Touching the $\mathbb{Z}_2$ in Three-Dimensional Rotations

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Rotations, belts, braids, spin-1/2 particles, and all that

The space of all three-dimensional rotations is usually denoted by $SO(3)$. This space has a well-known and fascinating topological property—a complete rotation of an object is a motion which may or may not be continuously deformable to the trivial motion (i.e., no motion at all) but the composition of two motions that are not deformable to the trivial one gives a motion, which is. (Here and further down by "complete rotation" we will mean taking the object at time $t = 0$ and rotating it as t changes from 0 to 1 arbitrarily around a fixed point, so that at $t = 1$ the object is brought back to its initial orientation.) A rotation around some fixed axis by 360° cannot be continuously deformed to the trivial motion, but it can be deformed to a rotation by 360° around any other axis (in any direction). However, a rotation by 720° is deformable to the trivial one.

You may try to see some of this at home by performing a complete rotation of a box, keeping one of the vertices fixed. Let us first rotate the box around one of the edges and then try to deform this motion to the trivial one. If you follow a vertex on one of the non-fixed edges, it will trace a large circle on a sphere. Now, for any complete rotation of the box (around the same fixed vertex) the vertex we are following will have to trace some closed path on that sphere. So as you try to deform continuously the initial motion to the trivial one, the vertex you are tracking will have to trace smaller and smaller paths, starting from the large circle and ending with the constant path, which is just the initial and final point. As you do this, one of the other vertices, which was left fixed by the initial motion, will start tracing larger and larger paths approaching a large circle on a sphere. Thus in effect, trying to contract a rotation around one of the edges to the trivial one, you only managed to deform it to a rotation around a different edge. There is some intrinsic "topological obstacle" to contracting such motions. You would need a considerable imagination to see the second property—if your initial motion consists of two full rotations around some axis, it can be deformed to the trivial motion. There are a few famous "tricks" relying on this property, most notably "Dirac's belt trick" and "Feynman's plate trick." In the "belt trick" you fasten one end of the belt and rotate the other end (the buckle) by 720°. Then, without changing the orientation of the buckle, you untwist the belt, by passing it around the buckle. (See a nice animation on Greg Egan's web-page [**6**] and Java applets analyzing the "tricks" by Bob Palais [**9**].) The "plate trick" is essentially the same. You put a (full) plate onto your palm and, without moving your feet, rotate it by 720°, at the same time moving it under your armpit and then over your head. You will end up in your initial position, your arm and body untwisted.

These experiments should leave you with a few questions: Is the complete rotation around one axis really not contractible to the trivial motion? If you have two arbitrary motions that are not contractible, can you always deform one to the other? If you compose two of the latter do you always get a motion that is contractible? (The affirmative answer to the last question actually will follow from the affirmative answer to the previous one together with the "belt trick" effect.) We will describe an experiment, which could be called the "braid trick" and which will give us enough machinery to answer these questions rigorously. In the process, we exhibit an intriguing relation between three-dimensional rotations and braid groups.

Complete rotations of an object are in one-to-one correspondence with closed paths in $SO(3)$. Two closed paths in a topological space with the same initial and final point (base point) are called *homotopic* if one can be continuously deformed to the other. Since homotopy of paths is an equivalence relation, all paths fall into disjoint equivalence classes. The set of homotopy classes of closed paths becomes a group when one takes composition of paths as the multiplication and tracing a path in the opposite direction as the inverse. This group, noncommutative in general, is one of the most important topological invariants of a space and was first introduced by Poincaré. It is called the *fundamental group* or the *first homotopy group* and is denoted by π_1. Thus for the space of three-dimensional rotations the topological property discussed so far is written in short as $\pi_1(SO(3)) \cong \mathbb{Z}_2$. This means that all closed paths in $SO(3)$ starting and ending at the same point, e.g., the identity, fall into two homotopy classes—those that are homotopic to the constant path and those that are not. Composing two paths from the second class yields a path from the first class.

A topological space with a fundamental group $\mathbb{Z}_2$ is a challenge to the imagination—it is easy to visualize spaces with fundamental group $\mathbb{Z}$ (the punctured plane), or $\mathbb{Z} \star \mathbb{Z} \cdots \star \mathbb{Z}$ (plane with several punctures), or even $\mathbb{Z} \oplus \mathbb{Z}$ (torus), but there is no subspace of $\mathbb{R}^3$ whose fundamental group is $\mathbb{Z}_2$.

The peculiar structure of $SO(3)$ plays a fundamental role in our physical world. There are exactly two principally different types of elementary particles, bosons, having integer spin, and fermions, having half-integer spin, with very distinct physical properties. The difference can be traced to the fact that the quantum state of a boson is described by a (possibly multi-component) wave function, which remains unchanged when a full (360°) rotation of the coordinate system is performed, while the wave function of a fermion gets multiplied by -1 under a complete rotation. Somewhat loosely speaking, the second possibility comes from the fact that only the modulus of the wave function has a direct physical meaning. Mathematical physicists have realized long ago [**11, 2**] that the wave function has to transform properly only under the action of transformations that are in a small neighborhood of the identity. When a "large" transformation is performed on the wave function, like a rotation by 360°, it can be done by a sequence of "small" transformations, but the end point—the transformed wave function—need not coincide with the initial one. On the other hand, if you take a closed path in $SO(3)$ which remains in a small neighborhood of the identity, the transformed wave function at the end must coincide with the initial one. In fact what is important is whether the closed path is contractible to the identity or not. It is quite obvious from continuity considerations that the end-point wave function must coincide with the initial one if the path in $SO(3)$ is contractible. Thus when you do two full rotations, i.e., rotation by 720°, the wave function should come back to the initial one which implies that the transformation, corresponding to a 360°–rotation must be of order 2.

There are several standard ways of showing that $\pi_1(SO(3)) \cong \mathbb{Z}_2$. The one that is best known uses substantially Lie group and Lie algebra theory. The space $SO(3)$ can be thought of as the space of 3×3 real orthogonal matrices with determinant 1. It has the structure of a closed three-dimensional smooth manifold embedded in $\mathbb{R}^9$ (a higher-dimensional analog of a closed smooth surface embedded in $\mathbb{R}^3$). It is also a group and the group operations are smooth maps. Such spaces are called Lie groups. Another Lie group, very closely related to $SO(3)$ is $SU(2)$—the group of 2×2 complex unitary matrices with determinant 1. It is relatively easy to see that topologically $SU(2)$ is the three-dimensional sphere S^3. Locally the two groups are identical, i.e., one can find a bijection between open neighborhoods of the identities of both, which is a group isomorphism and a (topological) homeomorphism. Globally, however, this map extends to a 2–1 homomorphism $SU(2) \to SO(3)$, sending any two antipodal points on $SU(2)$ to a single point on $SO(3)$. In topological terms this map is called a double covering of $SO(3)$. The topology of $SO(3)$ can now be easily understood—it is the three-dimensional sphere S^3 with antipodal points identified.

In the present paper we describe an alternative way of "seeing" and proving that $\pi_1(SO(3)) \cong \mathbb{Z}_2$. It does not use Lie groups or even matrices. It is purely algebraic-topological in nature and very visual. It displays a simple connection between full rotations (closed paths in $SO(3)$) and braids. We believe that this is an interesting way of demonstrating a nontrivial topological result to students in introductory geometry and topology courses as well as a suitable way of sparking interest in braids and braid groups, which appear naturally in various mathematical problems, from algebraic topology through operator algebras to robotics and cryptography.

Relationships between braids and homotopy groups appear at different levels. To begin with, braid groups can be defined as the fundamental groups of certain configuration spaces. Braids have been applied (see, e.g., [**4**]) to determining homotopy groups of the sphere S^2. In this paper, we present yet another, simple connection between braid groups and a fundamental group.

The goal of this paper is mostly pedagogical—presenting in a self-contained and accessible way a set of results that are basically known to algebraic topologists and people studying braid groups. The fact that the first homotopy group of $SO(3)$ can be related to spherical braids is a special case (in disguise) of the following general statement [**7**]: "The configuration space of three points on an r-sphere is homotopically equivalent to the Stiefel manifold of orthogonal two-frames in $r+1$-dimensional Euclidean space." Fadell [**7**] considers a particular element of $\pi_1(SO(3))$ and uses the fact that it has order 2 to prove a similar statement for a corresponding braid. Our direction is the opposite—we analyze braids to deduce topological properties of $SO(3)$.

In the next section we describe a simple experiment that actually demonstrates the $\mathbb{Z}_2$ in three-dimensional rotations. Then in section 3 we give a formal treatment of that experiment. We construct a map from $\pi_1(SO(3))$ into a certain factorgroup of a subgroup of the braid group with three strands. We prove that this map is an isomorphism and that the image is $\mathbb{Z}_2$.

The braid trick

Take a ball (a tennis ball will do) and attach three strands to three different points on its surface. Attach the other ends of the strands to three different points on the surface of your desk (FIGURE 1). Perform an arbitrary number of full rotations of the ball around arbitrary axes. You will get a plaited "braid". (When you do the rotations, your strands will have to be loose enough. Still, if you are performing just rotations of the ball without translational motions, what you will get is a "braid" and not the more complicated object "tangle" in which each strand can be knotted by itself. Even though this more complicated situation can be handled easily, we prefer to avoid it.) Now keep the orientation of the ball fixed. If the total number of full rotations is even, you can always unplait the "braid" by flipping strands around the ball. If the number of rotations is odd you will never be able to unplait it, but you can always reduce it to one simple configuration, e.g., the one obtained by rotating the ball around the first point and twisting the second and third strands around each other.

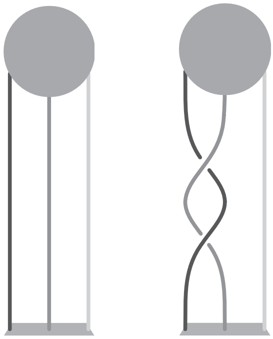

Figure 1 Rotating a ball with strands attached.

As we might expect, rotations that can be continuously deformed to the trivial rotation (i.e., no rotation) lead to trivial braiding. At this point we can only conjecture from our experiment that the fundamental group of $SO(3)$ contains $\mathbb{Z}_2$ as a factor.

Relating three-dimensional rotations to braids

With each closed path in $SO(3)$ we associate three closed paths in $\mathbb{R}^3$ starting at the sphere with radius 1 and ending at the sphere with radius $1/2$. We may think of continuously rotating a sphere from time $t = 0$ to time $t = 1$ so that the sphere ends up with the same orientation as the initial one. Simultaneously we shrink the radius of the sphere from 1 to $1/2$ (see FIGURE 2). Any three points on the sphere will trace three continuous paths in $\mathbb{R}^3$, which do not intersect each other. Furthermore, for fixed t the three points on these paths lie on the sphere with radius $1 - t/2$. To formalize things, let $\omega(t)$, $t \in [0, 1]$ be any continuous path in $SO(3)$ with $\omega(0) = \omega(1) = I$. $\omega(t)$ acts on vectors (points) in $\mathbb{R}^3$. Take three initial points in $\mathbb{R}^3$, e.g., $\mathbf{x}_0^1 = (1, 0, 0)$, $\mathbf{x}_0^2 = (-1/2, \sqrt{3}/2, 0)$, $\mathbf{x}_0^3 = (-1/2, -\sqrt{3}/2, 0)$. Define three continuous paths by

$$\mathbf{x}^i(t) := (1 - t/2)\omega(t)(\mathbf{x}_0^i), \quad t \in [0, 1], \quad i = 1, 2, 3.$$

In this way we get an object that will be called a *spherical braid*—several distinct points on a sphere and the same number of points, in the same positions, on a smaller sphere, connected by strands in such a way that the radial coordinate of each strand is monotonic in t.

Note. One can multiply two spherical braids by connecting the ends of the first to the beginnings of the second (and rescaling the parameter). When one considers classes of isotopic spherical braids one obtains the so called *braid group of the sphere* [**8**], which algebraically is B_3/R (see below). This is known as the mapping-class group of the sphere (with 0 punctures and 0 boundaries) and has been studied by topologists.

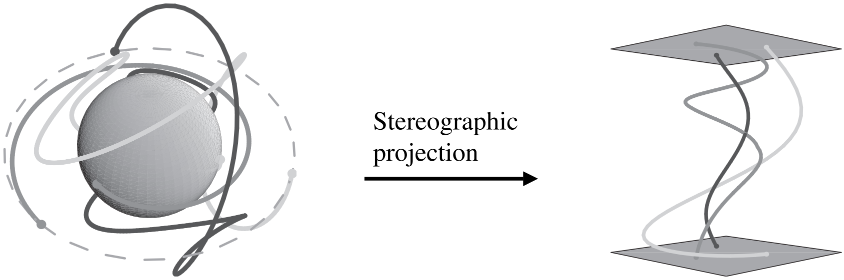

Figure 2 A "spherical braid" and a normal braid.

We can map our spherical braid to a conventional one using stereographic projection (FIGURE 2). First we choose a ray starting at the origin and not intersecting any strand. The ray intersects each sphere at a point, which we can consider as the "north pole". Then we map stereographically, with respect to its "north pole," each sphere with radius $1/2 \leq \rho \leq 1$ (minus its "north pole") to a corresponding (horizontal) plane. Finally we define the z-coordinate of the image to be $z = -\rho$.

Recall the usual notion of *braids*, introduced by Artin [**1**]. (See also [**4**] for a contemporary review of the theory of braids and its relations to other subjects.) We take two planes in $\mathbb{R}^3$, let's say parallel to the XY plane, fix n distinct points on each plane and connect each point on the lower plane with a point on the upper plane by a continuous path (strand). The strands do not intersect each other. In addition the z-coordinate of each strand is a monotonic function of the parameter of the strand and thus z can be used as a common parameter for all strands. Two different braids are considered equivalent or *isotopic* if there exists a homotopy of the strands (keeping the endpoints fixed), so that for each value of the homotopy parameter s we get a braid, for $s = 0$ we get the initial braid and for $s = 1$ the final one. When the points on the lower and the upper plane have the same positions (their x and y coordinates are the same), we can multiply braids by stacking one on top of the other. Considering classes of isotopic braids with the multiplication just defined, the *braid group* is obtained. Artin showed that the braid group B_n on n strands has a presentation with $n - 1$ generators and a simple set of relations—Artin's braid relations. We give them for the case $n = 3$ since this is the one we are mostly interested in. In this case the braid group B_3 is generated by the generators σ_1, corresponding to twisting of the first and the second strands, and σ_2, corresponding to twisting of the second and the third strands (the one to the left always passing behind the one to the right) (FIGURE 3). These generators are subject to a single braid relation (FIGURE 4):

$$\sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1 \tag{1}$$

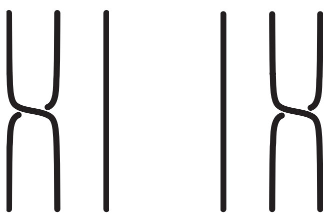

Figure 3 The generators σ_1 and σ_2 of B_3.

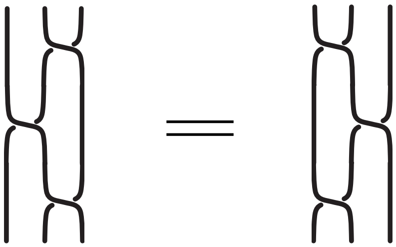

Figure 4 The braid relation for B_3.

We say that B_3 has a *presentation* with generators σ_1 and σ_2 and defining relation given by Equation 1, or in short:

$$B_3 = \langle \sigma_1, \sigma_2; \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \rangle \tag{2}$$

In our case, since a full rotation of the sphere returns the three points to their original positions, we always get *pure braids*, i.e., braids for which any strand connects a point on the lower plane with its translate on the upper plane. Pure braids form a subgroup of B_3 which is denoted by P_3. Note that intuitively there is a homomorphism π from B_3 to the symmetric group S_3 since any braid from B_3 permutes the three points. Formally we define π on the generators by

$$\pi(\sigma_1)(1, 2, 3) = (2, 1, 3), \quad \pi(\sigma_2)(1, 2, 3) = (1, 3, 2) \tag{3}$$

and then extend it to the whole group B_3 (it is important that π maps Equation 1 to the trivial identity). Pure braids are precisely those that do not permute the points and therefore we can give the following algebraic characterization of P_3:

$$P_3 := \operatorname{Ker} \pi.$$

Alternatively, S_3 is the quotient of B_3 by the additional equivalence relations $\sigma_i^2 \sim I$, $i = 1, 2$ and if N is the minimal normal subgroup containing σ_i^2, then $\pi : B_3 \to B_3/N$ is the natural projection. It is then easy to see that the kernel of π has to be a product of words of the following type:

$$\sigma_{i_1}^{\pm 1}\sigma_{i_2}^{\pm 1}\cdots\sigma_{i_k}^{\pm 1}\sigma_{i_{k+1}}^{\pm 2}\sigma_{i_k}^{\pm 1}\cdots\sigma_{i_2}^{\pm 1}\sigma_{i_1}^{\pm 1}.$$

The whole subgroup P_3 can in fact be generated by the following three *twists* (FIGURE 5)

$$a_{12} := \sigma_1^2, \quad a_{13} := \sigma_2\sigma_1^2\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^2\sigma_1, \quad a_{23} := \sigma_2^2. \tag{4}$$

In our construction so far we mapped any closed path in $SO(3)$ to a spherical braid and then, using stereographic projection, to a conventional pure braid. The last map, however, depends on a choice of a ray in $\mathbb{R}^3$ and, what is worse, spherical braids that are isotopic (in the obvious sense) may map to nonisotopic braids. To mend this, we will identify certain classes of braids in P_3. Namely, we introduce the following equivalence relations (see FIGURE 6):

$$r_1 := \sigma_1\sigma_2^2\sigma_1 \sim I, \quad r_2 := \sigma_1^2\sigma_2^2 \sim I, \quad r_3 := \sigma_2\sigma_1^2\sigma_2 \sim I. \tag{5}$$

In our model with the tennis ball the elements r_i, $i = 1, 2, 3$ correspond to *flips* of the ith strand above and around the ball. Such motions lead to isotopic spherical braids, as will be shown later. (The choice of these particular three flips given in Equation 5 is

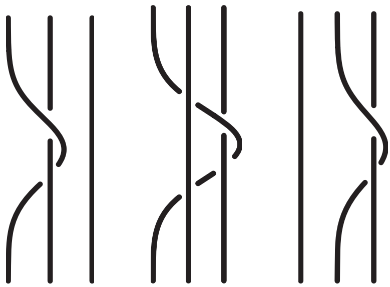

Figure 5 The generators a_{12}, a_{13}, and a_{23} of P_3.

based on the following intuition, coming from the experiment—thinking of the three strands of the trivial braid as arranged in a circle, we pull one of them out and flip it above and around the ball clockwise to obtain one of the r_i or counterclockwise to obtain its inverse. Thus in FIGURE 6 the middle strand is in the background, while the first and third are in the foreground. We do not take "more complicated" elements, like e.g., $\sigma_2^2\sigma_1^2$ which would correspond to first pulling the middle strand between the other two to the foreground and then performing the flip r_1, i.e., $\sigma_2^2\sigma_1^2$ is obtained from r_1 by conjugating it with σ_1 and its inverse.)

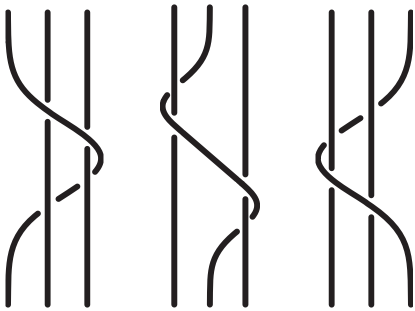

Figure 6 The flips r_1, r_2, and r_3.

Note. When any strand in any part of the spherical braid crosses the ray which we use for the stereographic projection, that projection will map the spherical braid to a different (Artin) braid, which we should consider as identical with the initial one. This means that we have to factorize by the normal closure in B_3 (not in P_3!) of the generators r_i, $i = 1, 2, 3$, i.e., the smallest normal subgroup in B_3 containing these three generators. This would then allow us to set to I any r_i (or its inverse) in any part of a word. We see easily that only one of the generators is needed then, since the other two will be contained in the normal closure of the first. We noticed experimentally, however, that we managed to untie any trivial braid just by a sequence of the three flips r_i defined in Equation 5 and their inverses, performed at the end of the braid. At the same time a nontrivial braid, corresponding to an odd number of rotations, cannot be untied even if we allow flips in any part of the braid. This can only be true if the flips r_i generate a normal subgroup in B_3 (which of course then coincides with the normal closure of any of the r_i and is also normal in P_3).

LEMMA 1. *The subgroup $R \subset P_3$, generated by r_1, r_2, r_3 is normal in B_3.*

Proof. We need to show that we can represent all conjugates of r_i with respect to the generators of B_3 and their inverses as products of the r_i and their inverses.

Straightforward calculations, using repeatedly Artin's braid relation (Equation 1) give the following identities:

$$\begin{aligned}
\sigma_1 r_1 \sigma_1^{-1} &= r_2, & \sigma_2 r_1 \sigma_2^{-1} &= \sigma_2^{-1} r_1 \sigma_2 = r_1, \\
\sigma_1 r_2 \sigma_1^{-1} &= r_2 r_1 r_2^{-1}, & \sigma_2 r_2 \sigma_2^{-1} &= r_3, \\
\sigma_1 r_3 \sigma_1^{-1} &= \sigma_1^{-1} r_3 \sigma_1 = r_3, & \sigma_2 r_3 \sigma_2^{-1} &= r_1^{-1} r_2 r_1 = r_3 r_2 r_3^{-1}, \\
\sigma_1^{-1} r_1 \sigma_1 &= r_1^{-1} r_2 r_1, & \sigma_1^{-1} r_2 \sigma_1 &= r_1, \\
\sigma_2^{-1} r_2 \sigma_2 &= r_1 r_3 r_1^{-1} = r_2^{-1} r_3 r_2, & \sigma_2^{-1} r_3 \sigma_2 &= r_2.
\end{aligned} \tag{6}$$

We demonstrate as an example the proof of the first identity in the second line. We have

$$\begin{aligned}
\sigma_1\sigma_2\sigma_1 &= \sigma_2\sigma_1\sigma_2 \\
\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1 \\
\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1 \\
\sigma_1\sigma_2^2\sigma_1\sigma_2^2 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1 \\
\sigma_1\sigma_2^2\sigma_1 &= \sigma_2^2\sigma_1\sigma_2^2\sigma_1\sigma_2^{-2} \\
\sigma_1^3\sigma_2^2\sigma_1^{-1} &= \sigma_1^2\sigma_2^2\sigma_1\sigma_2^2\sigma_1\sigma_2^{-2}\sigma_1^{-2},
\end{aligned}$$

and therefore

$$\sigma_1 r_2 \sigma_1^{-1} = \sigma_1 \cdot \sigma_1^2\sigma_2^2 \cdot \sigma_1^{-1} = \sigma_1^2\sigma_2^2 \cdot \sigma_1\sigma_2^2\sigma_1 \cdot \sigma_2^{-2}\sigma_1^{-2} = r_2 r_1 r_2^{-1}. \quad \blacksquare$$

By suitable full rotations we obtain all generators of P_3. For example, a_{12} is obtained by rotating around the vector $\mathbf{x}_0^3 = (-1/2, -\sqrt{3}/2, 0)$ and it twists the first and the second strand. Furthermore, homotopies between closed paths in $SO(3)$ correspond to isotopies of the spherical braids and thus homotopic closed paths in $SO(3)$ will be mapped to the same element in the factorgroup P_3/R. Hence we have a surjection $\pi_1(SO(3)) \to P_3/R$.

PROPOSITION 1. *The factorgroup P_3/R is isomorphic to $\mathbb{Z}_2$.*

Proof. To make notation simpler we use the same letter to denote both a representative of a class in P_3/R and the class itself, hoping that the meaning is clear from the context. In P_3/R we have

$$\sigma_1\sigma_2^2 = \sigma_1^{-1} = \sigma_2^2\sigma_1,$$

and

$$\sigma_2\sigma_1^2 = \sigma_2^{-1} = \sigma_1^2\sigma_2.$$

The following sequence of identities follow one from another:

$$\begin{aligned}
\sigma_2\sigma_1^2 &= \sigma_1^2\sigma_2, & \sigma_1\sigma_2\sigma_1^2 &= \sigma_1^3\sigma_2, \\
\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_1^3\sigma_2, & \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 &= \sigma_1^4\sigma_2, \\
\sigma_1\sigma_2^2\sigma_1\sigma_2 &= \sigma_1^4\sigma_2, & I &= \sigma_1^4.
\end{aligned}$$

We have used twice the braid relation (Equation 1) and the first equivalence relation in Equation 5. In a completely analogous way we prove

$$\sigma_2^4 = I.$$

Combining the last two results with the equivalence relations (Equation 5) we finally get

$$\sigma_1^2 = \sigma_1^{-2} = \sigma_2^2 = \sigma_2^{-2}. \tag{7}$$

It is now clear that in P_3/R the three generators, defined in Equation 4 reduce to one element of order 2. Therefore they generate $\mathbb{Z}_2$. This completes the proof. ∎

So far we have constructed a map $\pi_1(SO(3)) \to P_3/R$, which is onto by construction, and we have shown that the image is isomorphic to $\mathbb{Z}_2$. To show that this map is actually an isomorphism, we only need:

PROPOSITION 2. *The map $\pi_1(SO(3)) \to P_3/R$ is a monomorphism.*

Proof. It suffices to show that if a closed continuous path in $SO(3)$ is mapped to a braid in R, then this path is homotopic to the constant path. The proof basically reduces to the following observation — any spherical braid which is pure (the strands connect each point on the outer sphere with the same point on the inner sphere) determines a closed path in $SO(3)$. Two isotopic spherical pure braids determine homotopic closed paths in $SO(3)$. Indeed, recall that for a spherical braid we can parametrize the points on each strand with a single parameter t and that for a fixed t all three points lie on a sphere with radius $1 - t/2$. These three ordered points $\mathbf{x}^i(t)$, $i = 1, 2, 3$ give for every fixed t a nondegenerate triangle, oriented somehow in $\mathbb{R}^3$. Let $\mathbf{l}(t)$ be the vector, connecting the center of mass of the triangle with the vertex $\mathbf{x}^1(t)$, i.e., $\mathbf{l}(t) = \mathbf{x}^1 - (\mathbf{x}^1(t) + \mathbf{x}^2(t) + \mathbf{x}^3(t))/3$ and define $\mathbf{e}^1(t) := \mathbf{l}(t)/||\mathbf{l}(t)||$. Let $\mathbf{e}^3(t)$ be the unit vector, perpendicular to the plane of the triangle, in a positive direction relative to the orientation $(1, 2, 3)$ of the boundary. Finally, let $\mathbf{e}^2(t)$ be the unit vector, perpendicular to both $\mathbf{e}^1(t)$ and $\mathbf{e}^3(t)$, so that the three form a right-handed frame. Then there is a unique element $\omega(t) \in SO(3)$ sending the vectors $\mathbf{e}_0^1 = (1, 0, 0)$, $\mathbf{e}_0^2 = (0, 1, 0)$, $\mathbf{e}_0^3 = (0, 0, 1)$ to the triple $\mathbf{e}^i(t)$. According to our definitions, $\omega(0) = \omega(1) = I$ and we get a continuous function $\omega : [0, 1] \to SO(3)$, where continuity should be understood relative to some natural topology on $SO(3)$, e.g., the strong operator topology.

Recall that for any spherical braid the ith strand ($i = 1, 2, 3$) starts at the point $\mathbf{x}_0^i$ and ends at the point $\mathbf{x}_0^i/2$. If we have two isotopic spherical braids, by definition there are continuous functions $\mathbf{x}^i(t, s)$, $i = 1, 2, 3$, such that $\mathbf{x}^i(t, s)$ is a braid for any fixed $s \in [0, 1]$, $\mathbf{x}^i(0, s) = \mathbf{x}_0^i$, $\mathbf{x}^i(1, s) = \mathbf{x}_0^i/2$, $\mathbf{x}^i(t, 0)$ give the initial braid and $\mathbf{x}^i(t, 1)$ give the final braid. By assigning an element $\omega(t, s)$ to any triple $\mathbf{x}^i(t, s)$ as described, we get a homotopy between two closed paths in $SO(3)$.

Let $\omega'(t)$ be a closed path in $SO(3)$ which is mapped to a braid b in the class $r_1 \in R$. We can construct a spherical braid, whose image is isotopic to that braid. Let $\mathbf{z}$ be the point on the unit sphere with respect to which we perform the stereographic projection. This can always be chosen to be the north pole or a point very close to the north pole (in case a strand is actually crossing the axis passing through the north pole). Note that the points $\mathbf{x}_0^i$, $i = 1, 2, 3$ are on the equator. Construct a simple closed path on the unit sphere starting and ending at $\mathbf{x}_0^1$ and going around $\mathbf{z}$ in a negative direction (without crossing the equator except at the endpoints). Thus we have two continuous functions $\varphi(t)$, $\theta(t)$, $t \in [0, 1]$—the spherical (angular) coordinates describing this path. Let $\mathbf{x}^1(t)$ be the point in $\mathbb{R}^3$ whose spherical coordinates are $\rho(t) := 1 - t/2$, $\varphi(t)$, $\theta(t)$ and let $\mathbf{x}^i(t) := (1 - t/2)\mathbf{x}_0^i$, $i = 2, 3$. These three paths give the required spherical braid. It is isotopic to the trivial braid, coming from the constant path in $SO(3)$, and at the same time it is isotopic to the preimage of b under the stereographic projection. In this way we see that $\omega'(t)$ must be homotopic to the constant path. Obviously a similar argument holds with r_1 replaced by r_2 and r_3 or the inverses. Since any element in R

is a product of these generators, and since products of isotopic braids give isotopic braids, this completes the proof. ■

Further discussion, results, and generalizations

When we look at a complicated braid that has been plaited by numerous different rotations of our ball, it may seem difficult to tell whether it can be untied (by performing flips r_i) or not. Actually, there is a simple criterion to determine this. Assume that the braid is represented as some word in the Artin generators:

$$b = \sigma_1^{m_1}\sigma_2^{n_1}\sigma_1^{m_2}\sigma_2^{n_2}\dots\sigma_1^{m_k}\sigma_2^{n_k}. \tag{8}$$

Define the following invariant, called the *length* of the braid:

$$p(b) := m_1 + n_1 + m_2 + n_2 + \dots + m_k + n_k. \tag{9}$$

Note that m_i and n_i can be any integers (positive, negative or zero). We observe that the number $p(b)$ is invariant for Artin's braid, since applying the braid relation (Equation 1) inside any word does not change $p(b)$ of that word. Next, since we know that our braid is pure, it can be written as a product of the generators a_{12}, a_{13}, and a_{23} defined in Equation 4 and their inverses. Note that each of these generators has $p(b) = 2$. So we conclude that $p(b)$ is even. Now, if $p(b) = 0 \pmod 4$ this means that b is a product of even number of the generators a_{ij} (and their inverses). We saw in the proof of Proposition 1 that in P_3/R the three generators a_{ij} reduce to one element of order 2, so $p(b) = 0 \pmod 4$ implies that b is trivial in P_3/R or can be untied by performing flips. On the other hand, if $p(b) = 2 \pmod 4$, then b is a product of odd number of generators a_{ij} (and their inverses) and thus reduces to the single nontrivial element of P_3/R. In this way we have provided a (simple) algorithm solving the so-called *word problem* for P_3/R, i.e., one can decide in a finite number of steps algorithmically whether two words represent the same group element or not.

There is a more intriguing aspect of our "puzzle"—given a complicated braid which is trivial in P_3/R, can we provide a recipe for a sequence of flips r_i that will untie it? (When one experiments with the tennis ball one usually intuitively finds a sequence of flips, but can we program a computer to do it?) Mathematically the problem reduces to the following: given an element $b \in R \subset B_3$, which is written in terms of the generators of B_3, can we give an algorithm to rewrite this element in terms of the generators of R? The authors don't know the answer to this question, though it may be simple. We should point out that such questions about the braid group, its subgroups and factor-groups have sparked considerable interest, especially in connection with their possible use in cryptography (see, e.g., [**5**] for examples).

Figure 7 The full twist d in the case $n = 3$.

We can easily understand the "belt trick" or the "plate trick" using algebra. In our experiment with the ball let's perform two full rotations (full twists) around a vertical axis (FIGURE 8). A single full twist, as in FIGURE 7 leads to the braid $d := (\sigma_1\sigma_2)^3$. For two full twists, using twice Artin's braid relation, we get:

$$d^2 = (\sigma_1\sigma_2)^6 = (\sigma_2\sigma_1)^6 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1(\sigma_2\sigma_1)^3 = \sigma_2\sigma_1\sigma_1\sigma_2\sigma_1\sigma_1(\sigma_2\sigma_1)^3$$
$$= r_3\sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 = r_3\sigma_1^2\sigma_2\sigma_2\sigma_1\sigma_2^2\sigma_1 = r_3r_2r_1$$

Therefore we can unplait the braid d^2 by applying the sequence of flips $r_3^{-1}, r_2^{-1}, r_1^{-1}$ (in that order). Intuitively this is the same as flipping the whole bunch of three strands together above and around the ball. It is also obvious that it should not matter with which strand we start, so cyclic permutations of the above sequence of flips should also unplait the braid. If we look at some of the identities in Equations 6 we see indeed that $r_3r_2r_1 = r_2r_1r_3 = r_1r_3r_2$.

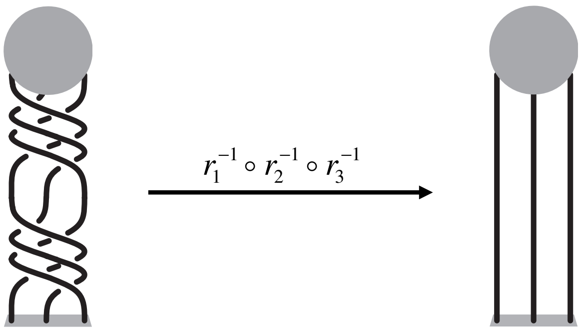

Figure 8 The "belt trick."

There is an obvious generalization of some of the results of the previous sections to the case $n > 3$. The minimal number of strands that is needed to capture the nontrivial fundamental group of $SO(3)$ is $n = 3$. When $n > 3$ any full rotation will give rise to a pure spherical braid but the whole group of pure braids will not be generated in this way. It is relatively easy to see that in this way, after projecting stereographically, we will obtain a subgroup of P_n, generated by a single *full twist d* of all strands around an external point and a set of n flips r_i:

$$d := (\sigma_1\sigma_2\cdots\sigma_{n-1})^n,$$
$$r_1 := \sigma_1\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1,$$
$$r_2 := \sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_2,$$
$$r_i := \sigma_{i-1}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_i, \quad i = 2, 3, \ldots n-1,$$
$$r_n := \sigma_{n-1}\sigma_{n-2}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{n-2}\sigma_{n-1}.$$

FIGURE 7 shows a full twist for the case with 3 strands while FIGURE 9 shows a generic flip. Straightforward calculations give the following generalization of Lemma 1:

LEMMA 1′. *The subgroup $R \subset P_n$, generated by r_i, $i = 1, \ldots n$, is normal in B_n.*

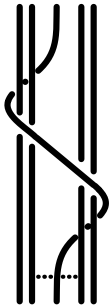

Figure 9 The flip r_i.

Proof. As in the proof of Lemma 1 we exhibit explicit formulas for the conjugates of all flips r_i:

$$\begin{aligned} \sigma_j r_i \sigma_j^{-1} &= \sigma_j^{-1} r_i \sigma_j = r_i, \quad i - j > 1 \text{ or } j - i > 0, \\ \sigma_{i-1} r_i \sigma_{i-1}^{-1} &= r_i r_{i-1} r_i^{-1}, \\ \sigma_{i-1}^{-1} r_i \sigma_{i-1} &= r_{i-1}, \\ \sigma_i r_i \sigma_i^{-1} &= r_{i+1}, \quad i \leq n-1 \\ \sigma_i^{-1} r_i \sigma_i &= r_i^{-1} r_{i+1} r_i, \quad i \leq n-1. \end{aligned}$$

■

Let us denote by S the subgroup, generated by d and r_i. Using purely topological information, namely that $\pi_1(SO(3)) \cong \mathbb{Z}_2$, we can deduce the following generalization of Proposition 1:

PROPOSITION 1′. *The factorgroup S/R is isomorphic to $\mathbb{Z}_2$.*

An equivalent statement is that $d^2 \in R$.

Given a braid with more than 3 strands it is generally not simple to determine whether or not it belongs to the group S, or in other words whether or not it can be plaited when its strands are tied together at each end, starting from the trivial braid and performing flips r_i and twists d and their inverses (to the upper end). It turns out that this question is of importance for the construction of knitting machines and has been solved explicitly in [**10**]. The braid in FIGURE 10 for example can be obtained by a sequence of flips. Since the strands in this case stay in pairs we can think of them as representing *ribbons*. You can play around with this example by taking a paper strip, cutting two slits parallel to the long sides and trying to plait the shown configuration or you can look at Bar-Natan's gallery of knotted objects [**3**] from which the example was borrowed. In fact the "braided theta" in FIGURE 10 can be obtained by perform-

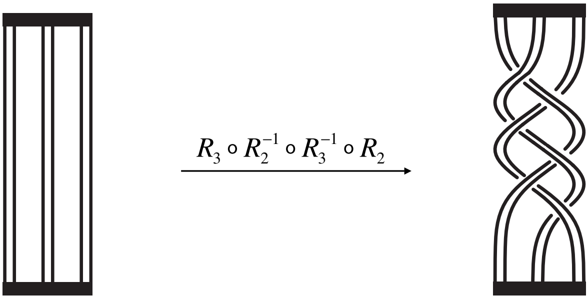

Figure 10 "Braided theta."

ing a sequence of *ribbon flips* R_1, R_2, R_3 and their inverses, which are similar to the ones in FIGURE 6 but performed on the 3 ribbons. By definition we have $R_i := r_{2i}r_{2i-1}$ and the effect of a flip R_i is similar to that of the usual flip r_i except that it twists the ith ribbon by 720° (counterclockwise). It is easier to find experimentally, rather than doing the algebra, that the "braided theta" in FIGURE 10 is the product $R_3R_2^{-1}R_3^{-1}R_2$.

If one tries to generalize the main result of this paper to higher dimensions, one would notice immediately that the isomorphism fails. On the one hand braids in higher than three-dimensional space can always be untangled. On the other hand the fundamental groups of $SO(n)$ are nontrivial. The reason for this failure is that we are able to attribute a path in $SO(3)$ to any spherical braid with 3 strands but this is not the case for $n > 3$ (4 points on S^3 may not determine an orientation of the orthonormal frame in $\mathbb{R}^4$.)

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