ARTICLES

The Discovery of Ceres: How Gauss Became Famous

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"The Duke of Brunswick has discovered more in his country than a planet: a super-terrestrial spirit in a human body."

These words, attributed to Laplace in 1801, refer to the accomplishment of Carl Friedrich Gauss in computing the orbit of the newly discovered planetoid *Ceres Ferdinandea* from extremely limited data. Indeed, although Gauss had already achieved some fame among mathematicians, it was his work on the Ceres orbit that "made Gauss a European celebrity—this a consequence of the popular appeal which astronomy has always enjoyed..." **[2]**. The story of Gauss's work on this problem is a good one and is often told in biographical sketches of Gauss (e.g., **[2]**, **[3]**, **[6]**), but the mathematical details of how he solved the problem are invariably omitted from such historical works. We are left to wonder, how did he do it? *Just how did Gauss compute the orbit of Ceres?* This is the question that we shall answer in this paper!

As the reader will observe, Gauss's work offers a rare instance of solving an historically great problem in applied mathematics using only the most modest mathematical tools. It is a complicated problem, involving over 80 variables in three different coordinate systems, yet the tools that Gauss uses are largely high school algebra and trigonometry! Gauss achieves greatness in this work not through deep, abstract mathematical thinking, but rather through an incredible vision of how the various quantities in the problem are related, a vision that guides him through extraordinary computations that others would likely abandon as futile.

Thus the description of Gauss's work that follows involves much algebraic and trigonometric computation. We hope the reader can appreciate Gauss's genius by observing how difficult it is to see how the various computational steps he undertakes might reasonably lead to the final goal. We hope also to have provided enough details so that the interested reader can follow Gauss's work from start to finish.

We begin with a brief introduction to reacquaint the reader with the historical background of the Ceres orbit problem.

Historical background

The asteroid Ceres was first observed by the Italian astronomer Joseph Piazzi in Palermo on New Year's Day, 1801. Within the European scientific community at the time there had been considerable discussion of the possibility that a major planet remained to be discovered on an orbit lying between those of Mars and Jupiter. Indeed, a group of 24 astronomers including Piazzi had formed to make a systematic search for such a planet, led by Baron Xavier von Zach, director of the Seeburg observatory and editor of the astronomical journal *Monatliche Correspondenz zur Beförderung der Erd- und Himmelskunde*.

Piazzi observed Ceres until February 11, 1801, when its position in the sky became too near that of the sun for any further observation. Meanwhile, on January 24, Piazzi had sent letters reporting his discovery to his colleagues Bode in Berlin, Oriani in Milan, and Lalande in Paris. In these letters, which reached Lalande in February but took until April to reach the others, Piazzi variously referred to the new object as a comet and as a planet **[3]**, **[7]**.

Reports of Piazzi's discovery soon reached von Zach, who published in the June 1801 issue of the *Monatliche Correspondenz* a long article "on a long supposed, now probably discovered, new major planet of our solar system between Mars and Jupiter." Though Piazzi requested that the publication of his observations be delayed, they were quickly shared among the leading European astronomers of the day; thus the July issue of the *Monatliche Correspondenz* contains a preliminary orbit for Ceres computed by the astronomer Burckhardt. In the September issue, von Zach finally published Piazzi's complete observations, and in the October issue, he reported that astronomers had looked carefully during August and September for the re-emergence of Ceres, but without success **[3]**.

It is at this point that Gauss became involved in the problem. At 24 years of age, Gauss had recently completed his doctoral degree and was living in relative obscurity in Brunswick, supported by an annual stipend from the Duke of Brunswick-Wölfenbuttel. Regarding the problem of computing planetary orbits from a short sequence of observations, Gauss writes in the preface to **[5]**,

> Some ideas occurred to me in the month of September of the year 1801,... which seemed to point to the solution of the great problem of [computing planetary orbits]... [T]hese conceptions... happily occurred at the most propitious moment for their preservation and encouragement that could have been selected. For just about this time the report of the new planet, discovered on the first day of January of that year with the telescope at Palermo, was the subject of universal conversation; and soon afterwards the observations made by that distinguished astronomer Piazzi from the above date to the eleventh of February were published. Nowhere in the annals of astronomy do we meet with so great an opportunity, and a greater one could hardly be imagined, for showing most strikingly, the value of this problem, than in this crisis and urgent necessity, when all hope of discovering in the heavens this planetary atom, among innumerable small stars after the lapse of nearly a year, rested solely upon a sufficiently approximate knowledge of its orbit to be based upon these very few observations. Could I ever have found a more seasonable opportunity to test the practical value of my conceptions, than now in employing them for the determination of the orbit of the planet Ceres, which during these forty-one days had described a geocentric arc of only three degrees, and after the lapse of a year must be looked for in a region of the heavens very remote from that in which it was last seen? This first application of the method was made in the month of October, 1801, and the first clear night, when the planet was sought for as directed by the numbers deduced from it, restored the fugitive to observation.

Gauss's earliest extant notes on Ceres were recorded in November of 1801, and it was in that month that he completed his first orbit determination. In the December 1801 issue of the *Monatliche Correspondenz*, von Zach published Gauss's predicted orbit for Ceres, writing that "Great hope for help and facilitation is accorded us by the recently shared investigation and calculation of Dr. Gauss in Brunswick" **[11]**. Although he pointed out that Gauss's orbit was significantly different from those of Burckhardt and other well known astronomers, von Zach gave arguments in its favor, concluding that "All this proves the Gaussian ellipse. What confidence it must thus awaken if astronomers recognize the precision with which it represents the collected Piazzi observations." Precision indeed, for on December 7, 1801, von Zach was able to relocate Ceres according to Gauss's predictions, and a few weeks later on New Year's Eve, the rediscovery was confirmed by Wilhelm Olbers, an amateur astronomer who later became a close friend of Gauss. And almost immediately, Gauss's reputation as a young genius was established throughout Europe.

Just how did Gauss compute the orbit of Ceres? Though his *Theoria motus corporum coelestium in sectionibus conicis solem ambientium (Theory of the motion of the heavenly bodies moving about the sun in conic sections)* of 1809 is clearly his crowning achievement in the area of planetary motion, Gauss writes in his preface to that work that "scarcely any trace of resemblance remains between the method in which the orbit of Ceres was first computed, and the form given in this work." Dunnington **[3]**, in his monumental biography of Gauss, writes that "His earliest notes on Ceres... lack clearness," and in **[8]** we find that "there is some controversy regarding precisely how he did it." Fortunately, Gauss sent a manuscript summarizing his methods in a letter to Olbers dated August 6, 1802, just seven months after the rediscovery of Ceres. The manuscript, entitled *Summarische Übersicht der zur Bestimmung der Bahnen der beiden neuen Hauptplaneten angewandten Methoden (Summary Survey of the Methods Applied in the Determination of the Orbits of Both New Planets)* was published years later in the September, 1809 issue of the *Monatliche Correspondenz*. Though the *Summarische Übersicht* had apparently already undergone certain refinements compared to the earliest methods, it is by far the most complete record of Gauss's early work on the computation of planetary orbits, and is therefore the work upon which we base our answer to the question of how Gauss computed the orbit of Ceres.

We close this portion of the narrative by recommending that the interested reader consult **[3]** for a more complete historical account of the discovery of Ceres, and **[7]** for a comparison of Gauss's earliest (unpublished) methods to those of the *Summarische Übersicht* and *Theoria Motus*.

The fundamentals of planetary orbits

To understand Gauss's work, we must first introduce the basic terminology of planetary orbits. According to *Kepler's First Law* the planet's orbit is an ellipse with the sun at one focus. As illustrated in Fig. 1, it is convenient to choose a standard rectangular coordinate system with the sun at the origin. (Typically the xy-plane is chosen to be the plane of the Earth's orbit, the so called *ecliptic* plane.) The angle i between the positive z-axis and the vector $\mathbf{n}$ normal to the planet's orbital plane is called the *inclination* of the orbit, with $0° \leq i \leq 90°$. The planet's orbital plane and the ecliptic plane intersect in the *line of nodes*, and, assuming that the direction of motion is as indicated by the arrow, the point N on this line is known as the *ascending node*. The angle Ω measured from the positive x-axis counterclockwise to the line of nodes is the *longitude of the ascending node*. Letting ω represent the angle between the line of nodes and the major axis of the planet's elliptical orbit, we define the *longitude of aphelion* $\pi = \Omega + \omega + 180°$ (the sum of angles in two

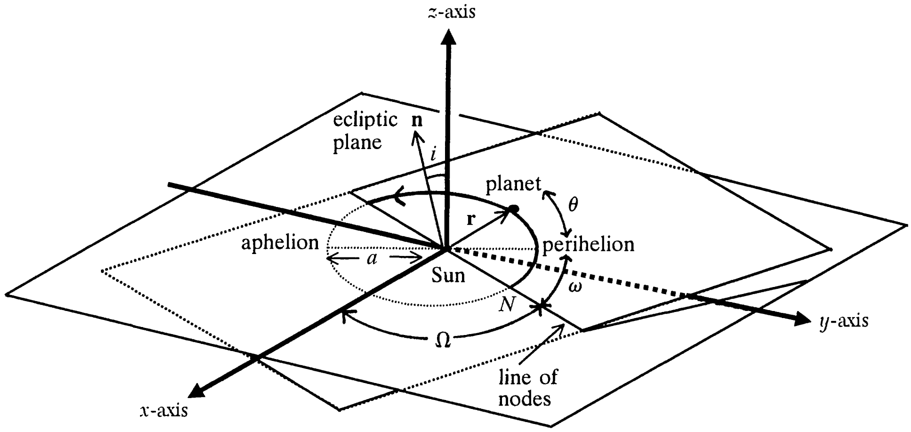

FIGURE 1
Parameters describing the planetary orbit

different planes!), which determines the orientation of the ellipse within the orbital plane. Note that *aphelion* is the point on the orbit furthest from the sun, whereas *perihelion* is the point closest to the sun. The ellipse itself is determined by a, the length of its semimajor axis, and e, its eccentricity. Finally, the position of the planet on this elliptical orbit is determined by τ_p, the time of perihelion passage. Collectively, the six quantities i, Ω, π, a, e, and τ_p are referred to as the *elements* of the orbit. For future reference, we note that the angle θ in Fig. 1 is known as the *true anomaly*, and define $v = \Omega + \omega + \theta$, again a sum of angles in two different planes. (The word *anomaly* was apparently chosen because of the discrepancy between the actual and computed values of θ in early studies of planetary motion.)

Suppose now that we know two heliocentric (sun-centered) vectors $\mathbf{r}$ and $\mathbf{r}''$ describing the planet's position at times τ and τ''. It is straightforward to compute from the normal vector $\mathbf{n} = \mathbf{r} \times \mathbf{r}''$ the inclination i, the equations of the orbital plane and the line of nodes, and Ω, the longitude of the ascending node.

Within the orbital plane, the elliptical orbit is given by the polar equation

$$r = \frac{k}{1 + e\cos\theta} = \frac{k}{1 - e\cos(v - \pi)},$$

where $k = a(1 - e^2)$. (Consistent with our previous notation, we use θ and θ'' to denote the true anomalies at times τ and τ'', respectively; similarly for v and v''.) With this notation, we note that the area of the ellipse is $\pi a^{\frac{3}{2}}\sqrt{k}$ or $\pi a^2\sqrt{1 - e^2}$, letting the context distinguish between our two uses of the symbol π.

According to *Kepler's Second Law*, the vector from the sun to a planet sweeps out area at a constant rate. If α denotes the area of the elliptical sector determined by the two vectors $\mathbf{r}$ and $\mathbf{r}''$, $\Delta\tau$ denotes the elapsed time between observations, and t denotes the period of the planet's orbit, Kepler's Second Law gives us

$$\frac{\alpha}{\Delta\tau} = \frac{\pi a^{\frac{3}{2}}\sqrt{k}}{t}.$$

If we denote the period and semimajor axis for the Earth's orbit by T and A, respectively, *Kepler's Third Law* tells us that $\frac{t^2}{a^3} = \frac{T^2}{A^3}$. Using $T = 365.25$ (days) and

$A = 1$ (astronomical unit), and approximating α from $\mathbf{r}$ and $\mathbf{r}''$ with the trapezoidal rule applied to a polar area integral, k can be determined.

The value of the ratio $\dfrac{\cos\theta''}{\cos\theta} = \dfrac{\cos(\theta + \theta'' - \theta)}{\cos\theta}$ can be obtained from the ellipse equations

$$e\cos\theta = \frac{k}{\|\mathbf{r}\|} - 1 \quad \text{and} \quad e\cos\theta'' = \frac{k}{\|\mathbf{r}''\|} - 1,$$

and the angle $\theta'' - \theta$ can be found from $\mathbf{r}\cdot\mathbf{r}'' = \|\mathbf{r}\|\,\|\mathbf{r}''\|\cos(\theta'' - \theta)$. Together, these can be solved for θ using the cosine addition formula. Next, ω can be obtained from $\mathbf{r}\cdot\langle\cos\Omega, \sin\Omega, 0\rangle = \|\mathbf{r}\|\cos(\theta + \omega)$.

Once θ and ω are known, π, e and a can easily be determined from the preceding relationships.

The elements i and Ω determine the orbital plane, and π, a, and e determine the shape and orientation of the elliptical orbit within that plane. Thus the geometry of the orbit in space is completely determined, but we still do not know *where* on the orbit the planet is located at a particular time. For this we need an initial condition, namely τ_p, the time (i.e., date and hour) of perihelion passage. To determine τ_p, we introduce yet another term from astronomy: in Fig. 2, a circle of radius a is

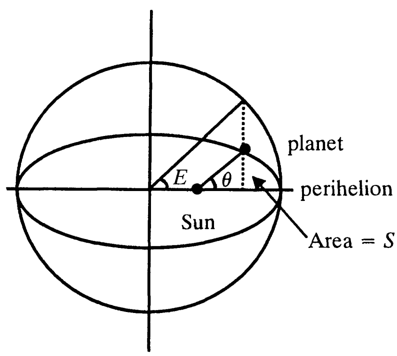

FIGURE 2
The true anomaly θ and eccentric anomaly E

circumscribed around the elliptical orbit with semimajor axis of length a, with the centers of the two coinciding. As noted above, θ is the true anomaly, whereas the angle E is known as the *eccentric anomaly*. These are related by

$$\tan\left(\frac{\theta}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\left(\frac{E}{2}\right),$$

which one derives by applying the half angle formula for tangent to angles E and θ in Fig. 2. The eccentric anomaly E can in turn be used to compute τ_p via *Kepler's equation* $E - e\sin E = \dfrac{2\pi}{t}(\tau - \tau_p)$. This latter equation is an immediate consequence of Kepler's second law: in Fig. 2, the area S of the elliptical sector is related to τ_p by

$$\frac{S}{\tau - \tau_p} = \frac{\pi a^2\sqrt{1-e^2}}{t}.$$

Substituting the computed area $S = \frac{1}{2}a^2\sqrt{1-e^2}\,(E - e\sin E)$ (found by integration), one obtains Kepler's equation.

The interested reader will find more details on the preceding computations in any orbital mechanics text such as [**8**] or [**9**]. For a concise, elementary treatment of computing the orbital elements from **r** and **r**″, including derivations of all of the preceding formulas as well as computational examples of their use, see [**10**].

Gauss's method in computing the orbit of Ceres

The computation of the orbital elements as described above was well known in 1801. The problem, of course, is that through telescopic observations alone we cannot determine the vectors **r** and **r**″; instead, we are only able to determine the geocentric (Earth-centered) longitude and latitude of the planet, and no information whatsoever about distance. Thus, the problem that confronted Gauss in 1801 was the following: from three geocentric observations (longitude and latitude) of a planet, determine two heliocentric vectors approximating the planet's position at two different times. From these two heliocentric vectors, the six orbital elements for the planet can be determined as previously outlined. We now describe the method that Gauss used in the *Summarische Übersicht* to solve this problem.

We begin by introducing two coordinate systems as shown in Fig. 3. The following notation is that of the *Summarische Übersicht*, with some exceptions which will be

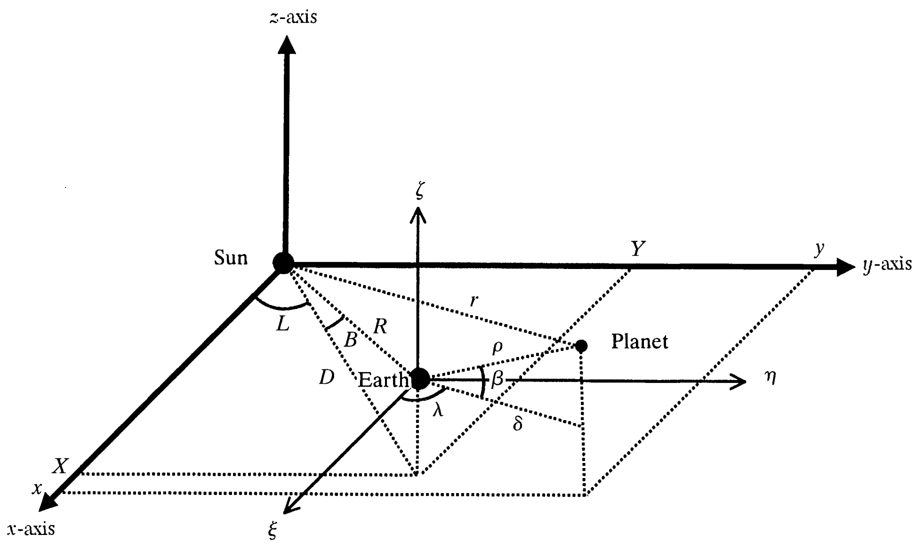

FIGURE 3
The xyz (heliocentric) and $\xi\eta\zeta$ (geocentric) coordinate systems

noted as they occur. Uppercase letters will consistently refer to the Earth, and lowercase letters to the planet. In the heliocentric coordinate system, the positions of the Earth and the planet at time τ are (X, Y, Z) and (x, y, z), respectively. The planet's geocentric coordinates are (ξ, η, ζ), and the two coordinate systems are related by $\xi = x - X$, $\eta = y - Y$, and $\zeta = z - Z$. Alternatively, one can describe the Earth's position using the heliocentric longitude L, latitude B, and distance R; similarly the planet's position is described by the geocentric longitude λ, latitude β, and distance $\rho = \delta \sec\beta$. (It is the values of λ and β that one determines by telescopic observation.) By adding a single prime (X', x', L', λ', etc.) or double prime

(X'', x'', L'', λ'', etc.), these symbols represent the corresponding quantities at times τ' and τ''.

To complete the notation, we let f denote the area of the triangle formed by the sun, the planet at time τ', and the planet at time τ'', and let g denote the area of the corresponding sector of the elliptical orbit. Similarly, $-f'$ and $-g'$ denote the areas of the triangle and sector corresponding to times τ and τ'', and f'' and g'' denote the areas corresponding to times τ and τ'. Finally F, $-F'$, F'', G, $-G'$, and G'' are analogously defined for the Earth.

Though Gauss did not have modern matrix notation at his disposal, it will be convenient for us to set

$$\phi = \begin{pmatrix} x & x' & x'' \\ y & y' & y'' \\ z & z' & z'' \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}.$$

Since the columns $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$ of ϕ all lie in the orbital plane, there are constants c_1 and c_2 such that $\mathbf{r} = c_1\mathbf{r}' + c_2\mathbf{r}''$. Then $\mathbf{r} \times \mathbf{r}' = c_2(\mathbf{r}'' \times \mathbf{r}')$ and $\mathbf{r} \times \mathbf{r}'' = c_1(\mathbf{r}' \times \mathbf{r}'')$, with $c_1 > 0$ and $c_2 < 0$. (These cross products are all perpendicular to the orbital plane with $\mathbf{r}'' \times \mathbf{r}'$ directed opposite the others.) Using $\|\mathbf{r}' \times \mathbf{r}''\| = 2f$, $\|\mathbf{r} \times \mathbf{r}''\| = -2f'$, and $\|\mathbf{r} \times \mathbf{r}'\| = 2f''$, we have $f'' = -c_2 f$ and $-f' = c_1 f$, so that $\mathbf{r} = -\frac{f'}{f}\mathbf{r}' - \frac{f''}{f}\mathbf{r}''$. Therefore $f\mathbf{r} + f'\mathbf{r}' + f''\mathbf{r}'' = 0$, or equivalently, $\phi\mathbf{f} = 0$. Analogously, for

$$\Phi = \begin{pmatrix} X & X' & X'' \\ Y & Y' & Y'' \\ Z & Z' & Z'' \end{pmatrix} \quad \text{and} \quad \mathbf{F} = \begin{pmatrix} F \\ F' \\ F'' \end{pmatrix},$$

we have $\Phi\mathbf{F} = 0$, from which it follows that

$$(F + F'')(\phi - \Phi)\mathbf{f} = \Phi((f + f'')\mathbf{F} - (F + F'')\mathbf{f}). \qquad (*)$$

Next, we transform equation $(*)$ by introducing spherical coordinates. Referring again to Fig. 3, define $\pi = \langle \cos\lambda, \sin\lambda, \tan\beta \rangle$. (Here Gauss presents a severe handicap to the reader: he defines π in his carefully laid out list of symbols just as we previously defined it, i.e., as the longitude of aphelion, but his first use of π is as we are using it here. Let the context distinguish among the various uses in the remainder of this paper!) Then

$$\delta\pi = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} x - X \\ y - Y \\ z - Z \end{pmatrix},$$

which is the first column of $\phi - \Phi$. Similarly, the second and third columns are $\delta'\pi'$ and $\delta''\pi''$, respectively. In the same fashion, let $P = \langle \cos L, \sin L, \tan B \rangle$, so that the columns of Φ are DP, $D'P'$, and $D''P''$. If equation $(*)$ is left-multiplied by the 4×3 matrix whose rows are $\delta''\pi'' \times \delta\pi$, $D'P' \times \delta\pi$, $D'P' \times \delta'\pi'$, and $D'P' \times \delta''\pi''$

(a straightforward but tedious task!), the resulting set of four equations is

$$(F+F'')f'\delta'[\pi\pi'\pi''] = (Ff''-F''f)(D[\pi P\pi''] - D''[\pi P''\pi''])$$
$$+((f+f'')F' - (F+F'')f')D'[\pi P'\pi''] \tag{1}$$

$$(F+F'')(f'\delta'[\pi\pi'P'] + f''\delta''[\pi\pi''P']) = (Ff''-F''f)(D[\pi PP'] - D''[\pi P''P']) \tag{2}$$

$$(F+F'')(f\delta[\pi'\pi P'] + f''\delta''[\pi'\pi''P']) = (Ff''-F''f)(D[\pi'PP'] - D''[\pi'P''P']) \tag{3}$$

$$(F+F'')(f\delta[\pi''\pi P'] + f'\delta'[\pi''\pi'P']) = (Ff''-F''f)(D[\pi''PP'] - D''[\pi''P''P']). \tag{4}$$

Here Gauss's original notation $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$ denotes the determinant of the matrix whose columns are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; equivalently, it is the triple scalar product $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Common factors of $\delta\delta''$, $\delta D'$, $\delta' D'$, and $\delta'' D'$ have been divided out of equations (1)–(4), respectively.

Some comments are appropriate here. Equations (1)–(4) appear above precisely as in the *Summarische Übersicht*. Gauss does not use equation (3) in the remaining development. His derivation of these equations involves no matrices, and although he uses the determinants $[\pi\pi'\pi'']$ etc., it is interesting to note that much of the modern theory of determinants was developed after Gauss's paper appeared [**1**]. Gauss makes up for the cumbersome notation available to him by simply presenting the main results with little or no clue as to the computations behind them.

Next, Gauss writes, "we now want to examine these four equations, which are precisely true, more closely in order to build the first approximation on them." To this end, he argues that in equations (2) and (4), the left side is $\mathcal{O}(t^3)$, whereas the right side is $\mathcal{O}(t^5)$ (or in Gauss's words, "If we view the intervening times as infinitely small quantities of the first order, ... what stands on the right of the second, third, and fourth equations above is of fifth order..."). Thus, by setting the right side equal to zero, "one can, as the first approximation, set

from 2) $\quad f'\delta'[\pi\pi'P'] = -f''\delta''[\pi\pi''P']$

from 4) $\quad f\delta[\pi\pi''P'] = -f'\delta'[\pi'\pi''P']$."

Solving these equations for δ and δ'', and using Kepler's second law in the form

$$\frac{g}{\tau''-\tau'} = \frac{-g'}{\tau''-\tau} = \frac{g''}{\tau'-\tau},$$

Gauss obtains

$$\delta = \frac{g}{f}\frac{f'}{g'}\frac{\tau''-\tau}{\tau''-\tau'}\frac{[\pi'\pi''P']}{[\pi\pi''P']}\delta' \tag{5}$$

and

$$\delta'' = \frac{g''}{f''}\frac{f'}{g'}\frac{\tau''-\tau}{\tau'-\tau}\frac{[\pi\pi'P']}{[\pi\pi''P']}\delta'. \tag{6}$$

The ratios $\frac{f}{g}$, $\frac{f'}{g'}$, and $\frac{f''}{g''}$ can all be approximated by one, as Gauss apparently did in his earliest work (see [**4**], p. 156 and [**7**], pp. 16–17). All other right side quantities are known except δ'. Thus, if δ' can be found, we will have two complete geocentric positions $\langle \delta\cos\lambda, \delta\sin\lambda, \delta\tan\beta \rangle$ and $\langle \delta''\cos\lambda'', \delta''\sin\lambda'', \delta''\tan\beta'' \rangle$, from which two heliocentric positions $\mathbf{r}$ and $\mathbf{r}''$ can easily be obtained.

Next, Gauss develops the main result of his paper from (1). Letting r, r', and r'' represent the lengths of the planet's heliocentric position vectors $\mathbf{r}$, $\mathbf{r}'$, and $\mathbf{r}''$, the polar equation of the elliptical orbit gives us

$$\frac{1}{r} = \frac{1}{k}(1 - e\cos(v - \pi)), \quad \frac{1}{r'} = \frac{1}{k}(1 - e\cos(v' - \pi)), \quad \text{and}$$
$$\frac{1}{r''} = \frac{1}{k}(1 - e\cos(v'' - \pi)).$$

After multiplying these equations by $\sin(v'' - v')$, $\sin(v - v'')$, and $\sin(v' - v)$, respectively, then adding, the result can be rewritten using the triangle area formulas $f = \frac{1}{2}r'r''\sin(v'' - v')$, $f' = \frac{1}{2}rr''\sin(v - v'')$, and $f'' = \frac{1}{2}rr'\sin(v' - v)$ and the identity $\sin\gamma + \sin\psi - \sin(\gamma + \psi) = 4\sin\frac{\gamma}{2}\sin\frac{\psi}{2}\sin\frac{\gamma + \psi}{2}$. The terms of the form $e\cos\gamma\sin\psi$ add to zero, producing

$$\frac{f + f' + f''}{f'} = \frac{-2r'}{k}\,\frac{\sin\frac{1}{2}(v'' - v')\sin\frac{1}{2}(v' - v)}{\cos\frac{1}{2}(v'' - v)}. \tag{7}$$

Next, using the fact that the planet's elliptical orbit has area $\pi a^{\frac{3}{2}}\sqrt{k}$, Kepler's second law gives us

$$\frac{\pi a^{\frac{3}{2}}\sqrt{k}}{t} = \frac{g}{\tau'' - \tau'} = \frac{g''}{\tau' - \tau},$$

from which Kepler's third law gives

$$\frac{\pi^2 A^3 k}{T^2} = \frac{\pi^2 a^3 k}{t^2} = \frac{gg''}{(\tau'' - \tau')(\tau' - \tau)}. \tag{8}$$

If M, M', and M'' describe the Earth's angular displacement from perihelion at times τ, τ', and τ'' under the assumption of *constant angular velocity*, then

$$\frac{2\pi}{T} = \frac{M' - M}{\tau' - \tau} = \frac{M'' - M'}{\tau'' - \tau'},$$

and so we obtain from (8)

$$k = \frac{T^2 gg''}{\pi^2 A^3(\tau'' - \tau')(\tau' - \tau)} = \frac{4gg''}{A^3(M'' - M')(M' - M)}. \tag{9}$$

(M is actually defined by $M = \frac{2\pi}{T}(\tau - \tau_p)$ and is called the *mean anomaly*.) Equation (9) and the small-angle approximations

$$\cos\tfrac{1}{2}(v'' - v) \approx 1, \quad g \approx r'r''\sin\tfrac{1}{2}(v'' - v'), \quad g'' \approx rr'\sin\tfrac{1}{2}(v' - v), \quad \text{and} \quad rr'' \approx r'^2$$

allow us to transform (7) into the approximation

$$\frac{f + f' + f''}{f'} = -\frac{A^3}{2r'^3}(M' - M)(M'' - M'). \tag{10}$$

Analogously, one obtains the approximation

$$\frac{F+F'+F''}{F'} = -\frac{A^3}{2R'^3}(M'-M)(M''-M'). \tag{11}$$

To complete the method, Gauss now turns to equation (1), arguing that the left side is $\mathscr{O}(t^5)$, while the two terms on the right side are $\mathscr{O}(t^7)$ and $\mathscr{O}(t^5)$, respectively. The first term on the right side of (1) is dropped, and (10) and (11) are used to transform (1) into

$$(F+F'')f'\delta'[\pi\pi'\pi''] = f'F'\frac{A^3}{2}(M'-M)(M''-M')\left(\frac{1}{R'^3}-\frac{1}{r'^3}\right)D'[\pi P'\pi''].$$

Using an approximation which is effectively $\frac{F'}{F+F''} \approx -1$, Gauss rewrites this equation as

$$\frac{[\pi\pi'\pi'']}{[\pi P'\pi'']}\frac{2}{A^3(M'-M)(M''-M')} = -\left(\frac{1}{R'^3}-\frac{1}{r'^3}\right)\frac{D'}{\delta'}, \tag{12}$$

which he describes as "the most important part of the entire method and its first foundation." Finally, by taking the xy-plane to be the ecliptic plane (so that $D' = R'$), approximating A by R', and computing the given determinants, (12) becomes

$$\left(1-\left(\frac{R'}{r'}\right)^3\right)\frac{R'}{\delta'} = \frac{-2}{(M'-M)(M''-M')} \times \frac{\tan\beta'\sin(\lambda''-\lambda) - \tan\beta\sin(\lambda''-\lambda') - \tan\beta''\sin(\lambda'-\lambda)}{\tan\beta\sin(L'-\lambda'') - \tan\beta''\sin(L'-\lambda)}. \tag{13}$$

(Equation (12), as published in the Gauss *Werke*, differs by a minus sign and has an R instead of a D' on the right side. Also, in the *Werke*, (13) differs by a minus sign and has an A^3 in the denominator of the right side. These errors are corrected without comment in **[7]**, where equations equivalent to (12) and (13) are given.)

The right side of (13) is computed from observational data, and the left side can be reduced to the single variable $\frac{R'}{\delta'}$ by means of

$$\frac{R'}{r'} = \frac{R'}{\delta'}\left(1+\tan^2\beta' + \left(\frac{R'}{\delta'}\right)^2 + 2\frac{R'}{\delta'}\cos(\lambda'-L')\right)^{-\frac{1}{2}},$$

which is obtained by applying the law of cosines to the triangle with vertices at $(0,0,0)$, $(x', y', 0)$, and $(X', Y', 0)$. Solving (13) (numerically) for $\frac{R'}{\delta'}$, and using the known value of R', one obtains a value for δ'; equations (5) and (6) now give values for δ and δ''. The geocentric position vectors $\langle \delta\cos\lambda, \delta\sin\lambda, \delta\tan\beta\rangle$ and $\langle \delta''\cos\lambda'', \delta''\sin\lambda'', \delta''\tan\beta''\rangle$ of the planet can now be added to the Earth's position vectors to obtain two complete heliocentric vectors $\mathbf{r}$ and $\mathbf{r}''$ describing the planet's positions at times τ and τ'', as desired. With this, the problem is solved.

What a magnificent achievement! Though the mathematical tools used are not particularly sophisticated, all sense of the motivating geometry is lost very early in this work, leaving one to wonder what led Gauss through the rather extraordinary computations needed to achieve his goal. Perhaps no less than a "super-terrestrial spirit in a human body" could have done it!

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