REFERENCES

4. R. F. Ryan, Results concerning uniqueness for $\frac{\sigma(x)}{x} = \frac{\sigma(p^n q^m)}{(p^n q^m)}$ and related topics, *Int. Math. J.* 2 (2002), 497–514.

A Circle-Stacking Theorem

**ADAM BROWN**
Havergal College
Toronto, Ontario M5N 2H9
Canada
adam.brown@havergal.on.ca

By way of introduction, we first consider a preliminary problem—the wine rack problem. We have a wine rack and, fortunately, 13 bottles of wine. Three bottles are placed on the bottom row so that the outermost two are tangent to the vertical sides as depicted in the figure below. Then we place rows of two, three, two, and finally three more bottles on top of these. Surprisingly, no matter where the middle bottle is in the bottom row, the top row is always perfectly horizontal! Can you see why? This problem can be found in *Which Way Did the Bicycle Go?*, a nifty problems book published by the MAA. The problem was discovered by Charles Payan of the *Laboratoire de Structures Descretes et de Didactique* in France, one of the creators of CABRI geometry. He discovered this result while using CABRI, an incredibly powerful tool for experimenting with Euclidean geometry.

![Diagram](image)

We will not present a proof here of this engaging result, but will offer a companion instead. After playing around with this figure for a little while, one suspects that the middle circle in the middle row is half-way between the bottom two extreme circles. In fact, more turns out to be true: start with a row of equal circles on a line, and place a row of one fewer equal circles tangent to circles of the bottom row two at a time. Carry on in this way until one circle at the summit is reached. Then, provided the circles on
the bottom are positioned so that none of the succeeding circles can slip through any gaps, the center of the top circle lies half-way between the centers of the outermost bottom two.

The figure on the left shows the naked truth of the theorem, and the one on the right displays constructions and labeling. Notice that it suffices to show that $O_1O_2$ is equal in length to $O_1O_3$ for then triangle $O_1O_2O_3$ will be isosceles and the vertex $O_1$ will lie directly above the midpoint of the base. By joining the centers of tangent circles we obtain chains of rhombi, that is, quadrilaterals each of whose edge lengths equal twice the common radius of all the circles. Thus $O_1O_5$ is equal and parallel to $O_6O_7$, and so on. To transfer this segment all the way down to $O_2$, we reflect $O_4$ in $O_2O_3$ yielding $O_4'$. Thus $O_2O_4'$ is parallel and equal to $O_1O_3$ which implies that $O_1O_2$ is parallel and equal to $O_2O_4'$. So, this sequence of translations has enabled us to transfer $O_1O_2$ onto a new equal and parallel segment. By carrying on this reasoning, we will transfer $O_1O_2$ onto $O_3O_1'$, where $O_1'$ is the reflection of $O_1$ in $O_2O_3$. Then, by reflection in $O_2O_3$, $O_3O_1'$ is carried onto $O_3O_1$. Hence we have carried $O_1O_2$ onto $O_1O_3$ by a sequence of translations, followed by a reflection, and $O_1O_2 = O_1O_3$!

REFERENCE


**Editor's Note:** Alert reader Dwayne Jennings of Union University in Jackson, Tennessee, while enjoying our June cover, noticed that one of the lines indicating bunny ancestry is drawn in the wrong place. Please find the error and correct your copy accordingly by removing one line and adding another. (Hint: In the current version, the original pair is missing a pair of offspring.)