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A Surprising but Easily Proved Geometric Decomposition Theorem

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Introduction

The whole is the sum of its parts—what might those parts look like? If we have two very different-looking sets in the plane, when can their corresponding separate parts look alike? It is a question with some surprising answers.

In FIGURE 1, two closed sets A and B are composed of disjoint subsets—$A = A_1 \cup A_2$ and $B = B_1 \cup B_2$—in such a way that A_1 is similar to B_1 and A_2 is similar to B_2. For the "summands" to be truly disjoint, we must also account for the boundaries. To obtain the desired similarities, we assign the bottom edge of the square A_1 to the rectangle A_2 and the top edge of the square B_1 to the rectangle B_2. Could the same sort of decomposition be obtained if, say, the set A was replaced by a circular disk? A glance ahead to FIGURE 3 might affect your answer. And look at FIGURE 4—can each of those sets be partitioned into two disjoint subsets so that the corresponding parts of each set look alike? How would you bet?

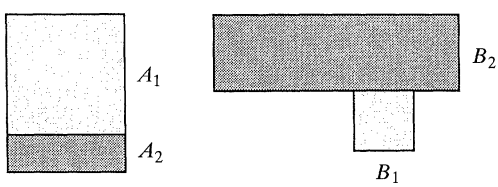

FIGURE 1

A remarkable result

Two sets A and B in the plane are *homothetic*, denoted $A \sim B$, if they are similar and similarly oriented. For example, in FIGURE 2, the sets A, B, and C are homothets of each other, but not of set D (even though D is congruent to A) because "similarly oriented" does not permit rotations or reflections. Thus in FIGURE 1, with A_1 missing its bottom edge and B_1 missing its top edge, the sets A_1 and B_1 are similar but they are not homothetic because the similarity mapping A_1 onto B_1 involves a 180° rotation. A *homothetic transformation* (or *homothety*) of the plane onto itself is a mapping of the form $f(\mathbf{v}) = k\mathbf{v} + \mathbf{a}$, where $\mathbf{a}$ is a constant vector and k is a positive scalar constant. When $k = 1$, f is a *translation*. When $\mathbf{a} = 0$ and $k = 1$, f is the identity mapping. When $\mathbf{a} = 0$ and $k \neq 1$, f is a *contraction* toward the origin or an *expansion* about the origin, according as $k < 1$ or $k > 1$. When $k \neq 1$, we may set $m = 1/(1-k)$ and note that

$$f(m\mathbf{a} + (\mathbf{v} - m\mathbf{a})) = f(\mathbf{v}) = k\mathbf{v} + \mathbf{a} = m\mathbf{a} + k(\mathbf{v} - m\mathbf{a}),$$

thus representing f as a contraction toward or expansion about the point m**a**.

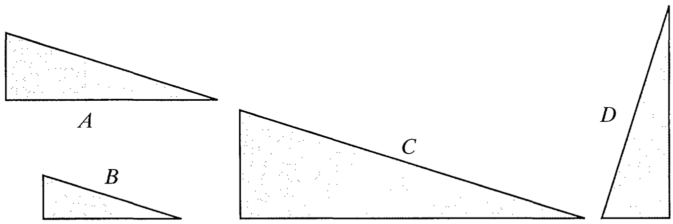

FIGURE 2

Using the definition, the reader will readily verify that the composition of two homotheties is again a homothety, that the inverse of a homothety is a homothety, and that each line is mapped by a homothety onto a parallel line.

We will say that two sets A and B are 2-*homothetic*, denoted $A \approx B$, if each of them can be partitioned into two disjoint sets ($A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with $A_1 \cap A_2 = \varnothing = B_1 \cap B_2$) in such a way that $A_1 \sim B_1$ and $A_2 \sim B_2$.

In FIGURE 1, if square B_1 were on top of rectangle B_2 rather than below, then A and B would be 2-homothetic, since the bottom edges of squares A_1 and B_1 could be assigned to A_2 and B_2 respectively, and then no forbidden rotation would be needed to establish the similarities. But when B_1 is tacked onto the bottom of B_2, as in FIGURE 1, it becomes an interesting exercise to try to show that A and B are 2-homothetic by finding the required partitions, remembering to take care of the boundaries.

Another example is found in FIGURE 3, which suggests an infinite nesting of inscribed squares and disks that might show the square and the disk to be 2-homothetic!!! Of course, we must always be careful of what is happening on the boundaries of the subsets. Is it really true that a square and a disk can be built from the same two pieces if we are allowed just expansions and contractions?

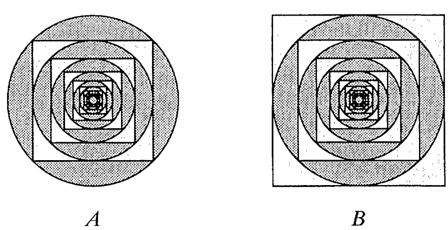

FIGURE 3

It is certainly not obvious that the two sets in FIGURE 4 are 2-homothetic, since the sets include isolated points, whiskers, random curves, components that may not be Lebesgue measurable (the shaded eye in B), and are generally as badly behaved as we could draw them. However, their 2-homotheticity is a consequence of the following remarkable result.

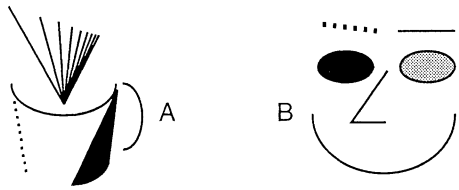

FIGURE 4

THEOREM 2HOM. *Two sets in the plane are 2-homothetic provided each of them is bounded and has nonempty interior.*

Although Theorem 2HOM seems surprising, it turns out to be an easy corollary of the following strengthened form of the famous Cantor–Bernstein theorem, and thus is a nice example to show the geometric power of abstract set theory.

THEOREM CBB. *If f: $A \to B$ is a function that maps a set A one-to-one* ***into*** *a set B (i.e., onto a subset of B) and g: $B \to A$ is a function that maps B one-to-one* ***into*** *A, then there are partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $f(A_1) = B_1$ and $g(B_2) = A_2$. Setting $h(a) = f(a)$ for all $a \in A_1$, and $h(a) = g^{-1}(a)$ for all $a \in A_2$, we have a one-to-one mapping h of A* ***onto*** *B.*

Proof of Theorem 2HOM. Suppose that A and B are both bounded, and each has an interior point. Since A has an interior point, A contains an entire circular disk C, and since B is bounded, a sufficiently great expansion of C about its center produces a larger disk D that contains B. The inverse of this expansion is a contraction (hence a homothety) that maps B into A. Similarly, there is a contraction that maps A into B. Since these contractions are clearly one-to-one, an application of Theorem CBB immediately yields the stated conclusion. ∎

Under the hypotheses of Theorem 2HOM, there are infinitely many contractions that pull set A into set B, and infinitely many that pull B into A, so there are infinitely many partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ for showing that A and B are 2-homothetic. Nevertheless, it is an interesting exercise to try to draw (or even imagine) such a partition in specific cases such as the one provided by FIGURE 4.

The original Cantor–Bernstein theorem asserts the existence of a one-to-one mapping h of A onto B, without specifying the relationship of h to the original mappings f and g. According to Fraenkel **[8]**, the stronger form stated above is due to Banach **[1]**, so we think of it as the Cantor–Bernstein–Banach (CBB) theorem. (The name of Schröder is often associated with the Cantor–Bernstein theorem. However, according to **[8]**, the theorem was conjectured by Cantor, the first complete published proof was due to Bernstein, and an independent proof of Schröder turned out to be defective.) See **[5]** for an extension of the CBB theorem.

The first explicit statement of Theorem 2HOM may have been the one in **[12]**, but Banach in **[1]** had already mentioned the possibility of geometric applications of the CBB theorem, and Theorem CBB was used in **[2]** to establish the famous Banach–Tarski paradox (see (7) below).

Two proofs of the CBB theorem

With such a strong corollary, you might expect that CBB has a difficult proof, but the classic proof of Banach [**1**] (found also in Birkhoff and MacLane [**4**]) is short and easy. It is the second proof below. Another nice proof of the CBB theorem uses a fixed-point theorem of Birkhoff [**3**]. To set this up, we need a quick review of complete lattices.

A *partial order* for a set S is a binary relation $\leq$ on S (i.e., a subset of the Cartesian product $S \times S$) with these properties:

1) *Reflexivity*: For each $a \in S$ the pair (a, a) is an element of the subset $\leq$ of $S \times S$. (We usually write $a \leq b$ to mean $(a, b) \in \leq$. Thus reflexivity is the condition that $a \leq a$ for all $a \in S$.)
2) *Anti-symmetry*: If $a \leq b$ and $b \leq a$ then $a = b$.
3) *Transitivity*: If $a \leq b$ and $b \leq c$ then $a \leq c$.

The pair $(S, \leq)$ is called a *partially ordered set*, or *poset*. For example, the real numbers form a poset with their usual ordering. But the reals have the additional property that every two elements are comparable, and hence we say that they form a *totally ordered set*. However, in posets it may happen that two elements a and b are not comparable—i.e., neither $a \leq b$ nor $b \leq a$ is true.

An element $s \in S$ is a *lower bound* for the set $T \subseteq S$ if $s \leq t$ for each $t \in T$. Similarly $u \in S$ is an *upper bound* for T if $t \leq u$ for each $t \in T$. The (necessarily unique) *least upper bound* for a subset T is an upper bound m for T such that $m \leq u$ for every upper bound u. *Greatest lower bounds* are similarly defined. A *lattice* is a non-empty poset in which each set of two elements (and hence each nonempty finite subset) has a least upper bound and a greatest lower bound. A *complete lattice* is a lattice in which every nonempty subset has a least upper bound and a greatest lower bound. The upper bound for the whole set S is usually denoted 1 and the lower bound for S is denoted 0.

Some examples might help.

Example 1. Let L denote the integer lattice in the Cartesian plane—the set of all points with both coordinates integers. If we define $(x, y) \leq (u, v)$ to mean $x \leq u$ and $y \leq v$ (in the usual sense) then $(L, \leq)$ is a poset. Some pairs of points, such as $(5, 8)$ and $(9, 6)$, are not comparable. But the point $(5, 6)$ is the greatest of all their lower bounds. The finite part of L shown in FIGURE 5 is a complete lattice. The upper right point is the upper bound and the lower left point is the lower bound for the whole subset shown. However, the infinite set L is a lattice but not a complete lattice.

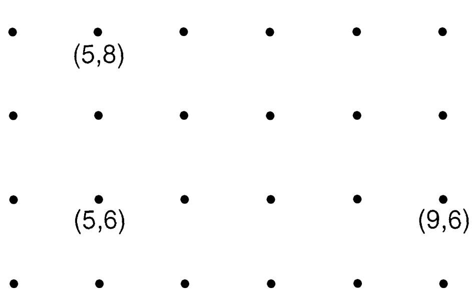

FIGURE 5

Example 2. Let $\mathscr{P}(R^2)$ denote the collection of all subsets of the plane R^2. Then $(\mathscr{P}(R^2), \subseteq)$ is a complete lattice. For any nonempty collection $\mathbb{C}$ of elements of $\mathscr{P}(R^2)$, the least upper bound (resp. greatest lower bound) of $\mathbb{C}$ is the union (resp. intersection) of all elements of $\mathbb{C}$.

When a function f maps a set S into itself, a point $a \in S$ is a *fixed point* for f if $f(a) = a$. Fixed-point theorems are among the most interesting and useful tools in mathematics. Theorem FP below is an all-time favorite that will be used to give a proof of the CBB theorem. A mapping f of a poset $(S, \leq)$ into a poset $(W, \preccurlyeq)$ is *order-preserving* if $x \leq y$ in S implies $f(x) \preccurlyeq f(y)$ in W.

THEOREM FP [3]. *Every order-preserving function f of a complete lattice $(S, \leq)$ into itself has a fixed point.*

Proof of Theorem FP. Let $T = \{a \in S \mid a \leq f(a)\}$. Clearly $0 \in T$ so $T \neq \emptyset$. Let m be the least upper bound of T. Since $t \leq m$ for every $t \in T$, and f is order-preserving, $t \leq f(t) \leq f(m)$, so $f(m)$ is also an upper bound of T. Hence $m \leq f(m)$ because m is the least upper bound of T. Thus $f(m) \leq f(f(m))$, so $f(m) \in T$ and $f(m) \leq m$. Since $m \leq f(m)$ and $f(m) \leq m$, it follows from anti-symmetry that $f(m) = m$ and m is the desired fixed point. ■

Fixed-point Proof of the CBB Theorem. Assuming without loss of generality that the sets A and B are disjoint, we will use the given one-to-one into functions $f: A \to B$ and $g: B \to A$ to define a function φ from the complete lattice $(\mathscr{P}(A), \subseteq)$ into itself. For each subset of A, let $C' = \{a \in A \mid a \notin C\}$ denote the *complement* of C in A. Similarly if $D \subseteq B$ let D' denote the complement of D in B. Then for each $C \subseteq A$ define $\varphi(C) = g((f(C'))')$. That is, we take the complement of C in A, map it into B by f, take the complement in B, and map this complement back into A by g. Since $C_1 \subseteq C_2$ implies $f(C_1) \subseteq f(C_2)$ and $C_1' \supseteq C_2'$, it is easily seen that φ is an order-preserving mapping of $\mathscr{P}(A)$ into itself. Hence by Theorem FP, φ has a fixed point. Call this fixed point A_2, set $A_1 = A_2'$, and set $B_1 = f(A_1) = B_2'$. Then the restrictions $f: A_1 \to B_1$ and $g: B_2 \to A_2$ are one-to-one and onto, and the partitions $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are the ones desired for the CBB Theorem. ■

Classic Proof of the CBB Theorem [1, 4]). We again assume that the sets A and B are disjoint. A point $x \in A \cup B$ is a *parent* of a point $y \in A \cup B$ if $x \in A$ and $f(x) = y$, or $x \in B$ and $g(x) = y$. Since A and B are disjoint and the mappings f and g are one-to-one, each point of $A \cup B$ has at most one parent. That parent (if it exists) has at most one parent, etc. This sequence of parents forms the *ancestral chain* of y. The sequence may be empty, as would be the case if $y \in B \setminus f(A)$ or $y \in A \setminus g(B)$. It may be infinite, as would be the case if $y = g(f(y))$ or $y = f(g(y))$. If the ancestral chain is neither empty nor infinite, it terminates in a point that has no parent. (See FIGURE 6).

Now let A_{even}, A_{odd}, and A_∞ denote the points of A for which the length of the ancestral chain is respectively even, odd, or infinite. This partitions A, and B has a similar partitioning. It is clear that f maps A_∞ into B_∞, A_{even} into B_{odd}, and A_{odd} into B_{even}. Further, since each point of $B_\infty \cup B_{\text{odd}}$ has a parent, the first two mappings are onto; that is, $f(A_\infty) = B_\infty$ and $f(A_{\text{even}}) = B_{\text{odd}}$. Similarly, $g(B) = A_\infty$ and $g(B_{\text{even}}) = A_{\text{odd}}$. Setting

$$A_1 = A_{\text{even}} \cup A_\infty, \quad A_2 = A_{\text{odd}}, \quad B_1 = B_{\text{odd}} \cup B_\infty, \text{ and } B_2 = B_{\text{even}},$$

we have the partitions whose existence is asserted by the CBB Theorem. ■

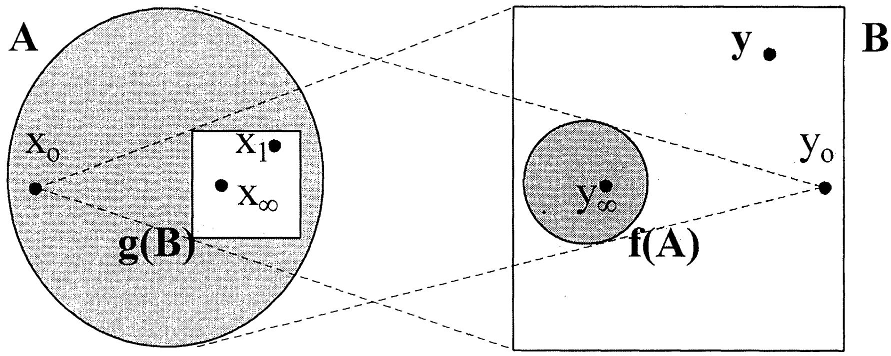

FIGURE 6

EXAMPLE 3. In FIGURE 6, the contraction f about the point y_o in the interior of B maps set A homothetically and one-to-one into set B. Similarly, the contraction g about the point $x_o \in A$ is a homothety which maps B one-to-one into A. Clearly, each of the points x_o, y_o, and y is an *orphan* (i.e., has no parent). Thus the ancestral chain of $x_1 = g(y)$ is just $\{y\}$, of (odd) length 1. Since $x_\infty = g(y_\infty) = g(f(x_\infty)$, the ancestral chain of $y_\infty \in B$ is $\{x_\infty, y_\infty, x_\infty, y_\infty, \dots\}$, of infinite length.

Remarks and open problems

1) The setting for Theorem 2HOM was the plane R^2, but the definitions (2-homothetic, bounded, interior) and the proof of Theorem 2HOM are all valid in an arbitrary (even infinite-dimensional) normed vector space.
2) When two subsets A and B of d-space are 2-homothetic and are both geometrically "nice" in some sense, it is interesting to ask how nice their summands (the sets A_1, A_2, B_1, B_2 in the partitions) can be made. Of course, niceness is in the eye of the beholder, and in any case the answer must depend on geometric or topological properties of the sets A and B. In particular, if the set A is connected, then it is impossible for A_1 and A_2 both to be closed (or both to be open) relative to A unless one of A_1 or A_2 is empty. However, one might hope to have A_1 closed and A_2 open relative to A, and then of course B_1 closed and B_2 open relative to B. FIGURE 7 shows that this can happen in some cases. In FIGURE 7a, the sets A and B are both bounded and convex, but neither is compact. In FIGURE 7b, the sets A and B are both compact, but neither is convex. However, it seems that the following problems are open for each $d > 2$:
   (a) Is there an example of two d-dimensional compact convex subsets A and B of d-space such that A and B are not homothetic but they are 2-homothetic by means of convex summands, $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, in such a way that the sets A_1 and B_1 are not only convex but also closed?
   (b) If A and B are both d-dimensional compact convex sets in Euclidean d-space, must they be 2-homothetic by means of summands A_i and B_i that are connected?

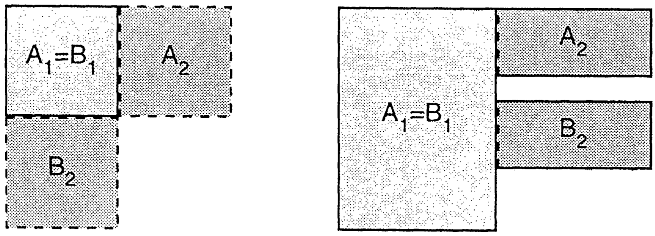

FIGURE 7

In both cases, A_1 and B_1 are closed relative to the sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ respectively. In 7a, A and B are convex but not compact, and in 7b they are compact but not convex.

3) In connection with problem 2(a), note that if A and B are compact subsets of d-space, each with nonempty interior, and $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are the partitions constructed in the proof of Theorem 2HOM, then each A_i and each B_i is both an F_σ-set (the union of countably many closed sets) and a G_δ-set (the intersection of countably many open sets). This follows from Banach's proof of the CBB Theorem. For let f and g be homotheties which, respectively, carry A into B and B into A. Define $A_0 = A$, $B_0 = B$, and having defined A_i and B_i, set $A_{i+1} = g(B_i)$ and $B_{i+1} = f(A_i)$. Then each A_i and each B_i is compact, hence is a G_δ set, and

$$A_0 \supseteq A_1 \supseteq \cdots, \qquad B_0 \supseteq B_1 \supseteq \cdots .$$

It follows that each set $A_i \setminus A_{i+1}$ is σ-compact, as is each set $B_i \setminus B_{i+1}$. Now define

$$A_{\text{even}} = (A_0 \setminus A_1) \cup (A_2 \setminus A_3) \cup \cdots \cup (A_{2j-2} \setminus A_{2j-1}) \cup \cdots$$
$$A_{\text{odd}} = (A_1 \setminus A_2) \cup (A_3 \setminus A_4) \cup \cdots \cup (A_{2j-1} \setminus A_{2j}) \cup \cdots$$
$$A_\infty = A_0 \cap A_1 \cap \cdots ,$$

and

$$B_{\text{even}} = (B_0 \setminus B_1) \cup (B_2 \setminus B_3) \cup \cdots \cup (B_{2j-2} \setminus B_{2j-1}) \cup \cdots$$
$$B_{\text{odd}} = (B_1 \setminus B_2) \cup (B_3 \setminus B_4) \cup \cdots \cup (B_{2j-1} \setminus B_{2j}) \cup \cdots$$
$$B_\infty = B_0 \cap B_1 \cap \cdots .$$

Then each of the sets A_∞ and B_∞ is compact, each of the sets A_{even}, A_{odd}, B_{even}, B_{odd} is σ-compact, and we have already seen that the desired partition is obtained by setting

$$A_1 = A_{\text{even}} \cup A_\infty, \quad A_2 = A_{\text{odd}}, \quad B_1 = B_{\text{odd}} \cup B_\infty, \quad \text{and} \quad B_2 = B_{\text{even}}.$$

Since the disjoint sets A_1 and A_2 are both F_σ-sets and their union is the compact set A, A_1 and A_2 are both also G_δ-sets. Similarly, B_1 and B_2 are both F_σ-sets and G_δ-sets.

4) It is an easy exercise to show that for any two homotheties f and g, the *commutator* $fgf^{-1}g^{-1}$ is merely a translation. Thus, although the group of

homotheties is not commutative, its first commutator subgroup is commutative. This (the fact that the group of homotheties is *solvable*) is a key to showing that Lebesgue measure in d-space can be extended to a finitely additive measure that is defined for all bounded sets and is not merely invariant under translation but multiplies properly under all homotheties. When $d = 2$, a similar conclusion applies to the group of transformations of the plane generated by the rotations and the homotheties. (See **[20]**, Chapter 10.)

5) It is easy to see that the homothety relation $\sim$ is reflexive, symmetric, and transitive. In particular, if $B = kA + \mathbf{a}$ and $C = mB + \mathbf{b}$, then $C = (km)A + (m\mathbf{a} + \mathbf{b})$, so $A \sim C$. The 2-homothety relation $\approx$ is reflexive and symmetric, but it is not transitive. FIGURE 8 shows sets A, B, and C, made up of parallel half-open intervals in the plane, with $A \approx B$ and $B \approx C$, but it is not true that $A \approx C$.

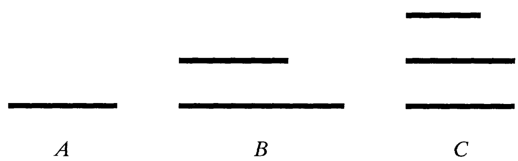

FIGURE 8

6) For any integer r with $2 \le r \le |A| = |B|$ we may define sets A and B to be *r-homothetic* in the obvious way: there exist partitions $A = A_1 \cup \cdots \cup A_r$ and $B = B_1 \cup \cdots \cup B_r$ and homotheties $f_i(\mathbf{x}) = k_i\mathbf{x} + \mathbf{a_i}$ such that $f_i(A_i) = B_i$ for each i. If, in addition, each A_i and each B_i has at least two points and the scalars $k_1, \dots, k_r$ are all different, we say that the sets A and B are *nontrivially r-homothetic*. In FIGURE 8, A is nontrivially 3-homothetic to C but A and C are not 2-homothetic. Other aspects of r-homothety make easy exercises.

7) A new family of problems arises when the group of homothetic transformations is replaced by some other group of transformations such as the rigid motions. The most famous result in this direction is the *Banach–Tarski paradox* **[2]**, asserting that if $d \ge 3$ and A and B are subsets of d-space each of which is bounded and has nonempty interior, then A and B are *equidecomposable* in the sense that for some finite n, A can be partitioned into n sets $A_1, \dots, A_n$ and B can be partitioned into n sets $B_1, \dots, B_n$ such that A_i is congruent to B_i for $1 \le i \le n$. See **[18]** and **[9]** for expositions of some aspects of the Banach–Tarski result, and see Wagon's book **[20]** for an extensive study of the "paradox" and related material.

8) In connection with the questions in 2), see **[17]** and **[11]** for some results and problems that involve decomposing two convex sets into a finite number of respectively congruent convex parts. And see **[6]** for a proof that in partitioning a ball of unit radius (in 3-space) into five sets that can be rearranged to form a partition of the union of two such balls, it can be arranged that each of the five sets is both connected and locally connected (of course, they cannot all be measurable).

9) Because of the measure-extension result mentioned in 4), if two subsets of the plane are both bounded and Lebesgue measurable, they cannot be equidecomposable unless they have the same measure. In 1925, Tarski **[19]** posed the

following modern version of the problem of squaring the circle: If D is a circular disk and S is a square of the same area, are D and S equidecomposable? Dubins, Hirsch, and Karush **[7]** showed that a circle and a square cannot be decomposed into respectively congruent parts that could (intuitively speaking) be cut out with a pair of scissors. However, Tarski's question did not restrict the nature of the sets in the partitions, and a brilliant affirmative solution to the question was given by Laczkovich **[15]** in 1990. His partitions involve a very large number of sets, but he requires only translations rather than the full group of rigid motions to move these sets from a disk-filling position to a square-filling position. For an excellent exposition of his work, see the article by Gardner and Wagon **[10]**. See also **[14]** and the 1994 survey article by Laczkovich **[16]**.

10) Even though Theorem CBB has the remarkable decomposition result Theorem 2HOM as an easy consequence, neither proof of CBB used the axiom of choice. This is in contrast to the situation for the measure-extension result mentioned in 4), for the Banach–Tarski paradox in 7), and for the theorem of Laczkovich in 9).

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