Who Solved the Bernoulli Differential Equation and How Did They Do It?

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Everyone loves a mystery; mathematicians are no exception. Since we seek out puzzles and problems daily, and spend so much time proving things beyond any reasonable doubt, we probably enjoy a whodunit more than the next person.

Here's a mystery to ponder: Who first solved the Bernoulli differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n?$$

The name indicates it was a Bernoulli, but which? Aren't there 20 Bernoulli mathematicians? (Twenty is probably an exaggeration but we could reasonably count nine!) Or, as is so often the case in mathematics, perhaps the name has nothing to do with the solver. The culprit could be anyone! Like every good mystery, the clues contradict each other.

Here are the prime suspects.

Was it Gottfried Leibniz—the German mathematician, philosopher, and developer of the calculus? According to Ince [**12**, p. 22] "The method of solution was discovered by Leibniz, *Acta Erud.* 1696, p.145."

Or was it Jacob (James, Jacques) Bernoulli—the Swiss mathematician best known for his work in probability theory? Whiteside [**21**, p. 97] in his notes to Newton's papers, states, "The 'generalized de Beaune' equation $dy/dx = py + qy^n$ was given its complete solution in 1695 by Jakob Bernoulli."

Or was it Johann (Jean, John) Bernoulli—Jacob's acerbic and brilliant younger brother? Varignon [**11**, p. 140] wrote to Johann Bernoulli in 1697 that "In truth, there is nothing more ingenious than the solution that you give for your brother's equation; and this solution is so simple that one is surprised at how difficult the problem appeared to be: this is indeed what one calls an elegant solution."

Was it all three? Kline [**14**, p. 474] says, "Leibniz in 1696 showed it can be reduced to a linear equation by the change of variable $z = y^{1-n}$. John Bernoulli gave another method. In the Acta of 1696 James solved it essentially by separation of variables."

These are the suspects. Bring them in for questioning. Let's examine the evidence and close this case.

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Background

No matter who solved the Bernoulli equation, it was certainly first proposed in print in 1695 by Jacob Bernoulli [**3**]. He had been stuck on this problem for several months and decided to organize a competition to solve it. He published an article in the December 1695 issue of the journal *Acta Eruditorum*, the preeminent scientific publication in Germanic lands, though written exclusively in Latin. The article had a mouthful of a title: *Explicationes, annotationes et additiones ad ea quæ in actis superiorum annorum de curva elastica, isochrona paracentrica, & velaria, hinc inde memorata, & partim controversa leguntur; ubi de linea mediarum directionum, aliisque novis*. At the end of this article, what we call the Bernoulli differential equation is proposed. (See Figure 1.)

> *Problema*: Æquationem $ady = ypdx + by^n qdx$ (ubi a & b quantitates datas & conſtantes, n poteſtatem quamvis ſit, y, p & q quantitates utcunque datas per x denotant) conſtruere, ſaltem per quadraturas, hoc eſt, ſeparare in illa literas indeterminatas x & y cum ſuis differentialibus a ſe invicem.

Figure 1. Jacob proposes the Bernoulli differential equation [**3**, p. 553].

However, this differential equation didn't spring fully-formed from Jacob, but it is part of the evolution of the de Beaune equation, proposed to Decartes in 1638 by Florimond de Beaune. Geometrically, de Beaune essentially asked for a curve with constant subtangent, equivalent to solving $\frac{dy}{dx} = \frac{y}{\alpha}$, not achieved analytically for several decades. De Beaune actually asked for the solution in a system with axes skewed 45°. Lenoir [**19**, p. 360] gives a translation of de Beaune's original problem and both [**19**] and [**2**] show that his geometric question can be expressed analytically by $\frac{dy}{dx} = \frac{\alpha}{y-x}$. De Beaune wrote to Mersenne on March 5, 1639 [**9**] that he was interested in these inverse-tangent problems for "only one precise aim: to prove that the isochronism of string vibrations and of pendulum oscillations was independent of the amplitude."

Goldstine in [**2**] explains how a variety of similar differential equations, such as

$$\frac{dy}{dx} = \frac{1}{2}x^{-1}y - \frac{1}{2}y^{-1}x,$$

can be obtained by generalizing the de Beaune equation. These equations evolved into

$$\frac{dy}{dx} = ayx^m + by^r x^v,$$

which were studied by Jacob Bernoulli in his notebooks *Meditationes* CCXXXII and *Varia Posthuma* XII. They were further generalized to

$$\frac{dy}{dx} = P(x)y + Q(x)y^r,$$

which is what concerns us today. The story of this evolution is fascinating. It involves l'Hopital sending Jacob's solution of the original de Beaune equation to Huygens, then publishing it himself under a pseudonym. (See [**2**] or [**8**].)

To place the problem in context, it is helpful to know what techniques were available for solving ordinary differential equations in 1695. Newton had used series to solve differential equations for years. Separation of variables was communicated from Leibniz to Huygens, and James Bernoulli utilized the technique in print, coining the phrase "separation of variables." Leibniz had also solved homogeneous differential equations using a substitution. In 1694, Leibniz communicated to l'Hopital how to reduce first-order, linear differential equations to quadratures, though the technique hadn't appeared in print. Sporadic other equations had been solved via substitutions, change of variables, or other minor techniques. In sum, the techniques available—series, separation of variables, and substitution—were those same techniques often taught today in a first course in ordinary differential equations. Interested readers can see [**13**] or [**12**] for more details.

Jumping forward 300 years, let's review how we solve the Bernoulli equation now. Starting with

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

and substituting $w = y^{1-n}$, the equation becomes a first-order, linear equation

$$\frac{dw}{dx} + (1-n)P(x)w = (1-n)Q(x).$$

Multiplying both sides by the integrating factor $\mu(x) = e^{\int (1-n)P(x)\,dx}$ gives

$$e^{\int (1-n)P(x)\,dx}\frac{dw}{dx} + e^{\int (1-n)P(x)\,dx}(1-n)P(x)w = e^{\int (1-n)P(x)\,dx}(1-n)Q(x).$$

The left side is a total differential $d(w\mu)$; integrating gives

$$w\mu = \int e^{\int (1-n)P(x)\,dx}(1-n)Q(x)\,dx,$$

giving us w and therefore y.

What, if anything, did each of our suspects contribute to the solution?

Leibniz's solution

Jacob Bernoulli was a mathematician of the first class. Solving his differential equation was a hard problem. Nonetheless, solutions to his challenge appeared almost immediately. The first was by Leibniz. Three months after the problem was published, Leibniz published a solution in the *Acta* [**16**].

In the passage in Figure 2, we see the statement of the problem and the claim that changing variables into "z" reduces the Bernoulli equation to one of the form $\cdots dv + \cdots v\,dz + \cdots dz = 0$. This is a linear differential equation, and Leibniz describes exactly the technique that we use today. However, we should notice a few peculiar things about this passage.

Leibniz doesn't provide the substitution that reduces the problem to a linear differential equation. He doesn't even give the coefficient functions, instead leaving ellipses, so a reader could not figure out the substitution. Leibniz is being deliberately vague.

Leibniz is also guarded about his technique later in this passage. He gives no indication of how to solve the linear differential equation. Remember that the solution to linear differential equations was far from well known. Leibniz omits the details on purpose, saying "Such a general equation is reduced to quadrature by me, and has already been communicated to friends, which I do not think it necessary to explain here..."

dis, separandisve ab invicem indeterminatis. *Problema* de eo præstando circa æquationem differentialem $ady = ypdx + by^n . qdx$ solvere possum, & reduco ad æquationem, cujus forma est ... dv + ... vdz + ... dz $= o$, ubi per punctata intelliguntur quantitates utcunque datæ per z. Talis autem æquatio generaliter per me reducta est ad quadraturas, ratione jam dudum amicis communicata, quã hic exponere necessarium non puto, contentus effecisse, ut acutissimus Autor problematis agnoscere possit methodum (ut opinor) non dissimilem suæ. Neque enim dubito & hoc ipsi innotuisse. Et sunt a me in istis multa olim tentata,

Figure 2. Leibniz's solution of Bernoulli's equation [**16**, p. 147].

The friend that Leibniz refers to is l'Hopital; the technique is in a letter from Leibniz to him dated November 27, 1694 [**17**, p. 257] (see Figure 3). Leibniz defines a new variable p by the equation $dp/p = n\,dx$. Substitution into the linear differential equation gives $pm\,dx + y\,dp + p\,dy = 0$. The second two terms are a product rule $d(py)$ and so we can integrate to get

$$\int pm\,dx = -py,$$

which gives the solution for y. It wasn't uncommon for people to guard their results in this way, sometimes even hiding their results in ciphers or anagrams!

raux et peuvent estre poussés plus loin: Soit m + ny + dy : dx = 0, ou m et n signifient des formules rationelles, ou irrationelles mais qui ne dependent que de la seule indeterminée x, je dis qu'on la peut resoudre generalement par $\int$mpdx + py = 0, posito $\int$dp : p = $\int$ndx. Nam differentiando fit mpdx + ydp + pdy = 0, sed dp = pndx, ergo fit mpdx + npydx + pdy = 0 seu mdx + nydx + dy = 0, ut desiderabatur.

Figure 3. Leibniz's solution to first-order, non-homogenous, linear, differential equations [**17**, p. 257].

Finally, it is unclear if Leibniz could give an analytic solution to the resulting linear differential equation (even if he desired one). The fact that Leibniz used the word "quadrature" seems to indicate that he was satisfied giving the solution as the area under a curve.

Despite these issues, in July of 1696 Jacob Bernoulli published a second article in the *Acta* [**4**] announcing that his problem has been solved. *Problema beaunianum universalius conceptum, sive solutio æquationis nupero Decembri propositæ,* $a\,dy = yp\,dx + by^n q\,dx$*; cum aliis quibusdam annotatis* clearly references the Bernoulli differential equation. Bernoulli writes that Leibniz has solved his challenge and connects his differential equation with the de Beaune equation.

Johann gives details and a second solution

Less than a year later, in March of 1697, Johann Bernoulli published *De conoidibus et spaeroidibus quaedam. Solutio analytica æquationis in Actis A. 1695, pag. 553 propositæ (A Fratre Jac. Bernoullio)* [**7**]. The title tells us that Johann is solving his brother's equation. Johann actually gives *two* solutions!

His first solution is an elaboration of Leibniz's method (see Figure 4). He gives the explicit substitution $y = v^{1/(1-n)}$ that transforms a Bernoulli equation into a linear differential equation. (Warning: In Johann's collected works, this is misprinted as $y = v^{n/(1-n)}$.)

What we want to highlight is Johann's second solution (see Figure 5). Johann suggests we write the solution as $y = mz$. Notice that he substitutes y into the differential equation $a\,dy = yp\,dx + by^n q\,dx$, meaning that y solves the original differential equation. Then, he states that $adz/z = p\,dx$. In other words, z satisfies

$$a\frac{dy}{dx} = yp,$$

the *homogeneous* portion of the Bernoulli equation

$$a\frac{dy}{dx} = yp + by^n q.$$

What Johann has done is write the solution in two parts $y = mz$, introducing a degree of freedom. The function z will be chosen to solve the homogeneous differential equation, while mz solves the original equation. Bernoulli is using variation of parameters 78 years before Lagrange's famous paper [**15**] on the subject in 1775! Let's follow his argument a bit further.

Æquatio propoſita eſt hæc: $ady = ypdx + by^n qdx$ (ubi a & b quantitates datas & conſtantes, n poteſtatem quamvis literæ y; p & q quantitates utcunque datas per x denotant) ſeparandæ ſunt in illa literæ indeterminatæ x & y cum ſuis differentialibus a ſe invicem, ut ſaltem per quadraturas conſtrui poſſit, idquod ſic facio: Ut poteſtas n deprimatur ponendum eſt $y^n = v^{\frac{n}{1-n}}$, unde propoſita mutatur in hanc ulterius reſolvendam $\frac{1}{1-n} adv = vpdx + bqdx$, quæ reſpondet formulæ Leibnitianæ in Martio 1696 traditæ. Sed hac de-

Figure 4. Johann gives Leibniz's explicit substitution [**7**, p. 115].

First, z is a solution of the homogeneous equation $a\,dz = zp\,dx$. This is separable, hence we can solve for z as a function of "x & constantes." Second, since $y = mz$ solves the Bernoulli differential equation, we have that $a\,dy = a(m\,dz + z\,dm) = mzp\,dx + bq\,dx$. Since $a\,dz = zp\,dx$, we have $az\,dm = bq\,dx$. Substitution of the z found above into this differential equation leads to another separable equation that we can solve for m. Finally, writing $y = zm$ gives the solution to the linear differential equation.

sponder formulæ Leibnitianæ in Martio 1696 traditæ. Sed hac e
pressione potestatis mihi non opus est; immediate enim consequ
finem ponendo $y = mz$, ideoque $dy = m\,dz + z\,dm$; quibus subs
tutis in æquatione proposita habebitur $azdm + amdz = mzpdx + bm$
$z^n qdx$. Nunc ut hæc æquatio quatuor terminorum ad duos red
gatur, pono $amdz = mzpdx$ id est $\frac{adz}{z} = pdx$, unde cum habe
tur z per x aut algebraice aut saltem transcendenter esto, $z = \xi$ (p
ξ intelligo quantitatem datam per x & constantes.) Quoniam ve
destructis mdz & $mzpdx$ in æquatione transmutata, remanet azd
$= bm^n z^n qdx$ seu surrogato valore ipsius z, $a\xi dm = bm^n \xi^n qdx$ id
$am^{-n}dm = b\xi^{n-1}qdx$, hinc pariter habetur m per x, nimirum
$m^{-n+1} = b\int \xi^{n-1}qdx$, posito ergo $m = X$ (quantitati itidem ex x
constantibus compositæ) proveniet $y = (zm) = \xi X =$ quantitati p
dependenti ab x & constantibus. Q. E. F.

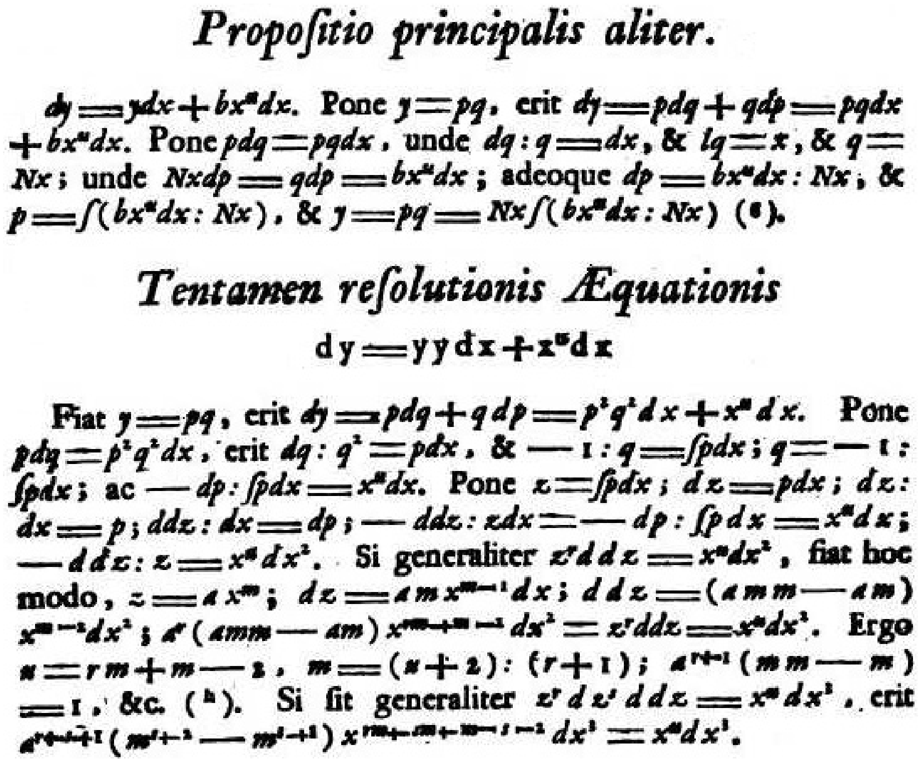
Figure 5. Johann introduces variation of parameters 78 years before Lagrange [**7**, p. 115].

Figure 5 is verbatim from an August 1696 letter from Johann to Leibniz [**18**, p. 323]. It shows that Johann knew of this technique at least eight months before he published it. Indeed, Johann wrote to l'Hospital in December 1696 that the equation "ne m'a donné aucune peine" (it didn't give me any trouble) [**1**, p. 265].

Not so fast, my friends!

The story doesn't end here. After Jacob's death, his papers and notebooks were collected and published together as the *Basiliensis Omnia*, along with extensive notes and annotations by the editor, Cramer. In the footnotes to the 1696 *Problema beaunianum universalius conceptum, sive solutio æquationis nupero Decembri propositæ,* $a\,dy = yp\,dx + by^n q\,dx$*; cum aliis quibusdam annotatis*, wherein Jacob announced that Leibniz has solved his problem, Cramer makes several clarifying comments. Some we've seen (such as giving the explicit substitution for Leibniz's trick). He also points the reader to Jacob's *Varia Posthuma*, Chapter XII [**5**], referred to above, where Jacob attacks the differential equation $dy = ayx^m\,dx + by^r x^v\,dx$, which can be thought of as an easier version of the Bernoulli differential equation.

Jacob's notes show that he solved many differential equations by writing the solution as a product of two functions. Figure 6 gives two examples, $dy = y\,dx + bx^v\,dx$ and $dy = yy\,dx + x^v\,dx$, both solved by supposing that $y = pq$. In the first example, q solves the homogeneous differential equation $dy = y\,dx$ [**5**, p. 1053].



Propositio principalis aliter.

$dy = ydx + bx^n dx$. Pone $y = pq$, erit $dy = pdq + qdp = pqdx + bx^n dx$. Pone $pdq = pqdx$, unde $dq : q = dx$, & $lq = x$, & $q = Nx$; unde $Nxdp = qdp = bx^n dx$; adeoque $dp = bx^n dx : Nx$, & $p = \int(bx^n dx : Nx)$, & $y = pq = Nx\int(bx^n dx : Nx)$ (g).

Tentamen resolutionis Æquationis

$$dy = yydx + x^n dx$$

Fiat $y = pq$, erit $dy = pdq + qdp = p^2q^2dx + x^n dx$. Pone $pdq = p^2q^2dx$, erit $dq : q^2 = pdx$, & $-1 : q = \int pdx$; $q = -1 : \int pdx$; ac $-dp : \int pdx = x^n dx$. Pone $z = \int pdx$; $dz = pdx$; $dz : dx = p$; $ddz : dx = dp$; $-ddz : zdx = -dp : \int pdx = x^n dx$; $-ddz : z = x^n dx^2$. Si generaliter $z^r ddz = x^n dx^2$, fiat hoc modo, $z = ax^m$; $dz = amx^{m-1}dx$; $ddz = (amm - am)x^{m-2}dx^2$; $a^r(amm - am)x^{rm+m-2}dx^2 = z^r ddz = x^n dx^2$. Ergo $n = rm + m - 2$, $m = (n+2) : (r+1)$; $a^{r+1}(mm - m) = 1$, &c. (h). Si fit generaliter $z^r dz^s ddz = x^n dx^3$, erit $a^{r+s+1}(m^{s+2} - m^{s+1})x^{rm+sm+m-s-2}dx^3 = x^n dx^3$.

Figure 6. Jacob's *Varia Posthuma* shows he has the idea of variation of parameters at the time of his brother Johann [**5**, p. 1053].

The interesting part is that Chapter XII of the *Varia Posthuma* was written sometime between September 1694 and June 1696 [**6**, p. 298]. While Jacob may not have used his technique to solve his own 'Bernoulli' differential equation, nor grasped the power of applying it generally, it appears that he had the seed of variation of parameters at least as early as his brother.

It is the how, not the who, that matters

Kenneth May in [**20**] warns of the dangers of "priority chasing," because we rarely will know who was first to have an idea. Instead, our purpose should be to "find out, relate, and explain these events." In this vein, our goal wasn't to say who solved the Bernoulli differential equation first. Rather, we hope in the process of studying priority questions to learn some mathematics that may influence our research or teaching.

Using variation of parameters to solve Bernoulli equations is rarely taught. Leibniz's substitution method is preferred. There is no reason for this. Certainly variation of parameters will be covered eventually in any ordinary differential equations course. Introducing it for first-order differential equations early in the course better motivates the technique for higher-order equations. Repetition reinforces learning. It adds to the generality of variation of parameters if students see it solve other equations. Perhaps we do a disservice by teaching a substitution that is memorized for one particular type of equation.

Not only does variation of parameters give an analytic solution to the Bernoulli differential equation, but it can also be used to solve first-order, non-homogeneous, linear differential equations, which are a special case of Bernoulli differential equations with $n = 0$. While Leibniz's trick is useless when $n = 0$, Johann's variation of parameters works perfectly well. Ince [**12**] says that this is the first analytical solution to linear differential equations, "but the solution by quadratures was known to Leibniz several years earlier" as we've seen above.

Here's another opportunity to re-evaluate our teaching. Today, we think of first-order, non-homogenous, linear differential equations as being "almost exact" and use an integrating factor, a method due to Johann's student Leonhard Euler [**10**] in *De integrationa aequationum differentialium* almost 70 years later in 1763, where Euler gave the first systematic study of integrating factors.

Today the integrating factor technique is by far the most common way taught in introductory courses. Again, there is no reason for this. It is simpler to use variation of parameters to convert a linear equation into two separable equations than to convert the equation into an exact differential equation, especially since we don't explain where the integrating factor comes from until later in the course. Students just memorize a formula. Applying variation of parameters shows the power of the technique, motivates its later appearance, and provides an elegant theme winding through the whole course.

And, it is historically accurate.

Conclusion

All three suspects are guilty of contributing to the solution of the Bernoulli differential equation. Jacob is convicted for proposing it in print, while Leibniz and Johann each supplied important ideas. Leibniz knew the technique that we teach today, though he chose to sequester most of the details. Johann's solution was variation of parameters years before Lagrange studied the technique.

Of the who, what, when, where, and why in the history of mathematics, who is sometimes least important. We hope that re-discovering how to apply variation of parameters to the Bernoulli and first-order linear equations will influence our teaching.

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Summary. The Bernoulli brothers, Jacob and Johann, and Leibniz: Any of these might have been first to solve what is called the Bernoulli differential equation. We explore their ideas and the chronology of their work, finding out, among other things, that variation of parameters was used in 1697, 78 years before 1775, when Lagrange introduced it in general.

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