Another Way to Sum a Series: Generating Functions, Euler, and the Dilog Function

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Abstract. It is tempting to try to reprove Euler's famous result that $\sum 1/k^2 = \pi^2/6$ using power series methods of the sort taught in calculus 2. This leads to $\int_0^1 -\frac{\ln(1-t)}{t}\,dt$, the evaluation of which presents an obstacle. With two key identities the obstacle is overcome, proving the desired result. And who discovered the requisite identities? Euler! Whether he knew of this proof remains to be discovered.

It is by now a familiar story: the young Leonhard Euler stuns the mathematical world in 1735 by announcing that $\sum 1/k^2 = \pi^2/6$. (Here and throughout, the indices in summations are assumed to run from 1 to ∞.) As Dunham [**2**] explains, the problem had been open since 1644, "and anyone capable of summing the series was certain to make a major splash." Sandifer [**13**] gives a detailed account of Euler's several derivations of the $\pi^2/6$ result, but our favorite retelling is the succinct statement of Erdős and Dudley [**3**], "In 1731 he obtained the sum accurate to 6 decimal places, in 1733 to 20, and in 1734 to infinitely many."

Today many different proofs of Euler's result are known. Weisstein [**15**] lists nearly a dozen references to proofs, and Kalman's survey [**9**] gives six proofs in detail. So, between the historical accounts of Euler's solution and all of the known proofs, can there be anything more to say about the subject?

We say there is. In this paper we will present yet another method for finding $\sum 1/k^2$, which deserves consideration for several reasons. First, the analysis starts with methods of elementary calculus, of the sort that any second-semester calculus student might think to try. Second, after what appears to be encouraging preliminary success, the method runs into a roadblock. It turns out that there is a way 'round, relying on two key identities. And whom may we thank for those identities? None other than Euler. Indeed, the proof possesses that familiar Eulerian flourish, with delicious manipulations and a breezy disregard for technicalities. But don't worry. All the manipulations can be justified, leaving us with a rigorous determination of $\sum 1/k^2$.

We are also left with a historical puzzle. Did Euler know this proof? As we will show, though the historical evidence is inconclusive, he might well have known it. Given Euler's genius, it is hard to imagine that he did *not* know it. He certainly had all the necessary steps well in hand. But in the paper containing the key identity, there is a hint that Euler either was not aware of this proof or did not consider the proof worth mentioning.

Here we will take up each of these matters: a second-semester calculus approach to evaluating $\sum 1/k^2$, the roadblock that arises, the Euler-style manipulations that get us past the roadblock, and the rigorous justification for the manipulations, as well as a brief look at our historical puzzle.

Before we proceed a few comments are in order. First, we acknowledge Lewin [**12**, p. 4] as our source for the derivation to follow. In fact, we shall refer to it as *Lewin's argument*, although we are not certain that it originated with him. Second, in citing

http://dx.doi.org/10.4169/amer.math.monthly.119.01.042
MSC: Primary 11M06, Secondary 33B30

Euler's publications, we provide references to the most readily available editions of his works—i.e., the scanned images available online at the Euler Archive [**4**], as well as the *Opera Omnia*, the modern reprinting of his collected work. For each publication, we also provide the Eneström index number, a valuable aid regardless of which source one uses. In those cases where it is necessary to cite a specific passage in one of Euler's works, paragraph or section numbers are used, as these are fairly consistent across different printings and translations.

In retracing the history of Euler's work, it is important to distinguish between a date of publication (which is included in the bibliographical citations) and the times when his discoveries were made. In some cases the publications themselves indicate a date of presentation before a learned society, frequently far in advance of the publication date. The Eneström index also specifies dates of completion for some works, and Euler's correspondence provides another means for dating his discoveries. In considering whether Euler was aware of Lewin's argument, we will see that timing may be significant. Where we specify dates of particular results, they are generally as reported in [**1, 2, 13**].

The Calculus 2 Approach and Roadblock. The alert calculus 2 student catches a glimpse of a powerful idea: generating functions. It is one of the most important tools in enumerative combinatorics and a bridge between discrete mathematics and continuous analysis [**16**]. But even the tiny glimpse afforded our calculus 2 student is impressive enough. Turning a numerical series into a function, we apply the methods of calculus. This allows us to find not only the sum of the original series but the sums of an infinite number of related series.

Here is an example: Suppose we wish to sum the alternating series

$$\sum (-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Consider the closely related power series

$$f(z) = \sum \frac{1}{k} z^k = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots . \tag{1}$$

The general problem of evaluating f is, at first glance, much harder than our original problem, which only concerns $-f(-1)$. But if we differentiate (1), we obtain the geometric series

$$f'(z) = \sum z^{k-1} = 1 + z + z^2 + \cdots .$$

This tells us that $f'(z) = 1/(1-z)$, and hence

$$f(z) = \int 1/(1-z)\, dz = -\ln(1-z) + C.$$

Moreover, we know from the definition that $f(0) = 0$. Consequently, $C = 0$ and so $f(z) = -\ln(1-z)$. It follows that $-f(-1) = \ln 2$ and the original problem is solved. Not only that, we can evaluate f at any number of points to obtain sums of other series. For example, $\sum 1/(k3^k) = f(1/3) = -\ln(1/3) = \ln 3$.

There is a small fly in the ointment. The radius of convergence for f is 1, justifying term-by-term differentiation for $-1 < z < 1$. Our evaluation of $\sum 1/(k3^k)$ is perfectly valid. We have to work a little harder to justify applying our results when $z = -1$. But let us not lose heart. For the original series $1 - 1/2 + 1/3 - 1/4 + \cdots$, numerical investigation must surely convince the most determined skeptic that $\ln 2$ is correct. We can worry about the technical justification later.

Exhilarated by the success of this method, we charge forward. What other series might it sum for us? Nothing could be more natural than $\sum 1/k^2$. So, beginning as before, our goal is to sum the series

$$\sum \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .$$

The related power series

$$g(z) = \sum \frac{1}{k^2} z^k = z + \frac{z^2}{4} + \frac{z^3}{9} + \cdots \tag{2}$$

has derivative

$$g'(z) = \sum \frac{1}{k} z^{k-1} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots .$$

But this is evidently $f(z)/z$, with f defined as above. Therefore,

$$g'(z) = -\ln(1-z)/z. \tag{3}$$

Now all that remains is to integrate $-\ln(1-z)/z$. Our calculus 2 student knows several methods that appear promising. Perhaps integration by parts. Or what about a substitution? Unfortunately, although the integrand doesn't seem especially complicated, none of our efforts lead anywhere. We have hit a road block.

How frustrating! We have come tantalizingly close to summing the series. In fact, because we know that $g(0) = 0$, we can express the sum we seek as a definite integral:

$$\sum \frac{1}{k^2} = g(1) = \int_0^1 g'(z)\, dz = -\int_0^1 \frac{\ln(1-z)}{z}\, dz.$$

If only we could evaluate the integral.

Now we are confronted by two possibilities. Perhaps $\ln(1-z)/z$ is one of those functions that simply cannot be antidifferentiated in closed form. If so, our roadblock is probably impassable. On the other hand, it may be that we have just not found the proper trick.

Here today's students have resources not available to Euler. They can use mathematics software. If the required integral can be evaluated by elementary means, a symbolic integrator will almost certainly be able to show us the answer.

Let us try Quickmath [**11**], a free website powered by Mathematica. We ask it to integrate `-log(1-x)/x` for `x` going from 0 to 1. The answer comes back: $\pi^2/6$. That's encouraging. How did Quickmath do that? To find out, we ask for the indefinite integral of `-log(1-x)/x`, and quickmath responds `Li`$_2$`(x)`. Investigating further, we learn that Li_2 is the *dilogarithm* function, a special case of the polylogarithm function [**14**]. Alas, we also discover that dilog is defined by the very series we dubbed $g(z)$. The definite integral was not evaluated by some trick of antidifferentiation. On the contrary, Mathematica converted the definite integral into the series $\sum 1/k^2$, and then returned the known value of that series. As a means for summing the series, this argument is circular.

But all is not lost. It is true that we cannot antidifferentiate $\ln(1 - z)/z$. But there is more than one way to skin an integral. Actually, we do not need the definite integral formulation to evaluate $g(1)$. There is an alternative approach using an identity discovered by Euler.

Through the Roadblock in Eulerian Style. From now on, we will refer to the function defined in (2) as dilog and denote it Li_2. Our goal is to evaluate $\mathrm{Li}_2(1)$.

Euler's identity follows from the fact that

$$\mathrm{Li}_2'(z) = -\ln(1 - z)/z \tag{4}$$

as shown in (3). The identity says

$$\mathrm{Li}_2(-1/z) + \mathrm{Li}_2(-z) + \frac{1}{2}(\ln z)^2 = C, \tag{5}$$

where C is a constant. It can be verified by showing that the derivative of the left-hand side is zero, using the chain rule and (4). The details are left as an exercise for the reader.

Taking $z = 1$ in (5) leads to

$$C = 2\mathrm{Li}_2(-1) = 2(-1 + 1/4 - 1/9 + 1/16 - \cdots). \tag{6}$$

This can be related to $\mathrm{Li}_2(1)$ with a well-known trick. The even terms of $\sum 1/k^2$ have sum $E = \sum 1/(2k)^2 = (1/4)\mathrm{Li}_2(1)$. Therefore the odd terms must sum to $D = (3/4)\mathrm{Li}_2(1)$, and the alternating sum in (6) is $E - D = -(1/2)\mathrm{Li}_2(1)$. This shows that $C = -\mathrm{Li}_2(1)$. Hence, (5) becomes

$$\mathrm{Li}_2(-1/z) + \mathrm{Li}_2(-z) + \frac{1}{2}(\ln z)^2 = -\mathrm{Li}_2(1). \tag{7}$$

Next substituting $z = -1$, we find

$$2\mathrm{Li}_2(1) + \frac{1}{2}[\ln(-1)]^2 = -\mathrm{Li}_2(1)$$

so that

$$\mathrm{Li}_2(1) = -\frac{1}{6}[\ln(-1)]^2.$$

To complete the analysis, we recall another of Euler's identities:

$$e^{i\pi} = -1$$

and so

$$i\pi = \ln(-1).$$

This tells us that $[\ln(-1)]^2 = (i\pi)^2 = -\pi^2$, and thus

$$\mathrm{Li}_2(1) = \frac{\pi^2}{6}.$$

This is an argument that Euler might well have prized. The manipulation of series and appearance of identities have the familiar Eulerian flair, and like so many of his arguments, conceal technical difficulties that require more careful consideration. Also, as with most of Euler's arguments, all of the steps can be made rigorous by modern standards, as we will see in the next section.

We have cited Lewin [**12**] as our source for the foregoing analysis. He gives a reference for Euler's key identity (5), and mentions earlier work by Euler on Li_2, but Lewin does not credit Euler for the specific argument evaluating $\sum 1/k^2$. Neither does he claim the argument as his own, saying only that the result is well known but the derivation "is perhaps not so familiar." When Lewin wrote those words, maybe the argument was part of the folklore among specialists concerned with dilog and its brethren. If so, Lewin himself might not have known where the argument originated. Might Euler have known this proof? We will return to that intriguing question at the end of the paper.

Making the Proof Rigorous. Euler's key identity depends on a formula for $\mathrm{Li}_2'(z)$ derived from the power series using term-by-term differentiation. In the context of calculus 2, this can be justified on the interior of the interval of convergence of the power series. But we apply Euler's identity at $z = \pm 1$, which are on the boundary of the interval of convergence. The proof also strays into the realm of complex analysis with the evaluation of $\ln(-1)$. To address both of these points, we define $\mathrm{Li}_2(z)$ via integration in the complex plane.

As a preliminary step, let us consider the logarithm as a branch of the inverse of the exponential function. For $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$, take $\ln z$ to be $\ln r + i\theta$. This is an analytic function in the domain $\mathbb{C} \setminus (-\infty, 0]$, that is, the complement in the complex plane of the real interval $(-\infty, 0]$. With this understanding, we see that $\ln(1-z)$ is analytic in $\Omega = \mathbb{C} \setminus [1, \infty)$.

We shall obtain Li_2 by integrating $-\ln(1-z)/z$. An apparent difficulty at the origin evaporates when we realize that the integrand has a removable singularity there. In particular, if we define

$$F(z) = \begin{cases} 1 & \text{if } z = 0, \\ \dfrac{-\ln(1-z)}{z} & \text{otherwise,} \end{cases}$$

then F is analytic in Ω. Of course away from the origin, F is analytic throughout Ω because it is the product of analytic functions. On the other hand, for $|z| < 1$ we have the series representation

$$-\ln(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots.$$

This shows that

$$-\frac{\ln(1-z)}{z} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots \tag{8}$$

for $z \neq 0$. The function F agrees in a neighborhood of zero with the series on the right, and so is analytic there.

Now we can define $\mathrm{Li}_2(z)$.

Definition. For $z \in \Omega$,

$$\mathrm{Li}_2(z) = \int_0^z F(w)dw = \int_0^z \frac{-\ln(1-w)}{w}\,dw.$$

The integral in the definition is a complex path integral, and because the integrand is analytic in Ω, the integral is path independent and Li_2 is analytic in Ω. We also observe immediately that $\mathrm{Li}_2'(z) = F(z)$. Therefore, term-by-term integration of (8) gives us the series representation

$$\mathrm{Li}_2(z) = z + \frac{z^2}{4} + \frac{z^3}{9} + \frac{z^4}{16} + \cdots \tag{9}$$

for $|z| < 1$. The series converges absolutely for $|z| = 1$, as well, but we need something more than the standard methods of calculus 2 to conclude that the series and $\mathrm{Li}_2(z)$ agree for $|z| = 1$. Indeed, our definition of $\mathrm{Li}_2(z)$ does not even assign a value for $z = 1 \notin \Omega$.

How then do we justify our evaluation of $\sum 1/k^2$ as $\mathrm{Li}_2(1)$? The argument is a bit circuitous. First, by examining radial limits, we show that $\mathrm{Li}_2(z)$ and $\sum z^k/k^2$ agree not only in the interior of the unit disk, but also on the unit circle T, except at $z = 1$. Then we can define $\mathrm{Li}_2(1) = \sum 1/k^2$ and deduce that continuity obtains on T. Second, we show that Euler's identity (5) holds for $z \in \mathbb{C} \setminus (-\infty, 0]$. Third, (7) is justified as before, but now depends on both the integral definition of Li_2 and its power series representation in the closed unit disk. Finally, we extend the identity to $z = -1$ using continuity on T.

To fill in the details of the preceding outline, we first prove the following lemma.

Lemma 1 (Radial Limits). *Suppose $z \in T$. Then*

$$\lim_{t\to 1^-} \sum \frac{(tz)^k}{k^2} = \sum \frac{z^k}{k^2}.$$

Moreover, for any $\epsilon > 0$ there exists $\delta \in (0, 1)$ independent of z such that $\sum \frac{(tz)^k}{k^2}$ is within ϵ of $\sum \frac{z^k}{k^2}$ whenever $\delta < t < 1$.

Proof. For any real $t \in (0, 1)$ both series mentioned in the lemma converge absolutely. Their absolute difference is

$$\left| \sum \frac{(1-t^k)z^k}{k^2} \right| \le \sum \frac{(1-t^k)}{k^2}. \tag{10}$$

Split the sum on the right into two parts, corresponding to $k \le N$ and $k > N$, respectively. The first part is a polynomial in t and converges to 0 as t approaches 1. It can be made small by choosing $t > \delta$ for an appropriate $\delta < 1$. The second part is bounded above by a tail of $\sum 1/k^2$ irrespective of the value of t. Therefore, by choosing first N and then δ, we can make (10) arbitrarily small, establishing the limit asserted in the lemma. And because δ can be chosen without regard to the value of z, the second part of the lemma is also verified. ∎

The radial limits result implies the following lemma.

Lemma 2. *For all $z \neq 1$ in the closed unit disk,*

$$Li_2(z) = \sum \frac{z^k}{k^2}.$$

Proof. We already know that the equation holds for $|z| < 1$. So consider a fixed $z \in T$, $z \neq 1$. By continuity of Li_2 in Ω we have

$$\mathrm{Li}_2(z) = \lim_{t \to 1^-} \mathrm{Li}_2(tz) = \lim_{t \to 1^-} \sum \frac{(tz)^k}{k^2} = \sum \frac{z^k}{k^2}.$$

That is what we wished to show. ■

Next, we extend the definition of $\mathrm{Li}_2(z)$ to $z = 1$.

Lemma 3. *Define $Li_2(1) = \sum_1^\infty 1/k^2$. Then the restriction of $Li_2(z)$ to T is continuous.*

Proof. Because we already know that $\mathrm{Li}_2(z)$ is continuous (and in fact analytic) in Ω, we need only show continuity at $z = 1$. To that end, let $\epsilon > 0$ be given. We will show that on the unit circle $\mathrm{Li}_2(z)$ varies by no more than ϵ for z near 1. As illustrated in Figure 1, The idea is to go radially from z to t^*z (with t^* near 1), then along a circular arc from t^*z to t^*, and finally from t^* to 1 along the real axis, estimating the variation in Li_2 separately at each stage.

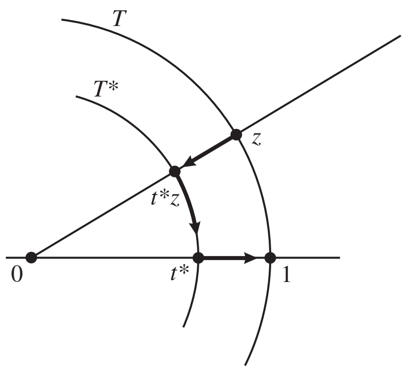

Figure 1. Going from z to 1 in three stages.

Applying Lemma 1, choose δ_1 so that for any $z \in T$, $\sum (tz)^k/k^2$ is within $\epsilon/3$ of $\sum z^k/k^2$ when $\delta_1 < t < 1$. In other words, $\mathrm{Li}_2(z)$ varies by less than $\epsilon/3$ along any radial line between the unit circle and the concentric circle of radius δ_1.

Fix t^* in the interval $(\delta_1, 1)$. The dilog function is continuous on the circle $T^* = t^*T = \{t^*z \mid z \in T\}$ and, in particular, is continuous at $t^* = t^* \cdot 1$. This implies that for some δ_2, at any $z^* \in T^*$ within δ_2 of t^*, $|\mathrm{Li}_2(z^*) - \mathrm{Li}_2(t^*)| < \epsilon/3$.

Now we claim that at all points z of the unit circle within δ_2 of 1, $|\mathrm{Li}_2(z) - \mathrm{Li}_2(1)| < \epsilon$. Indeed, we have

$$|\mathrm{Li}_2(z) - \mathrm{Li}_2(1)| \leq |\mathrm{Li}_2(z) - \mathrm{Li}_2(t^*z)| + |\mathrm{Li}_2(t^*z) - \mathrm{Li}_2(t^*)| + |\mathrm{Li}_2(t^*) - \mathrm{Li}_2(1)|.$$

On the right, the first and third terms measure radial variation of Li_2 between the unit circle and T^*. These terms are each less than $\epsilon/3$. The middle term measures variation along the circle T^* over a distance less than δ_2, so it too is less than $\epsilon/3$. Therefore we have shown that $|\mathrm{Li}_2(z) - \mathrm{Li}_2(1)| < \epsilon$, and that proves that the restriction of $\mathrm{Li}_2(z)$ to the unit circle is continuous at $z = 1$. ■

Having precisely defined dilog and established the behavior of its power series $\sum z^k/k^2$, let us turn to Euler's identity (5). We can justify the identity by differentiation only where $\mathrm{Li}_2(-1/z)$, $\mathrm{Li}_2(-z)$, and $\ln(z)$ are all analytic. That requires $-1/z$ and $-z$ both to be in Ω, and z to be in $\mathbb{C} \setminus (-\infty, 0] = \Lambda$. In fact, all three conditions hold for $z \in \Lambda$, so the left-hand side of the identity is analytic there. As before, we can infer that it is constant by verifying that its derivative vanishes. Moreover, since we know that neither $-z$ nor $-1/z$ is zero, we can differentiate their dilogs using (4). In this way we see that the identity does indeed hold in Λ.

In particular, the identity holds when $z = 1$, permitting the development we saw earlier leading up to (7). Now, though, Li_2 is defined as an integral, and we need our earlier result on power series representation to see that $\sum(-1)^k/k^2$ converges to $\mathrm{Li}_2(-1)$. Arguing as before, we see that $C = -\sum 1/k^2$, and by definition, that is $-\mathrm{Li}_2(1)$.

To complete the proof we would like to take $z = -1$ in (7). But we have to be more careful. Identity (5) has not been established for $z = -1 \notin \Lambda$. However, the identity does hold at every other point of T. By Lemma 3, the two Li_2 terms are continuous for $z \in T$. Meanwhile, although $\ln z$ jumps from $-\pi i$ to πi where T passes through -1, $(\ln z)^2$ is continuous there, with value $-\pi^2$. Therefore, the left side of (7) is continuous on all of T. But we already knew it to be the constant $-\mathrm{Li}_2(1)$ everywhere except at $z = -1$, so we can now conclude that the identity must hold at $z = -1$ as well.

This justifies at last applying (7) when $z = -1$, with the additional understanding that $[\ln(-1)]^2 = -\pi^2$. As argued earlier, that leads in turn to $\mathrm{Li}_2(1) = \pi^2/6$. Thus we have shown that Lewin's argument is valid.

Now we return to the historical question. Here we will be content to present a brief outline of the evidence we considered and conclusions we reached, such as they are. The interested reader is encouraged to see [**10**] for a more detailed discussion.

The Historical Puzzle. What did he know and when did he know it? Euler studied the function that we now call dilog as early as 1730 [**5**], when he discovered the identity

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) = C. \tag{11}$$

Note that this was before his first derivation of the $\pi^2/6$ result. In fact, he used (11) in the same paper to give his first estimate of $\sum 1/k^2$, correct to 6 decimal places. Although (11) is similar in appearance to (5), we found no evidence that Euler derived them during the same period. He returned to the study of dilog repeatedly over nearly 50 years, refining his methods of analysis in the process.

For example, dilog appears in Euler's correspondence with Daniel Bernoulli in 1742. It shows up again in 1768 in the *Institutionum calculi integralis* [**7**] (volume 1, chapter 4, paragraphs 196–200). Finally, in 1779, at the age of 72, Euler presented (5) in a paper whose primary focus is the dilog function [**6**]. In both of the two latest works, Euler evaluates a constant of integration (like the C in (11)) using the fact that $\mathrm{Li}_2(1) = \pi^2/6$. Apparently when these works were written, Euler considered the $\pi^2/6$ result to be settled fact, requiring no further substantiation. In the 1779 paper, in particular, if he did realize the Lewin argument could be used to evaluate $\mathrm{Li}_2(1)$, he might well have considered it insufficiently interesting to mention. So the question remains, did Euler know the Lewin argument?

With so prolific an author as Euler, determining if he published an argument like Lewin's is a daunting task. Our limited efforts in this direction produced no smoking gun. On the other hand, this is *Euler*. Is it conceivable that, with all of the necessary identities and methods at his fingertips, he failed to notice Lewin's argument? We think not. Either we failed to find where he wrote about it, or possibly it was something he knew but never published. If the latter is true, what were Euler's motivations?

In this regard timing seems to be vitally important. For example, when did Euler first discover (5)? As Sandifer [**13**] has explained, Euler was interested in derivations of the $\pi^2/6$ result for an extended period. His first proofs in 1735 used methods that drew some criticism. Over the next decade he continued to refine and develop these methods, deriving known results with them as one form of validation. But in 1741 he provided an additional derivation, this time using only elementary tools: Taylor series and integration by parts. After that, he no doubt considered the result to be beyond question. Consequently, if he discovered (5) (and along with it the Lewin argument) much later than 1741, there would have been little motivation for publishing an additional evaluation of $\mathrm{Li}_2(1)$.

Here, the methods Euler used in different periods are suggestive. For example, (11) is first derived in 1730 [**5**] and then rederived by simpler and more general methods in [**7**]. These same general methods are applied systematically in 1779 [**6**] to greatly expand Euler's supply of dilog identities. This suggests (5) was not discovered by Euler in his earlier work with dilog and raises the possibility that the identity was unknown prior to the 1779 paper.

Taking a different tack, Roy (personal communication, 2008) says that even if Euler knew the Lewin argument, he might have been reluctant to publish it because evaluating $\ln(-1)$ as $i\pi$ would be controversial. Again, timing is significant. As early as 1728, Euler and Johann Bernoulli corresponded about their divergent views of $\ln(-1)$. According to Bradley [**1**], Euler arrived at a complete understanding of the complex logarithmic function between 1743 and 1746. Apparently, at the time of Euler's 1741 definitive evaluation of $\sum 1/k^2$, he still harbored some uncertainty about the meaning of $\ln(-1)$. This may be evidence in favor of Roy's idea. If Euler was aware of (5) prior to 1746, his confidence in Lewin's argument would likely have been undermined by uncertainties about the logarithms of negative quantities. If his discovery of (5) came later than 1746, his interest in Lewin's argument might have been diluted by the feeling that $\sum 1/k^2$ was well established.

We may never know whether Euler was aware of Lewin's argument. As stated earlier, given Euler's amazing creativity and insight, once he had (5), it seems to us unlikely that he would not have thought of Lewin's argument. Our historical investigations lend weight to this position, suggesting that his discovery of (5) either came too early (and so while he was still uncertain about $\ln(-1)$) or too late (and so after he had provided an airtight evaluation of $\sum 1/k^2$). But these speculations are hardly conclusive. We hope that further research in Euler's papers and correspondence may throw additional light on this question.

ACKNOWLEDGMENTS. We thank Ari Mark Turner for bringing Lewin's argument to our attention, William Dunham for presenting a PREP workshop on Euler, and V. S. Varadarajan and Ranjan Roy for their contributions to our historical research.

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