ARTICLES

The Lost Calculus (1637–1670): Tangency and Optimization without Limits

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If we wished to find the tangent to a given curve or the extremum of a function, we would almost certainly rely on the techniques of a calculus based on the theory of limits, and might even conclude that this is the only way to solve these problems (barring a few special cases, such as the tangent to a circle or the extremum of a parabola). It may come as a surprise, then, to discover that in the years between 1637 and 1670, very general algorithms were developed that could solve virtually every "calculus type" problem concerning algebraic functions. These algorithms were based on the theory of equations and the geometric properties of curves and, given time, might have evolved into a calculus entirely free of the limit concept. However, the work of Newton and Leibniz in the 1670s relegated these techniques to the role of misunderstood historical curiosities.

The foundations of this "lost calculus" were set down by Descartes, but the keys to unlocking its potential can be found in two algorithms developed by the Dutch mathematician Jan Hudde in the years 1657–1658. In modernized form, Hudde's results may be stated as follows: Given any polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

1. if $f(x)$ has a root of multiplicity 2 or more at $x = a$, then the polynomial that we know as f' has $f'(a) = 0$, and
2. if $f(x)$ has an extreme value at $x = a$, then $f'(a) = 0$.

These results can, of course, be easily derived through differentiation, so it is tempting to view them as results that point "clearly toward algorithms of the calculus" [**2**, p. 375]. But Hudde obtained them from purely algebraic and geometric considerations that at no point rely on the limit concept. Rather than being heralds of the calculus that was to come, Hudde's results are instead the ultimate expressions of a purely algebraic and geometric approach to solving the tangent and optimization problems.

We examine the evolution of the lost calculus from its beginnings in the work of Descartes and its subsequent development by Hudde, and end with the intriguing possibility that nearly every problem of calculus, including the problems of tangents, optimization, curvature, and quadrature, could have been solved using algorithms entirely free from the limit concept.

Descartes's method of tangents

The road to a limit-free calculus began with Descartes. In *La Géométrie* (1637), Descartes described a method for finding tangents to algebraic curves. Conceptually, Descartes's method is the following: Suppose we wish to find a circle that is tangent to the curve OC at some point C (see FIGURE 1). Consider a circle with center P on some convenient reference axis (we may think of this as the x-axis, though in practice any clearly defined line will work), and suppose this circle passes through C. This circle may pass through another nearby point E on the curve; in this case, the circle is, of course, not tangent to the curve. On the other hand if C is the only point of contact between the circle and curve, then the circle will be tangent to the curve. Thus our goal is simple: Find P so that the circle with center P and radius CP will meet the curve OC only at the point C.

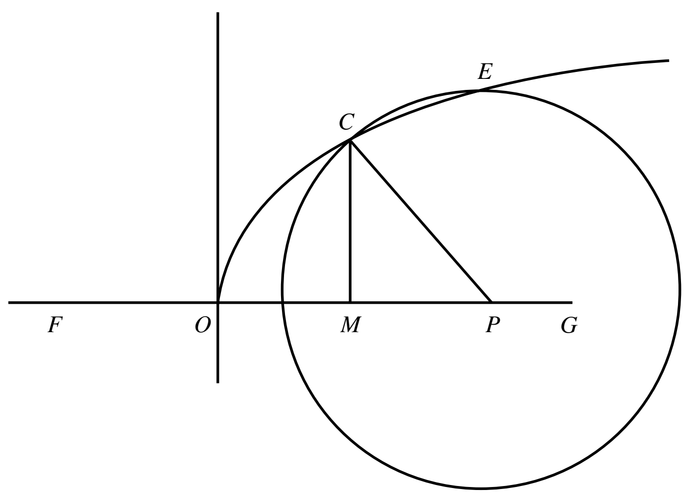

Figure 1 Descartes's method of tangents

Algebraically, any points the circle and curve have in common correspond to a solution to the system of equations representing the curve and circle. If there are two distinct intersections, this system will have two distinct solutions; thus, in order for the circle and curve to be tangent and have just a single point in common, the system of equations must have two equal solutions. In short, the system of equations must have a double root corresponding to the common point C.

Descartes, who wanted his readers to become proficient with his method through practice, never deigned to give simple examples. However, we will present a simple example of Descartes's method in modern form. Suppose OC is the curve $y = \sqrt{x}$, and let C be the point (a^2, a). Imagine a circle passing through the point (a^2, a), with radius r centered on the x-axis at the point $(h, 0)$, with h and r to be determined. Then the circle has equation

$$(x - h)^2 + y^2 = r^2.$$

Expanding and setting the equation equal to zero gives

$$y^2 + x^2 - 2hx + h^2 - r^2 = 0.$$

The points of intersection of the circle and curve correspond to the solutions to the system of equations:

$$y^2 + x^2 - 2hx + h^2 - r^2 = 0 \quad \text{and} \quad y = \sqrt{x}.$$

If we eliminate y using the substitution $y = \sqrt{x}$, we obtain

$$x^2 + (1 - 2h)x + (h^2 - r^2) = 0,$$

which is a quadratic and will in general have two solutions for x. By assumption, the circle and curve have the point (a^2, a) in common; hence $x = a^2$ is a root of this equation. In order for the circle and curve to be tangent, we want $x = a^2$ to be the only root. Thus it is necessary that

$$x^2 + (1 - 2h)x + (h^2 - r^2) = (x - a^2)^2.$$

Expanding the right-hand side and comparing coefficients we find that

$$1 - 2h = -2a^2$$

and thus $h = a^2 + 1/2$. Therefore the circle with center $(a^2 + 1/2, 0)$ will be tangent to the graph of $y = \sqrt{x}$ at the point (a^2, a).

This method of Descartes approached the problem of tangents by locating the center of the tangent circle. Today, we solve the problem by finding the slope of the tangent line. Fortunately there is a simple relationship between the two. From Euclidean geometry, we know that the radius through a point C on a circle will be perpendicular to the tangent line of the circle through C. In this case the radius PC will lie on a line with a slope $-2a$; hence the tangent line through C will have slope $1/(2a)$. This is, of course, what we would obtain using the derivative, but here we used only the algebraic properties of equations and the geometrical properties of curves.

The method in *La Géométrie* is elegant, and works very well for all quadratic forms. Unfortunately, it rapidly becomes unwieldy for all but the simplest curves. For example, suppose we wish to find the tangent to the curve $y = x^3$. As before, let the center of our circle be at $(h, 0)$; we want the system

$$x^2 - 2hx + h^2 + y^2 - r^2 = 0 \quad \text{and} \quad y = x^3$$

to have a double root at the point of tangency (a, a^3). Substituting x^3 for y gives

$$x^6 + x^2 - 2hx + h^2 - r^2 = 0. \tag{1}$$

If we wish to find the tangent at the point (a, a^3), this equation should have a double root at $x = a$; since the left-hand side is a 6th degree monic polynomial, it must factor as the product of $(x - a)^2$ and a fourth degree monic polynomial:

$$x^6 + x^2 - 2hx + h^2 - r^2 = (x - a)^2(x^4 + Bx^3 + Cx^2 + Dx + F).$$

Expanding the right-hand side gives us

$$\begin{aligned} x^6 &+ x^2 - 2hx + h^2 - r^2 \\ &= x^6 + (B - 2a)x^5 + (a^2 - 2aB + C)x^4 + (a^2B - 2aC + D)x^3 \\ &\quad + (a^2C - 2aD + F)x^2 + (a^2D - 2aF)x + a^2F. \end{aligned}$$

Comparing coefficients gives us the system

$$B - 2a = 0 \tag{2}$$

$$a^2 - 2aB + C = 0 \tag{3}$$

$$a^2B - 2aC + D = 0 \tag{4}$$

$$a^2C - 2aD + F = 1 \tag{5}$$

$$a^2D - 2aF = -2h \tag{6}$$

$$a^2F = h^2 - r^2. \tag{7}$$

From Equation 2 we have $B = 2a$. Substituting this into (3) we have

$$a^2 - 2a(2a) + C = 0;$$

hence $C = 3a^2$. Substituting the values for B and C into (4) gives us

$$a^2(2a) - 2a(3a^2) + D = 0;$$

hence $D = 4a^3$. Substituting these values into (5) gives

$$a^2(3a^2) - 2a(4a^3) + F = 1;$$

hence $F = 1 + 5a^4$. Substituting into (6) gives

$$a^2(4a^3) - 2a(1 + 5a^4) = -2h;$$

hence $h = a + 3a^5$ and the center of the tangent circle will be at $(a + 3a^5, 0)$. As before, the perpendicular to the curve will have slope $-a^3/(3a^5) = -1/(3a^2)$, and thus the slope of the line tangent to the curve $y = x^3$ at $x = a$ will be $3a^2$.

It is clear from this example that the real difficulty in applying Descartes's method is this: If $y = f(x)$, where $f(x)$ is an nth degree polynomial, then finding the tangent to the curve at the point where $x = a$ requires us to find the coefficients of $(x - a)^2$ multiplied by an arbitrary polynomial of degree $2n - 2$. The problem is not so much difficult as it is tedious, and any means of simplifying it would significantly improve its utility.

Descartes discovered one simplification shortly after the publication of *La Géométrie*. He described his modified method in a 1638 letter to Claude Hardy [**5**, vol. VII, p. 61ff]. Descartes's second method of tangents still relies on the system of equations having a double root corresponding to the point of tangency, but Descartes simplified the procedure by replacing the circle with a line and used the slope idea implicitly (as the ratio between the sides of similar triangles).

In modern terms, we describe Descartes's second method as follows: The equation of a line that touches the curve $f(x, y) = 0$ at (a, b) is $y = m(x - a) + b$, where m denotes a parameter to be determined. In order for the line to be tangent, the system of equations

$$f(x, y) = 0 \quad \text{and} \quad y = m(x - a) + b$$

must have a double root at $x = a$ (alternatively a double root at $y = b$).

For example, if we wish to find the tangent to $y = x^3$ at (a, a^3), the system of equations

$$y = x^3 \quad \text{and} \quad y = m(x - a) + a^3$$

can be reduced to

$$x^3 - mx + (ma - a^3) = 0$$

by substituting the first expression for y into the second equation. In order for the line to be tangent at $x = a$, it is necessary that $x = a$ be a double root, so $(x - a)^2$ is a factor of this polynomial; if we call the other factor $(x - r)$, we can write

$$\begin{aligned} x^3 - mx + (ma - a^3) &= (x-a)^2(x-r) \\ &= x^3 - (r+2a)x^2 + (a^2+2ar)x - a^2r. \end{aligned}$$

Comparing coefficients gives us the system

$$r + 2a = 0, \quad a^2 + 2ar = -m, \quad \text{and} \quad -a^2r = ma - a^3.$$

Solving this system gives us $m = 3a^2$. This is, of course, the same answer we would obtain by differentiating $y = x^3$, but obtained entirely without the use of limits.

Either of the two methods of Descartes will serve to find the tangent to any algebraic curve, even curves defined implicitly (since, as Descartes pointed out, an expression for y can be found from the equation of the circle or the tangent line and substituted into the equation of the curve). For example, during a dispute with Fermat over their respective method of tangents, Descartes challenged Fermat and his followers to find the tangent to a curve now known as the folium of Descartes [**5**, vol. VII, p. 11], a curve whose equation we would write as $x^3 + y^3 = pxy$.

The reader may be interested in applying Descartes's method to the folium. To find the line tangent to the folium at the point (x_0, y_0), we want the system

$$x^3 + y^3 = pxy \quad \text{and} \quad y = m(x - x_0) + y_0$$

to have a double root $x = x_0$. It should be pointed out that, contrary to Descartes's expectations, Fermat's method *could* be applied to the folium; Descartes subsequently challenged Fermat to find the point on the folium where the tangent makes a 45-degree angle with the axis (and again Fermat responded successfully).

Hudde's first letter: polynomial operations

The key to Descartes's methods is knowing when the system of equations that determine the intersection(s) of the two curves (whether a line and the curve, or a circle and the curve) has a double root, which corresponds to a point of tangency. An efficient algorithm for detecting double roots of polynomials would vastly enhance the usability of Descartes's method. Such a method was discovered by the Dutch mathematician Jan Hudde.

Hudde studied law at the University of Leiden, but while there he joined a group of Dutch mathematicians gathered by Franciscus van Schooten. At the time van Schooten, who had already published one translation of Descartes's *La Géométrie* from French into Latin, was preparing a second, more extensive edition. This edition, published in two volumes in 1659 and 1661, included not only a translation of *La Géométrie*, but explanations, elaborations, and extensions of Descartes's work by the members of the Leiden group, including van Schooten, Florimond de Beaune, Jan de Witt, Henrik van Heuraet, and Hudde.

Hudde (along with Jan de Witt) would soon leave mathematics for politics, and eventually became a high official of the city of Amsterdam. When Louis XIV invaded The Netherlands in 1672, Hudde helped direct Dutch defenses [**2, 6**]; for this, Hudde became a national hero. De Witt was less fortunate: his actions were deemed partially responsible for the ineptitude and unpreparedness of the Dutch army in the early stage of the war, and he and his brother were killed by a mob on August 20, 1672.

Hudde's return to politics may have saved The Netherlands, but mathematics lost one of its rising stars. Leibniz in particular was impressed with Hudde's work, and when Johann Bernoulli proposed the brachistochrone problem, Leibniz lamented:

> If Huygens lived and was healthy, the man would rest, except to solve your problem. Now there is no one to expect a quick solution from, except for the Marquis de l'Hôpital, your brother [Jacob Bernoulli], and Newton, and to this list we might add Hudde, the Mayor of Amsterdam, except that some time ago he put aside these pursuits [**9**, vol. II, p. 370].

As Leibniz's forecast was correct regarding the other three, one wonders what would have happened had Hudde not put aside mathematics for politics.

Hudde's work in the 1659 edition of Descartes consists of two letters. The first, "On the Reduction of Equations," was addressed to van Schooten and dated the "Ides of July, 1657" (July 15, 1657). In the usage of the time, to "reduce" an equation meant to factor it, usually as the first step in finding all its roots. Thus the letter begins with a sequence of rules (what we would call algorithms) that can be used to find potential factors of a polynomial. These factors have one of two types: those corresponding to a root of multiplicity 1, or those corresponding to a root of multiplicity greater than 1. Since Descartes's method of tangents (and Hudde's method of finding extreme values) relies on finding multiple roots, this has particular importance.

Key to Hudde's method of finding roots of multiplicity greater than 1 is the ability to find the greatest common divisor (GCD) of two polynomials. How can this be done? One way is to factor the two polynomials and see what factors they have in common. However this is impractical for any but the most trivial polynomials (and in any case requires knowing the roots we are attempting to find). A better way is to use the Euclidean algorithm for polynomials. For example, suppose we wish to find the GCD of $f(x) = x^3 - 4x^2 + 10x - 7$ and $g(x) = x^2 - 2x + 1$. To apply the Euclidean algorithm we would divide $f(x)$ by $g(x)$ to obtain a quotient (in this case, $x - 2$) and a remainder (in this case, $5x - 5$); we can express this division as

$$x^3 - 4x^2 + 10x - 7 = (x^2 - 2x + 1)(x - 2) + (5x - 5).$$

Next, we divide the old divisor, $x^2 - 2x + 1$, by the remainder $5x - 5$, to obtain a new quotient and remainder:

$$x^2 - 2x + 1 = (5x - 5)\left(\frac{1}{5}x - \frac{1}{5}\right) + 0.$$

The last nonzero remainder (in this case, $5x - 5$) corresponds to the GCD; in general, it will be a constant multiple of it.

While this is the way the Euclidean algorithm for polynomials is generally taught today, Hudde presented a clever variation worth examining. Since we are only interested in the remainder when the polynomials are divided, we can, instead of performing the division, find the remainder modulo the divisor. In particular, Hudde's steps treated the divisor as being "equal to nothing"; he set the two polynomials equal to zero and solved for the highest power term in each. In our example, we would have

$$x^2 = 2x - 1$$
$$x^3 = 4x^2 - 10x + 7.$$

The first gives us an expression for x^2 that can be used to eliminate the x^2 and higher degree terms of the other factor. Substituting and solving for the highest power remaining, we write the following sequence of steps:

$$\begin{aligned} x^3 &= 4x^2 - 10x + 7 \\ x(x^2) &= 4(2x-1) - 10x + 7 \\ x(2x-1) &= 4(2x-1) - 10x + 7 \\ 2x^2 - x &= 8x - 4 - 10x + 7 \\ 2(2x-1) - x &= -2x + 3 \\ 3x - 2 &= -2x + 3 \\ 5x &= 5 \\ x &= 1. \end{aligned}$$

Thus the second equation, $x^3 = 4x^2 - 10x + 7$, has been reduced to $x = 1$; we note that our result is equivalent to showing

$$x^3 - 4x^2 + 10x - 7 = x - 1 \bmod x^2 - 2x + 1.$$

Next, we can use $x = 1$ to eliminate all terms of the first or higher degree terms of the other equation:

$$\begin{aligned} x^2 &= 2x - 1 \\ 1^2 &= 2(1) - 1 \\ 1 &= 1 \end{aligned}$$

Since this is an identity, then the GCD is the factor corresponding to the last substitution: here, $x = 1$ corresponds to the factor $x - 1$.

The value of finding the GCD is made apparent in Hudde's tenth rule, which concerns reducing equations with two or more equal roots:

> If the proposed equation has two equal roots, multiply by whatever arithmetic progression you wish: that is, [multiply] the first term of the equation by the first term of the progression; the second term of the equation by the second term of the progression, and so forth; and set the product which results equal to 0. Then with the two equations you have found, use the previously explained method to find their greatest common divisor; and divide the original equation by the quantity as many times as possible [**4**, p. 433–4].

Hudde implies but does not state that the GCD will contain all the repeated factors; this is the first appearance of what we might call Hudde's Theorem:

HUDDE'S THEOREM. *Let* $f(x) = \sum_{k=0}^{n} a_k x^k$, *and let* $\{b_k\}_{k=0}^{n}$ *be any arithmetic progression. If* $x = r$ *is a root of* $f(x)$ *with multiplicity* 2 *or greater, then* $x = r$ *will be a root of* $g(x) = \sum_{k=0}^{n} b_k a_k x^k$.

We will refer to the polynomial $g(x)$ derived in this way from $f(x)$ as a *Hudde polynomial* (note that it is not unique). As an example, Hudde seeks to find the roots of $x^3 - 4x^2 + 5x - 2$. Using the arithmetic sequence 3, 2, 1, 0, we produce the Hudde polynomial $3x^3 - 8x^2 + 5x$ using a tabular array like this:

$$\begin{array}{cccc} x^3 - 4x^2 + 5x - 2 \\ 3 \quad\;\; 2 \quad\;\; 1 \quad\; 0 \\ \hline 3x^3 - 8x^2 + 5x \end{array}$$

The greatest common divisor of $x^3 - 4x^2 + 5x - 2$ and $3x^3 - 8x^2 + 5x$ is $x - 1$. Hence, if the original has a repeated factor, it can only be $x - 1$. Attempting to factor it out, we find $x^3 - 4x^2 + 5x - 2 = (x - 1)^2(x - 2)$. Thus the roots are 1, 1, and 2.

Hudde points out that *any* arithmetic sequence will work; indeed, he uses the same polynomial but a different arithmetic sequence:

$$\begin{array}{cccc} x^3 & -\,4x^2 & +\;5x & -\;\;2 \\ 1 & 0 & -1 & -2 \\ \hline x^3 & & -\;5x & +\;\;4 \end{array}$$

As before, the GCD of $x^3 - 5x + 4$ and $x^3 - 4x^2 + 5x - 2$ is $x - 1$.

Hudde further notes that the procedure can be repeated if there is a triple root, repeated twice if there is a quadruple root, and so on; in modern terms, we would say that if $f(x)$ has a root $x = a$ of multiplicity n, then a Hudde polynomial generated from $f(x)$ will have a root $x = a$ of multiplicity $n - 1$.

The value of the method is obvious: If the original polynomial has missing terms, the arithmetic sequence can be constructed to take advantage of these missing terms. In his example for an equation with three or more equal roots, Hudde takes advantage of this ability to choose the arithmetic sequence: Given the equation $x^4 - 6x^2 + 8x - 3 = 0$, Hudde applies his procedure to the polynomial twice, using an arithmetic sequence beginning with 0:

$$\begin{array}{ccccc} x^4 & *\,- & 6x^2\,+ & 8x\,- & 3 \\ 0 & 1 & 2 & 3 & 4 \\ \hline & & -\,12x^2\,+ & 24x\,- & 12 \\ & & 0 & 1 & 2 \\ \hline & & & 24x\,- & 24 \end{array}$$

(we follow Hudde and Descartes in the use of "$*$" to represent a missing term).

The GCD of the last polynomial, $24x - 24$ and the original polynomial $x^4 - 6x^2 + 8x - 3$ is $x - 1$; dividing the original by $x - 1$ repeatedly yields a factorization and consequently the roots: 1, 1, 1, and -3.

Hudde points out that this method can be used to solve the Cartesian tangent problem [**4**, p. 436], and solves one of Descartes's problems using his method. Let us apply Hudde's method to our earlier problem of finding the tangent to $y = x^3$. Recall that in this case we wished to find the center $(h, 0)$ of a circle that passed through the point (a, a^3). The corresponding system of equations

$$y = x^3 \quad \text{and} \quad (x - h)^2 + y^2 = r^2$$

could be reduced by substituting x^3 for y in the second equation; this results in

$$x^6 + x^2 - 2hx + h^2 - r^2 = 0.$$

In order for the circle to be tangent to the curve at (a, a^3), this equation must have a double root at $x = a$. Hence the corresponding Hudde polynomial will have a root at $x = a$. We can construct a Hudde polynomial by multiplying through by an arithmetic sequence ending in zero:

$$\begin{array}{ccccccc} x^6 & * & * & *\ + & x^2 - 2hx & & h^2 - r^2 \\ 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline 6x^6 & & & & +\,2x^2 - 2hx & & \end{array}$$

If this Hudde polynomial has a root at $x = a$, then h must satisfy

$$6a^6 + 2a^2 - 2ha = 0.$$

Hence $h = a + 3a^5$, and so the center of the tangent circle will be located at $(a + 3a^5, 0)$.

We can also apply Hudde's method to Descartes's second method of tangents (Hudde himself seemed unaware of this improved algorithm). By finding an expression for the slope m of the tangent line to a curve at a point, Hudde's methods would give us a tool equivalent to the derivative. In the case of $y = x^3$, we would want the system

$$y = x^3 \quad \text{and} \quad y = m(x - a) + a^3$$

to have a double root at $x = a$. Substituting $y = x^3$ into the second equation and setting it equal to zero gives us

$$x^3 - mx + am - a^3 = 0.$$

By assumption, $x = a$ is a double root; hence $x = a$ will be a root of any Hudde polynomial formed from this equation. For simplicity, we will form the Hudde polynomial by multiplying the kth degree term by k:

$$\begin{array}{cccc} x^3 & * & -\,mx & +\,(am - a^3) \\ 3 & 2 & 1 & 0 \\ \hline 3x^3 & & -\,mx & \end{array}$$

If $x = a$ is a root of the Hudde polynomial, then m must satisfy

$$3a^3 - ma = 0.$$

Hence $m = 3a^2$, which is precisely what the derivative of $y = x^3$ would give us. We leave it to the reader to show that all the standard rules for finding tangents to graphs of algebraic functions can be derived using Hudde's method.

Hudde's second letter: extreme values

In his first letter, Hudde also mentioned that his method could be used to find extrema, though details only appear in the second, much shorter letter dated "6 Calends of February 1658" (February 6, 1658). This letter has been translated from the original Latin into Dutch [**7**], though I am not aware of a translation into any other language.

The letter opens with a restatement of Hudde's Theorem, which he then proves. The proof is purely algebraic, rigorous by both contemporary and modern standards: hence, Hudde's methods, all based on Hudde's Theorem, neither make nor require any appeal to limits, infinitesimals, or any other ideas of calculus.

Hudde's proof, slightly modified for purpose of clarity, is the following: Suppose a polynomial $P(x)$ is the product of the third-degree polynomial $x^3 + px^2 + qx + r$ and a second-degree polynomial with a double root $x^2 - 2yx + y^2$ (whence $x = y$ is the double root). Hence the roots of $P(x)$ satisfy

$$\begin{aligned} P(x) = &\left(x^2 - 2yx + y^2\right)\, x^3 \\ &+ \left(x^2 - 2yx + y^2\right) px^2 \\ &+ \left(x^2 - 2yx + y^2\right) qx \\ &+ \left(x^2 - 2yx + y^2\right)\ \ r \quad = 0, \end{aligned}$$

where for convenience we designate the polynomial $\left(x^2 - 2yx + y^2\right)$ as the "coefficients" of the terms of the cubic polynomial.

Note that the x^2, $-2yx$, and y^2 terms of the coefficients correspond to terms of descending degree in $P(x)$; hence when a Hudde polynomial is formed from it, the coefficients will be multiplied by successive terms in the arithmetic sequence a, $a + b$, $a + 2b$, to become

$$ax^2 - (a + b)2yx + (a + 2b)y^2.$$

If $x = y$, this coefficient will be

$$ay^2 - (a + b)2y^2 + (a + 2b)y^2,$$

which is identically zero. Hence all coefficients of the Hudde polynomial will be zero when $x = y$, and thus $x = y$ will be a root of the Hudde polynomial. This proves Hudde's theorem (at least for fifth-degree polynomials; the extension of the proof to polynomials of arbitrary degree should be clear).

Geometrically, the application of Hudde's rule to finding the extreme value of a polynomial function is clear: Suppose $f(x)$ has an extremum at $x = a$, with $f(a) = Z$. Then $f(x) - Z$ will have a double root at $x = a$, and the corresponding Hudde polynomial will have a root of $x = a$.

It would seem that Hudde's method requires knowledge of the extreme value Z in order to find the extreme value. But if $f(x)$ is a polynomial function, the arithmetic sequence can be chosen so the constant term (and thus Z) is multiplied by 0 and eliminated. For example, consider the problem of finding the extreme values of $x^3 - 10x^2 - 7x + 346$. Suppose $x^3 - 10x^2 - 7x + 346 = Z$ is the extreme value, which occurs at $x = a$; then $x^3 - 10x^2 - 7x + 346 - Z$ will have a double root at $x = a$. By Hudde's Theorem, any corresponding Hudde polynomial will have a root at $x = a$. If we multiply by an arithmetic sequence ending in zero, we can eliminate the Z:

$$\begin{array}{cccc} x^3 & -\,10x^2 & -\,7x & +\,346 - Z \\ 3 & 2 & 1 & 0 \\ \hline 3x^3 & -\,20x^2 & -\,7x & \end{array}$$

(Modern readers will recognize this Hudde polynomial as $x \cdot f'(x)$.) Hudde gives no details, but presumably one would find the location of the extrema by setting the Hudde polynomial $3x^3 - 20x^2 - 7x$ equal to zero (giving an equation we will designate the Hudde equation). By assumption, $x = a$ is a double root of the original polynomial, so by Hudde's theorem, $x = a$ is a solution to $3x^3 - 20x^2 - 7x = 0$. The solutions are $x = 0$, $x = -1/3$, and $x = 7$. By assumption, at least one of the roots would have to correspond to a double root of the original for the appropriate value of Z; as with the corresponding calculus procedure, we must verify which (if any) correspond to an actual extremum. In this case, $x = -1/3$ corresponds to a local maximum, $x = 7$ to a local minimum, and $x = 0$ is extraneous.

Rational functions

Hudde also used his theorem to find extrema of rational functions, and we might compare Hudde's method with our own. Suppose we wish to find the extreme values of

$$f(x) = \frac{x^2 - 2x + 7}{x^2 + 4}.$$

Using calculus, we would find the critical points by solving $f'(x) = 0$; since

$$f'(x) = \frac{(x^2 + 4)(2x - 2) - (x^2 - 2x + 7)(2x)}{(x^2 + 4)^2},$$

then $f'(x) = 0$ if $(x^2 + 4)(2x - 2) - (x^2 - 2x + 7)(2x) = 0$; hence the solutions to this equation are the critical points.

In a like manner, Hudde used his theorem to obtain an equation whose roots correspond to the critical points of the rational function. As before, if $f(x)$ is a rational function with an extreme value Z at $x = a$, then $f(x) = Z$ will have a double root at $x = a$. To find the location of the extreme value, Hudde presents a rather more complex rule (though in fairness, it is not significantly more difficult than the quotient rule for differentiation): First, we are free to drop any constant terms (they can be subsumed into the extreme value Z). Next, break the denominator into individual terms and multiply each term of the denominator by a Hudde polynomial formed from the numerator polynomial using an arithmetic sequence whose terms are the difference between the power of the term in the numerator and the power of the term from the denominator. If the sum is set equal to zero, then a double root of the original rational expression will correspond to a root of this equation.

Using Hudde's method, we would first break the denominator of $f(x)$ into its component terms x^2 and 4. Then each term would be multiplied by a Hudde polynomial made from the numerator using an arithmetic sequence whose terms are the differences between the power of the numerator term and the power of the denominator term. Thus x^2 will be multiplied by $(2 - 2) \cdot x^2 - (1 - 2) \cdot 2x + (0 - 2) \cdot 7 = 2x - 14$, while 4 will be multiplied by $(2 - 0) \cdot x^2 - (1 - 0) \cdot 2x + (0 - 0) \cdot 7 = 2x^2 - 2x$; the Hudde polynomial will then be

$$x^2(2x - 14) + 4(2x^2 - 2x).$$

If $x = a$ corresponds to an extreme value Z, then $x = a$ will be a root of this polynomial. The reader may easily verify that the roots are $x = 0$, $x = 4$, and $x = -1$; a graph shows that $x = 0$ is extraneous, and the relative maximum occurs at $x = -1$ and the relative minimum at $x = 4$.

The method of Hudde for rational functions seems very complex, but Hudde shows how it derives naturally from the previous work. As an example, Hudde sought to find the extreme values of

$$\frac{ba^2x + a^2x^2 - bx^3 - x^4}{ba^2 + x^3} - a + x.$$

First, the constant term $-a$ can be dropped, and the expression can be rewritten as a single quotient:

$$\frac{2ba^2x + a^2x^2 - bx^3}{ba^2 + x^3}.$$

Suppose this has an extremum Z at $x = c$; we may write

$$\frac{2ba^2x + a^2x^2 - bx^3}{ba^2 + x^3} = Z \tag{8}$$

and note that, as before, $x = c$ corresponds to a double root of this equation. Multiply to convert this into a polynomial equation:

$$2ba^2x + a^2x^2 - bx^3 = Zba^2 + Zx^3.$$

In previous problems, we multiplied by an arithmetic sequence ending in zero to eliminate the term corresponding to the (unknown) extreme value Z. But here the extreme value Z appears in both the constant and third-degree terms, so how can we pick an arithmetic sequence that will eliminate it? The answer is remarkably simple: we can eliminate one of the terms including Z as before by choosing our arithmetic sequence appropriately. The remaining Zs can be eliminated using (8).

In this case, we can form a Hudde polynomial by multiplying the kth-degree term by k:

$$1 \cdot 2ba^2x + 2 \cdot a^2x^2 - 3 \cdot bx^3 = 3 \cdot Zx^3.$$

Solving this equation for Z gives us

$$Z = \frac{1 \cdot 2ba^2x + 2 \cdot a^2x^2 - 3 \cdot bx^3}{3 \cdot x^3}. \tag{9}$$

Substituting in the expression for Z from (8) yields

$$\frac{2ba^2x + a^2x^2 - bx^3}{ba^2 + x^3} = \frac{1 \cdot 2ba^2x + 2 \cdot a^2x^2 - 3 \cdot bx^3}{3 \cdot x^3}.$$

Cross-multiplying and collecting all the terms on one side gives us the equation

$$(1 \cdot 2ba^2x + 2 \cdot a^2x^2 - 3 \cdot bx^3)ba^2 + (-2 \cdot 2ba^2x + (-1) \cdot a^2x^2 - 0 \cdot bx^3)x^3 = 0,$$

which must have $x = c$ as a root.

We will prove Hudde's rule for the case of a rational function consisting of the quotient of two quadratics. Suppose

$$\frac{a + bx + cx^2}{d + ex + fx^2} = Z, \tag{10}$$

where Z is a local extremum. Clearing denominators we obtain

$$Z(d + ex + fx^2) - a - bx - cx^2 = 0.$$

One of the many possibilities for the Hudde equation is

$$Z(ex + 2fx^2) - bx - 2cx^2 = 0.$$

Solve this for Z:

$$Z = \frac{bx + 2cx^2}{ex + 2fx^2}.$$

Substituting this value of Z into (10) gives

$$\frac{a + bx + cx^2}{d + ex + fx^2} = \frac{bx + 2cx^2}{ex + 2fx^2}.$$

Cross-multiplying and rearranging the terms gives

$$\begin{aligned}&(0\cdot a + 1\cdot bx + 2\cdot cx^2)d\\&\quad + (-1\cdot a + 0\cdot bx + 1\cdot cx^2)ex\\&\quad + (-2\cdot a + -1\cdot bx + 0\cdot cx^2)fx^2 = 0,\end{aligned}$$

which gives a solvable form of the Hudde equation. In general:

HUDDE'S RULE FOR QUOTIENTS. Let $f(x) = g(x)/h(x)$, where

$$g(x) = \sum_{k=0}^{n} a_k x^k \quad \text{and} \quad h(x) = \sum_{j=0}^{m} b_j x^j,$$

and suppose $f(x)$ has an extremum Z at $x = a$. Then $x = a$ is a double root to

$$\sum_{j=0}^{m}\left(b_j x^j \sum_{k=0}^{n} a_k (k-j) x^k\right).$$

Constrained extrema

At the end of his second letter Hudde applied his method to a constrained extrema; this allows the method to be extended to functions defined implicitly, which means that all algebraic functions can be handled using his methods. For a simple example, suppose we wish to maximize the objective function xy subject to the constraint $x^3 + y^3 = 8xy$. Begin by assuming $Z = xy$ is the maximum. Solving the objective function for y we have $y = Z/x$, and substituting this into the constraint equation gives

$$x^3 + \frac{Z^3}{x^3} = 8Z.$$

Multiplying by x^3 and rearranging this gives

$$x^6 - 8Zx^3 + Z^3 = 0.$$

The corresponding Hudde equation might be

$$6x^6 - 24Zx^3 = 0.$$

Solving for Z gives $Z = x^3/4$. Since $Z = xy$, we can equate the two expressions for Z to find $y = x^2/4$. Substituting this last into the constraint equation, we get an equation in x alone:

$$x^3 + \frac{x^6}{64} = 2x^3$$

Hence $x = 4$. Since $y = x^2/4$, we also have $y = 4$.

Why does Hudde's procedure work? Consider the problem of maximizing an objective function $g(x, y)$ subject to the constraint $f(x, y) = 0$. The equation $g(x, y) = Z$ corresponds to a family of curves, and for any specific value of Z, the curve might or might not intersect the graph of $f(x, y) = 0$. The intersections, if they exist, correspond to points where the objective function has value Z. Provided f and g are sufficiently smooth, then the level curves $f(x, y) = 0$ and $g(x, y) = Z$, where Z is a maximum or minimum of $g(x, y)$, must be mutually tangent at points corresponding to extrema. Hence the corresponding system of equations must have a double root, and Hudde's procedure is applicable.

The lost calculus

Hudde's work shows that any problem involving the derivatives of algebraic functions, even those defined implicitly, could be handled using only algebra and geometry. Other developments suggested that limit-free calculus could go much farther, even to the point of being able to solve all the traditional problems of calculus (at least for algebraic functions). We will describe these developments briefly, as they are interesting enough to warrant separate treatment.

In addition to solving the tangent and optimization problems, the derivative is also used to find points of inflection. This problem may also be solved algebraically by noting that the points of inflection correspond to points where the system of equations for the tangent line and curve has a triple root; this was first pointed out by Claude Rabuel in his 1730 edition of Descartes [**3**]. We leave the application of Hudde's procedures to finding inflection points as an exercise for the reader.

Meanwhile another approach to finding tangents, developed by Apollonius of Perga and subsequently revived by John Wallis in his *Treatise on Conic Sections* (1655), would lay the groundwork for a useful link between the derivative and the integral. In modern terms, the method used by Wallis and Apollonius is the following: Suppose we wish to find the tangent to a curve $y = f(x)$ at the point $(a, f(a))$. The tangent line $y = T(x)$ may be defined as the line resting on one side of the curve; hence either $f(x) > T(x)$ for all $x \neq a$, or $f(x) < T(x)$ for all $x \neq a$. (This is generally true for curves that do not change concavity; if the curve does change concavity, we must restrict our attention to intervals where the concavity is either positive or negative.)

For example, if we wish to find the tangent to $y = x^2$ at (a, a^2), then we want the line $y = m(x - a) + a^2$ to always be below the curve; hence it is necessary that

$$m(x - a) + a^2 \leq x^2$$

for all x, with equality occurring only for $x = a$. Rearranging gives

$$m(x - a) \leq x^2 - a^2$$
$$m(x - a) \leq (x - a)(x + a).$$

First, consider the interval $x > a$; then $x - a > 0$, and we may divide both sides of the inequality to obtain the inequality $m \leq x + a$ for all $x > a$; hence $m \leq 2a$ is sufficient to guarantee the line is below the curve for $x > a$. Next, on the interval $x < a$, it is necessary that $m \geq x + a$; hence $m \geq 2a$ is sufficient to guarantee the line is below the curve for $x < a$. Thus if $m = 2a$, the line will be below the curve for all $x \neq a$; therefore, the line will lie on one side of the curve and be tangent at $x = a$.

A few years after Hudde's work Isaac Barrow proved a version of the Fundamental Theorem of Calculus using the same type of double-inequality argument used by Wallis. Barrow's proof appears in his *Geometrical Lectures* (1670, but based on lectures given in 1664–1666); we present a slightly modernized form.

Consider the curve $y = f(x)$, assumed positive and increasing, and an auxiliary curve $y = F(x)$ with the property that $F(a)$ is the area under $y = f(x)$ and above the x-axis over the interval $0 \leq x \leq a$. Now consider the line that passes through

$(a, F(a))$ and has slope $f(a)$; let the equation of this line be $T(x) = f(a)(x - a) + F(a)$. We seek to prove that this line will be tangent to the graph of $y = F(x)$.

First, take any b where $0 \le b \le a$, and note that $F(a) - F(b)$ is the area under $y = f(x)$ over the interval $b \le x \le a$. Since (by assumption) $f(x)$ is positive and increasing, this area is smaller than $f(a)(a - b)$; hence $F(a) - F(b) < f(a)(a - b)$; rearranging, we have $F(b) > f(a)(b - a) + F(a) = T(b)$. Thus for all b in $0 \le x \le a$, the line $T(x)$ lies below the curve $y = F(x)$. In a similar manner if $b > a$, then $F(b) - F(a)$ is the area below the curve over the interval $a \le x \le b$, and by assumption this area is greater than $f(a)(b - a)$; thus $F(b) - F(a) > f(a)(b - a)$ or $F(b) > f(a)(b - a) + F(a) = T(b)$, and the line $T(x)$ is again below the curve $y = F(x)$ for all $x > a$. Thus the line $T(x)$ passing through $(a, F(a))$ with slope $f(a)$ will be tangent to the curve $y = F(x)$. $T(x) = f(a)(x - a) + F(a)$ will be tangent to the curve $y = F(x)$. If we take the geometric interpretation of the integral as the area under a curve and the derivative as the slope of the tangent line, then Barrow has proven:

THE FUNDAMENTAL THEOREM OF CALCULUS. (BARROW'S VERSION) *Let $f(x)$ be positive and increasing function on $I = [0, b]$, and let $F(x) = \int_0^x f(t)\,dt$ for all x in I. Then $F'(x) = f(x)$ for all x in I.*

Together, the work of Descartes, Hudde, Wallis, and Barrow was on the verge of creating a calculus of algebraic functions that at no point required the use of limits or infinitesimal quantities. The main advantage of the limit-based calculus of Newton and Leibniz introduced in the 1670s is that it is capable of handling transcendental functions. Thus despite the lack of a theory of limits and concern over the use of infinitesimals, the calculus of Newton and Leibniz quickly supplanted the calculus of Descartes and Hudde, and the "lost calculus" vanished from the mathematical scene.

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