the corners). I wrote all of the code in *C++* using a new set of classes for representing positions, sequences, and symmetry. The program itself required only 1.3GB of RAM, and I posted the results to the Yahoo newsgroup speed-solving in January 2004. This was the largest such complete implicit graph search performed to date at that time, I believe, with 980 billion states (before symmetry and antisymmetry reductions).
>
> In November of 2005, I got an email from Silviu Radu that he had used these edges-with-centers results I had posted to lower the upper bound on God's number in the quarter turn metric to 38. This email started a collaboration that, in the end, led to all the rest of the work. My debt to Silviu is profound.

The mathematics of the algorithm used in [**14**] is too complicated to describe in detail here. The rough idea is as follows. Let $H = \langle U, D, F^2, B^2, L^2, R^2 \rangle$. This is a fairly large group: The coset space G/H has size 2,217,093,120. However, every cube position corresponding to an element of H has been solved optimally. (Of course, for each of these positions, at most 20 face turns are required.) The next goal is to do the same for each of the nontrivial cosets Hg. Such an approach is parallelizable, so the help provided by Sony and Google (described below) was invaluable. There is a more detailed explanation at Kociemba's website [**6**]. A full summary of results leading to God's number in the face-turn metric is given in Figure 2.

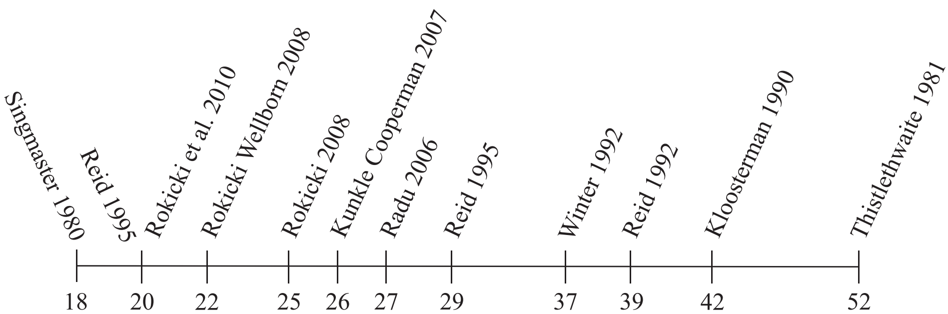

Figure 2. Names and dates of progress towards determining God's number for the Rubik's Cube under the face-turn metric, with two lower bound results and ten upper bound results over 30 years. Most of these results were reported only electronically; see [**13**] for links. See [**5, 8, 9, 14**] for the last three upper bound results.

Cochlear implants

While progress on the Rubik's Cube has been moving forward, so has research on hearing implants.

A cochlear implant tries to generate tiny electrical impulses in the nerve cells of the basilar membrane by locating small electrical contacts near where hair cells should be. Even though the cochlear implant is very small, it cannot match nature. In a normal human ear, there are about 20,000 hair cells. Currently it is far from possible to surgically implant 20,000 electrical nodes in the small area of the human skull where the inner ear is located. However, even 25 electrodes, which is the current state of the art, can transform a deaf person's life from noise (or silence) to meaningful sound. The implanted electrodes are "programmed" to nerve cells of the basilar membrane by an audiologist working closely with the patient. For some patients, the process of installing and mapping a functioning "bionic ear" goes fairly smoothly. For others, it can be a very frustrating experience.

How does the cochlear implant distinguish frequencies that are passed to the implanted electrodes? Sound waves can be decomposed into more basic sounds called harmonics. As a rough analogy, think of the sound from a piano chord, which can be decomposed into the sounds of its individual keys. Each of the individual piano keys makes a particular sound dominated by a particular frequency. Play them simultaneously, as in a piano chord, and you get a more complicated sound. Now, try to reverse this process. Listen to the sound of a complicated chord. How do you tell what the individual notes are? Roughly speaking, the general process of decomposing sound into frequencies uses the mathematics of Fourier analysis, a method physicists and mathematicians have known since the 1800s. However, this only touches on the theoretical difficulties involved. Besides that, there are practical matters: Solving a problem mathematically and creating a corresponding engineering solution are not the same. Implementing these procedures in a tiny hardware device small enough to be surgically implanted in an ear is still a very difficult, but actively researched, problem. Many engineers and scientists are developing new techniques, with improvements in hearing devices being discovered every year.

The final steps to the top

Tom was not working alone. Herbert Kociemba (high school mathematics teacher in Germany), Michael Reid (mathematician), Silviu Radu, Richard Korf, Gene Cooperman, Dan Kunkle (computer scientists), and others were also working on the problem of God's number. By early 2008, Rokicki's progress was so significant that Rubik's Cube aficionados started to think he was going to make a first ascent on their Mount Everest. In fact, it was soon afterwards that Sony's John Welborn contacted Tom out of the blue and offered CPU time on the farm of computers that Sony Pictures Imageworks uses to render animated movies such as *Spiderman 3*. As a result of this generous offer by Sony, Tom was able to extend the same method used in his original 25-move upper bound [**8**] to show that, in the face-turn metric, every position can be solved in at most 22 moves.

Then Google called.

Google donated time (35 CPU years, to be more precise) on their supercomputers to rule out the cases of positions at distance 21 and 22. This meant God's number is at most 20. As mentioned above, the superflip position is known to require 20 face turns. In other words, God's number in the face-turn metric is 20. With help from Sony and Google, Tom and his colleagues achieved an amazing milestone.

Tom's implants

At some point Tom's hearing became so bad that hearing aids no longer helped.

> I went in, and they said I might be a candidate for cochlear implants; I got implanted, and now I am hearing so much better! (If not normally.) It is hard to put into words how much better I hear now; it is like someone turned on the lights after I had spent years stumbling in the dark. Absolutely and truly a miracle in my life.

Tom Rokicki had his first surgery for a cochlear left-side implant in March 2008, just before he published his 25-move upper bound for God's number in the face-turn metric. His surgery in March 2011, about eight months after establishing the 20 move result, was for the right side. It was so successful that he listens to audio books in the car on the way to work in the morning.

Today

As this article was going to press, Tom Rokicki announced that he and Morley Davidson determined that God's number for the quarter-turn metric is 26, meaning that no position requires more moves than superflip-fourspot. This time computational resources came from the Ohio Supercomputer Center. It remains to determine the precise number of positions at large distances in each metric, and there is another Rubik's Cube metric Tom would like to study. See [**12**] in this issue for more information.

While Tom's hearing has been restored to a remarkable degree, not everyone responds as well as he did to cochlear implant surgery. Will the miraculous improvements in implants continue, making hearing loss a medically solved problem for everyone? Time will tell.

Acknowledgment. Many thanks to Tom Rokicki for generously sharing details of his life to help with the preparation of this article.

Summary. This is a tale of two problems. For years, Tom Rokicki worked to determine the exact value of God's number for the Rubik's Cube (the smallest number of moves needed to solve the cube in the worst case), a very difficult problem. By the time he solved this, Tom was completely deaf. Digitizing human hearing, and then implementing that into a medical device, is also a very difficult problem. Thanks to recent medical advances, Tom's hearing was restored about the same time that he discovered God's number.

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