Some Introductory Exercises in the Manipulation of Fourier Transforms*

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1. *Introduction.* In this paper we will not be so much interested in the intrinsic properties of Fourier transforms themselves as in what we can do with them. "What formal manipulations can we carry on and what problems can we solve by using the Fourier transforms as one of our tools?" will be the questions we try to answer. It might therefore be well at the outset before even telling what a Fourier transform is, to give a few samples of problems it can solve for us. Perhaps if the problems interest the reader and their formal solutions intrigue or mystify him, he will be willing to read further and find out how those queer looking solutions were obtained.

Let us take as one sample the non-linear integral equation

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt+2f(x)=g(x); \tag{1.1}$$

in which $g(x)$ is a given function of the real variable x, and $f(x)$ is the unknown function that we wish to find. To make the problem even more specific, let us assume that the given function is

$$g(x)=\frac{4x^2+10}{\pi(x^4+5x^2+4)}\cdot$$

Now I think the reader will agree that this is a problem to which no ordinary formal methods of approach apply. Linear integral equations are bad enough, but this is not even linear since $f(x-t)$ and $f(t)$ are multiplied together. Yet it is possible by means of Fourier transforms to write down a formal answer to this problem, and then by accurate analysis justify the formal process under certain conditions. In order (we hope) to whet the reader's curiosity, we shall write out the formal solution to (1.1) immediately, withholding the explanation of how it was obtained until a later part of the paper. Here it is:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\left\{\left[1+\int_{-\infty}^{\infty} e^{iu\xi}g(\xi)d\xi\right]^{\frac{1}{2}}-1\right\}du.$$

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If we use the specific function

$$g(x)=\frac{4x^2+10}{\pi(x^4+5x^2+4)}$$

and substitute in the above formula, we find (after carrying out the indicated operations with the aid of a table of definite integrals) that

$$f(x)=\frac{1}{\pi(x^2+1)}\ .$$

Moreover we can readily verify by substituting this function in the original equation (1.1) that it is actually a solution of the equation.

Perhaps the reader might be interested at the start to see a few other equations whose solutions will be found by means of Fourier transforms. In many cases the solutions have to be left in the form of definite integrals, since these integrals cannot be evaluated finitely in the terms of elementary functions. However, even being able to express the answer in such a form is better than not being able to express it at all. For instance, we shall see that the differential equation

$$\frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY=0 \tag{1.2}$$

has as its general solution

$$Y=A\int_0^\infty e^{-t^2/2}\cos\left(tx-(t^3/3)\right)dt$$
$$+B\int_0^\infty \left[e^{-t^2/2}\sin\left(tx-(t^3/3)\right)-e^{(t^2/2)-(t^3/3)-tx}\right]dt; \tag{1.3}$$

that the integral equation

$$\rho(x)+\int_0^\infty \rho(x-t)e^{-t}dt = \frac{1}{x^2+1} \tag{1.4}$$

has the bounded solution

$$\rho(x)=\int_0^\infty \frac{(2+u^2)\cos ux-u\sin ux}{4+u^2}\,e^{-u}du$$

and that the difference-differential equation

$$\frac{d}{dx}[f(x)]+f(x)+f(x+1)=\frac{1}{x^2+1} \tag{1.5}$$

has the bounded solution

$$f(x)=\int_0^\infty \frac{\cos xs+\cos(xs-s)+s\sin xs}{2+2\cos s+2s\sin s+s^2}\,e^{-s}ds.$$

2. *The formal definition of a Fourier transform.* The average paper on Fourier transforms or their applications is apt to present a rather forbidding aspect to the casual reader. One is assumed to have rather extensive knowledge of the Lebesgue integral and its properties; and in particular one is assumed to be very much at home in the spaces L_1, L_2, and L_p. Moreover it is usually taken for granted that the reader is well acquainted with the whole (very extensive) literature of Fourier Transforms, and that he is able to fit that particular paper right into the appropriate notch. Worst of all the theorems themselves are apt to merely deepen the mystery of the subject and completely discourage the reader; for in a large number of cases the hypotheses and conclusions seem to be entirely haphazard, having no relation whatsoever to anything else in mathematics, or even to each other. Guessing the behavior of the stock market five years in advance seems a small matter in comparison to guessing what conclusions will go with what hypotheses in the theory of Fourier transforms.

Now obviously a great deal goes on under the surface; for mathematicians do not go around making altogether haphazard guesses, and then pulling out of a hat chains of logic which prove these guesses to be correct. This underlying creative thinking is almost certain to be obscured in the mass of detailed work necessary to prove the theorems rigorously; and if the author does not give a preliminary sketch or final summary of his work, his most important ideas may be lost to the reader who is not a specialist. All this is unfortunate, because many men who might advantageously use Fourier transforms as a tool in their work are discouraged and prevented from doing so.

In order to avoid this difficulty and show the simplicity of the underlying ideas of the subject, we shall in this introductory paper lay aside all ideas of mathematical rigor. We shall not be explicit as to the type of integrals used or the sense in which infinite integrals are to be interpreted. These things are to exist in some reasonable sense; and just what is "reasonable" belongs to a later phase of the subject. In this spirit we make the following definition.

Definition: *If $f(x)$ is a given function, and $i=\sqrt{-1}$, the function*

$$(2.1)\qquad F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s)ds$$

is called the Fourier transform of $f(x)$; and we shall denote this relationship in the following way:

$$f(x)\rightrightarrows F(x).$$

This definition will seem more concrete if we actually apply it to a particular function and find its Fourier transform. Let us apply it to

$$f(x)=\exp(-a^2x^2)=e^{-a^2x^2},$$

(where a is a positive number). Replacing x by the variable of integration s, we have $f(s)=\exp(-a^2s^2)$; and substituting in (2.1), we have

$$(2.2)\qquad F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{-a^2s^2}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2/4a^2)-(as-(ix/2a))^2}ds.$$

Let $t=as-(ix)/(2a)$, and substitute in (2.2) then we have*

$$F(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x^2/4a^2)-t^2}dt/a$$

$$=\frac{e^{-(x^2/4a^2)}}{a\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt=\frac{e^{-(x^2/4a^2)}}{a\sqrt{2}},$$

since the probability integral

$$\int_{-\infty}^{\infty}\exp(-t^2)dt=\sqrt{\pi}$$

(see for instance B. O. Pierce's table of integrals or Wood's *Advanced Calculus*). We have thus obtained the result:

$$e^{-a^2x^2}\rightrightarrows\frac{1}{a\sqrt{2}}\,e^{-(x^2/4a^2)},$$

*Since x goes from $-\infty$ to $+\infty$ along the real axis, t really goes from

$$-\infty-(ix)/(2a)\quad\text{to}\quad+\infty-(ix)/(2a)$$

along a line parallel to the real axis and $x/(2a)$ units below it. But it is easy to see that this line can be moved up to the real axis without changing the value of the integral.

and in particular, when $a=1/\sqrt{2}$,

$$e^{-\frac{1}{2}x^2} \rightrightarrows e^{-\frac{1}{2}x^2}.$$

Thus we see that $\exp(-\frac{1}{2}x^2)$ is its own Fourier transform; and we anticipate that it will play an important role in the theory for this reason.

3. *Formal properties of Fourier transforms.* The concept we have just defined is a useful one because of its many useful formal properties. Some of these are:

(a) *It is linear.* This means that if

$$f(x) \rightrightarrows F(x) \quad \text{and} \quad g(x) \rightrightarrows G(x),$$

then

$$f(x)+g(x) \rightrightarrows F(x)+G(x)$$

and

$$cf(x) \rightrightarrows cF(x)$$

(where c is any real or complex constant). These facts are self evident when we write them out:

$$\int_{-\infty}^{\infty} e^{isx}[f(s)+g(s)]\,ds = \int_{-\infty}^{\infty} e^{isx}f(s)\,ds + \int_{-\infty}^{\infty} e^{isx}g(s)\,ds$$

$$\int_{-\infty}^{\infty} e^{isx}[cf(s)]\,ds = c\int_{-\infty}^{\infty} e^{isx}f(s)\,ds.$$

(b) *It replaces multiplication by ix by differentiation.* Symbolically

$$ixf(x) \rightrightarrows \frac{d}{dx}F(x);$$

or written out,

$$\frac{d}{dx}\int_{-\infty}^{\infty} e^{isx}f(s)\,ds = \int_{-\infty}^{\infty} e^{isx}[isf(s)]\,ds.$$

This formula can be verified by merely carrying out the indicated differentiation; and it holds whenever differentiation under the integral sign is permissable. We shall of course not worry about such a detail at present, but will operate with this formula as though it were universally true, and check up after all formal operations are completed.

(c) *It replaces differentiation by multiplication by* $-ix$. This property is practically the same as the preceding, except that the differentiation is applied to the original function and the multiplication is applied to the transform. Symbolically

$$\frac{d}{dx}f(x) \rightrightarrows -ix\,F(x)\;; \tag{3.1}$$

or written out

$$-ix\int_{-\infty}^{\infty} e^{isx}f(s)ds=\int_{-\infty}^{\infty} e^{isx}\left[\frac{d}{ds}f(s)\right]ds. \tag{3.2}$$

To see this formal relationship, integrate the right hand member by parts. We have

$$\int_{-\infty}^{\infty} e^{isx}f'(s)ds=\Big[(e^{isx})f(s)\Big]_{s=-\infty}^{s=+\infty}-\int_{-\infty}^{\infty} ixe^{isx}f(s)ds. \tag{3.3}$$

Now if $f(s)\rightarrow 0$ as $s\rightarrow\pm\infty$, the expression in brackets drops out, and the integral which remains equals the left member of (3.2). The condition that $f(s)\rightarrow 0$ as $s\rightarrow\pm\infty$ will usually hold for the functions with which we deal; and we will regard (3.1) as a formal identity for practical manipulative purposes.

4. *A second order differential equation.* Before going any further with our study of the properties of Fourier transforms, we shall see how the second example mentioned in the introduction can be partially solved by the use of properties (a), (b), (c) alone. Let us suppose that the solution $Y(x)$ of the differential equation

$$\frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY=0$$

is the Fourier transform of a function $y(x)$; and let us see what differential equation $y(x)$ must satisfy. Then since $y(x)\rightrightarrows Y(x)$, we have by (b):

$$ix\,y(x)\rightrightarrows\frac{d}{dx}\,Y(x),$$

and

$$(ix)^2y(x)\rightrightarrows\frac{d^2}{dx^2}\,Y(x).$$

Also by (c)

$$\frac{d}{dx}\, y(x) \rightrightarrows -ix\ Y(x),$$

or

$$\frac{1}{-i}\ \frac{d}{dx}\, y(x) \rightrightarrows x\,Y(x);$$

so that we obtain finally by using (a),

$$-x^2y(x)+ix\,y(x)+i\,\frac{d}{dx}\,y(x) \rightrightarrows \frac{d^2Y}{dx^2}+\frac{dY}{dx}+xY.$$

But the transform of zero is zero, so we expect $y(x)$ to satisfy the equation

$$-x^2y+ixy+i\,\frac{dy}{dx}=0$$

or

$$\frac{dy}{dx}=-(ix^2+x)y.$$

Moreover this equation is of the first order and the variables are separable, so we may write

$$\int\frac{dy}{y}=-\int(ix^2+x)dx$$

and

$$\log y=-i\,\frac{x^3}{3}+\frac{x^2}{2}+\log c$$

and

$$y(x)=ce^{-i(x^3/3)-(x^2/2)}.$$

Thus we have solved the transformed equation and found $y(x)$; and since $y(x) \rightrightarrows Y(x)$, we have

$$Y(x)=\frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{isx}e^{-i(s^3/3)-(s^2/2)}ds.$$

We can express this answer in terms of real quantities by using the fact that $\exp(iu)=\cos u+i\sin u$, and we obtain on putting

$$A=c\sqrt{2/\pi}\ ,$$

$$Y(x)=\frac{A}{2}\int_{-\infty}^{\infty}e^{i(sx-(s^3/3))}e^{-(s^2/3)}ds$$

$$= \frac{A}{2} \int_{-\infty}^{\infty} \cos\left(sx - (s^3/3)\right) e^{-(s^2/2)} ds$$

$$+ i \frac{A}{2} \int_{-\infty}^{\infty} \sin\left(sx - (s^3/3)\right) e^{-(s^2/2)} ds$$

$$= A \int_{0}^{\infty} \cos\left(sx - (s^3/3)\right) e^{-(s^2/2)} ds. \tag{4.1}$$

In the last step, the sine integral vanishes since its positive and negative parts cancel; and the cosine integral from $-\infty$ to $+\infty$ is twice its value from 0 to $+\infty$ since the cosine is an even function.

The reader will note that we have obtained only one part of the solution (1.3) and may wonder why. The answer is that the second part is not the Fourier transform of any function; so when we assumed that $Y(x)$ was the transform of $y(x)$, we ruled out the second part. The missing part can be obtained by modifying our definition of a Fourier transform and will be discussed later in section 12. If the coefficient of dY/dx in (1.2) had been negative (say -1) instead of positive, we would have formally obtained

$$Y(x) = A \int_{0}^{\infty} \cos\left(sx - (s^3/3)\right) e^{s^2/2} ds;$$

but the exponential now becomes infinite as $s \to \pm\infty$ and the integral diverges. Thus in this case we would get neither part of the solution by the present unmodified method. However, the modification of the method given in section 12 would give both solutions in this case.

It still remains to verify that (4.1) is actually a solution of (1.2), for we have just seen that the solution need not be a Fourier transform at all. We obtained the answer by purely formal manipulation of the properties (a), (b), (c); and we have already noted that these properties depend on certain extra conditions which are not necessarily satisfied in every case. Thus we must verify first that the integral (4.1) converges and second that it satisfies the equation (1.2). We therefore note that the factor $\exp(-s^2/2)$ goes to zero so rapidly as $s \to \infty$ that the integral converges and permits all necessary manipulations; and by direct substitution in (1.2) we find that it satisfies the equation.

5. *A non-homogeneous differential equation.* If we analyze the methods used in partially solving (1.2), we find that the steps are these:

(1) We let $y(x) \rightrightarrows Y(x)$ and see what equation $y(x)$ must satisfy when $Y(x)$ satisfies a given equation. We call this equation for $y(x)$ the transformed equation.

(2) We solve the transformed equation for $y(x)$.

(3) We calculate $Y(x)$ by taking the transform of $y(x)$.

Any one of these steps may be impossible to carry out; but consider the second particularly. If the transformed equation is no simpler than the original equation, the method is useless. Now since (ix) factors go into derivatives and vice versa, the order of the new equation must equal the highest degree of the coefficients of the original equation and vice versa; so the method improves the situation only when the given differential equation has polynomial coefficients of lower degree than the order of the equation. But first order linear differential equations are the only ones we can formally solve for coefficients which are general functions of x, and it thus appears that our method is likely to be useful only when the coefficients of the given equation are linear functions of x. However, such equations

$$(5.1)\qquad (a_0+b_0x)\frac{d^nY}{dx^n}+(a_1+b_1x)\frac{d^{n-1}Y}{dx^{n-1}}+\cdots+(a_n+b_nx)y=0$$

form an important class; and our method does formally apply to this type of equation.

We might next enquire whether we could solve (5.1) if the right hand side were a function of x instead of zero. For instance, apply the method to

$$\frac{d^3Y}{dx^3}-3xY=x^3e^{-\frac{1}{2}x^2}.$$

Letting $y(x) \rightrightarrows Y(x)$, we have

$$(ix)^3y(x) \rightrightarrows \frac{d^3}{dx^3}\ Y(x)$$

$$i\frac{d}{dx}y(x) \rightrightarrows x\ Y(x) \quad \text{and}$$

$$(5.2)\qquad -ix^3y(x)-3i\,\frac{d}{dx}\,y(x) \rightrightarrows \frac{d^3Y(x)}{dx^3}\ -3x\ Y(x).$$

But the right hand side equals $x^3 \exp(-\frac{1}{2}x^2)$, so we must find what function has this as its transform in order to know what the left side equals. Thus we see that it is necessary in working with Fourier transforms to be able to work backwards and forwards. We must not only know how to get the transform of a function, but also how to find the function corresponding to a given transform.

In the present case, this causes no difficulty, for we know that

$$e^{-\frac{1}{2}x^2} \rightrightarrows e^{-\frac{1}{2}x^2},$$

so that

$$\frac{d^3}{dx^3} e^{-\frac{1}{2}x^2} \rightrightarrows (-ix)^3 e^{-\frac{1}{2}x^2} \text{ ; or}$$

$$i^3(-x^3+3x)e^{-\frac{1}{2}x^2} \rightrightarrows x^3 e^{-\frac{1}{2}x^2}. \tag{5.3}$$

Since the right members of (5.2) and (5.3) are equal, we shall assume that the left members are also equal (the validity of such an assumption will be discussed later). Thus we have

$$-ix^3 y(x) - 3i\frac{d}{dx} y(x) = i(x^3-3x)e^{-\frac{1}{2}x^2}, \quad \text{or} \tag{5.4}$$

$$\frac{d}{dx} y(x) + \frac{x^3}{3} y(x) = -\tfrac{1}{3}(x^3-3x)e^{-\frac{1}{2}x^2}. \tag{5.5}$$

Being a first order linear differential equation, this has the integrating factor

$$e^{\int (x^3/3)dx} = e^{x^4/12};$$

and multiplying (5.5) by this, we have

$$e^{x^4/12}\left\{\frac{dy}{dx} + \frac{x^3}{3}y\right\} = -\tfrac{1}{3}(x^3-3x)e^{(x^4/12)-(x^2/2)},$$

or

$$\frac{d}{dx}[ye^{x^4/12}] = -\frac{d}{dx}[e^{(x^4/12)-(x^2/2)}].$$

Integrating, we have

$$ye^{x^4/12} = -e^{(x^4/12)-(x^2/2)} + c\ ;$$

or

$$y(x) = -e^{-\frac{1}{2}x^2} + ce^{-(x^4/12)}.$$

Finally, since $y(x) \rightrightarrows Y(x)$, we obtain

$$Y(x) = -e^{-\frac{1}{2}x^2} + \frac{c}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}e^{-(s^4/12)}ds.$$

The complex integral can be reduced to real form as in the preceding problem; and other parts of the solution involving other arbitrary constants can be found by the method of section 12.

6. *Fourier's theorem.* We have seen in the last problem that it is likely to be necessary not only to calculate Fourier transforms, but also their inverses. We need to know how to find the original function when its Fourier transform is given. Fortunately, this problem has a very simple formal solution; though the underlying theory is far from simple. This leads us to our fourth property of the Fourier transformation:

(d) *When repeated, it reproduces the original function with the sign of the independent variable changed.* Stated symbolically, this says that if

$$f(x) \rightrightarrows F(x),$$

then

$$F(x) \rightrightarrows f(-x);$$

or

$$f(x) \rightrightarrows F(x) \rightrightarrows f(-x).$$

Written out, this says that if

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(s)\,ds = F(x),$$

then (formally)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} F(s)\,ds = f(x). \tag{6.1}$$

Combining the two integrals and replacing x by s and s by t in the first, the statement is that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} \left[\int_{-\infty}^{\infty} e^{its} f(t)\,dt \right] ds = f(x)$$

or

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{is(t-x)} f(t)\,dt\,ds = f(x).$$

This can be (and originally was) stated in terms of real numbers. Thus, if we put $\exp(iu) = \cos u + i \sin u$, we find that sine cancels out by symmetry while the cosine doubles up, and we have

$$\frac{1}{2\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} \cos[s(t-x)] f(t)\,dt\,ds = f(x).$$

This is Fourier's theorem, which holds for a wide class of functions, though not by any means for all functions. We shall not however try to prove it in this form; but shall go back to Fourier transforms and our symbolic notation.

If we wish to indicate that $f(x)\rightrightarrows F(x)$ and $F(x)\rightrightarrows\mathfrak{F}(x)$, we will write

$$f(x)\rightrightarrows F(x)\rightrightarrows\mathfrak{F}(x).$$

However, if we are not interested in $F(x)$ and merely wish to indicate that $\mathfrak{F}(x)$ is the double transform of $f(x)$, we shall omit the $F(x)$ and merely write

$$f(x)\rightrightarrows\rightrightarrows\mathfrak{F}(x).$$

Thus property (d) says that

$$f(x)\rightrightarrows\rightrightarrows f(-x);$$

and we shall begin by verifying this for some simple functions.

We have already found that

$$e^{-\frac{1}{2}x^2}\rightrightarrows e^{-\frac{1}{2}x^2} ;$$

and of course if we apply the transformation again, we still get the same function; so

$$e^{-\frac{1}{2}x^2}\rightrightarrows\rightrightarrows e^{-\frac{1}{2}x^2} .$$

But $\exp(-\frac{1}{2}x^2)=\exp[-\frac{1}{2}(-x)^2]$, so $\exp(-\frac{1}{2}x^2)$ has the specified property (d). Let us also verify that $x^n\exp(-\frac{1}{2}x^2)$ has this property when n is a positive integer. Applying (b) n times, we have

$$(ix)^n e^{-\frac{1}{2}x^2}\rightrightarrows\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}) ,$$

and applying (c) n times, we obtain

$$\frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2})\rightrightarrows(-ix)^n e^{-\frac{1}{2}x^2} .$$

Thus

$$(ix)^n e^{-\frac{1}{2}x^2}\rightrightarrows\rightrightarrows(-ix)^n e^{-\frac{1}{2}x^2};$$

or

$$x^n e^{-\frac{1}{2}x^2}\rightrightarrows\rightrightarrows(-x)^n e^{-\frac{1}{2}(-x)^2} ;$$

and $x^n\exp(-\frac{1}{2}x^2)$ has property (d) when n is a positive integer. But since the Fourier transformation is linear, sums of functions of this type must also have the property (d) and it follows that if $P(x)$ is any polynomial, the product

$$P(x)e^{-\frac{1}{2}x^2}=(a_0x^n+a_1x^{n-1}+\cdots+a_n)e^{-\frac{1}{2}x^2} \tag{6.2}$$

has the same property. By approximating other functions by functions of the form (6.2), it is possible to show that a large class of these other functions have property (d). In his proof of Plancherel's theorem, Wiener applies limiting processes to sequences of functions of the form (6.2) and thus shows that (d) holds for the important and extensive class of functions known as *the class* L_2. A function $f(x)$ is said to belong to the class L_2 if $f(x)$ is Lebesgue integrable between every pair of finite limits a and b and $[f(x)]^2$ is absolutely integrable from $-\infty$ to $+\infty$. Of course this includes all functions $f(x)$ which are Riemann integrable on all finite intervals and for which

$$\int_{-\infty}^{+\infty} [f(x)]^2 dx$$

converges absolutely. In particular, it includes all continuous functions $f(x)$ which approach zero at $\pm\infty$ as fast or faster than $1/x$ does. Thus the functions

$$(6.3) \qquad \frac{1}{\sqrt{x^2+1}}, \quad e^{-x^4}, \quad \frac{\sin x}{x^2+1}, \quad \text{etc.}$$

belong to L_2 and so have property (d). Another class of functions having the property (d) is the class L_1 which consists of functions that are absolutely integrable from $-\infty$ to $+\infty$ in the Lebesgue sense. Such functions may have more violent discontinuities than those of L_2, but they have to approach zero at $\pm\infty$ somewhat faster. Thus,

$$(6.4) \qquad \frac{1}{x^{2/3}\sqrt{x^2+1}}, \quad e^{-x^4}, \quad \frac{\sin x}{x^2+1}, \text{ etc.}$$

belong to L_1 and so have the property (d). However, functions of L_1 and L_2 do not have identical properties with regard to their Fourier transforms, particularly in regard to the way the definition (2.1) is to be interpreted. Moreover the transform of a function of L_2 is again a function of L_2, while the transforms of functions of L_1 need not belong to either class. The first function of (6.3) does not belong to L_1 because it goes to zero too slowly at $\pm\infty$; while the first function of (6.4) does not belong to L_2 because its discontinuity at zero is too violent; (it approaches ∞ too fast).

Returning again to formal considerations, we find that the property (d) is very useful because it enables us to calculate many new definite integrals. For instance, if we denote the transform of* $\exp(-|x|)$ by $F(x)$, we have

*The symbol $|x|$ means the absolute value (numerical value) of x, and thus $|x|=x$ when x is positive, and $|x|=-x$ when x is negative. It is never negative, and $|x|=|-x|$ for all values of x.

$$\sqrt{2\pi}F(x)=\int_{-\infty}^{\infty} e^{isx}e^{-|s|}ds = \int_{0}^{\infty} e^{isx}e^{-s}ds + \int_{-\infty}^{0} e^{isx}e^{s}ds$$

$$=\left[\frac{e^{s(ix-1)}}{ix-1}\right]_0^{\infty} + \left[\frac{e^{s}(ix+1)}{ix+1}\right]_{-\infty}^{0}$$

$$=-\frac{1}{ix-1}+\frac{1}{ix+1}=\frac{-2}{(ix-1)(ix+1)}=\frac{2}{x^2+1}\ ;$$

so

$$e^{-|x|}\rightrightarrows\sqrt{\frac{2}{\pi}}\cdot\frac{1}{x^2+1}\ .$$

Now applying rule (d), we reverse the order and have

(6.5) $$\sqrt{\frac{2}{\pi}}\ \frac{1}{x^2+1}\rightrightarrows e^{-|-x|}\ ;$$

and this when written out gives us the new integration formula

$$\int_{-\infty}^{\infty} e^{isx}\ \frac{ds}{s^2+1}=\pi e^{-|x|}\ .$$

The correctness of this formula may be checked by means of a table of definite integrals.

7. *List of the formal properties of Fourier transforms.* To facilitate further formal calculations, it seems worth while to collect the various properties of Fourier transforms into one list; and we include in this list both the formulas already obtained and those which will be obtained later:

If c is a real or complex constant, τ a real constant, and

$$f(x)\rightrightarrows F(x)\quad\text{and}\quad g(x)\rightrightarrows G(x),$$

then it follows that

(a) $$f(x)+g(x)\rightrightarrows F(x)+G(x)$$

$$c\,f(x)\rightrightarrows c\,F(x)$$

(b) $$ix\,f(x)\rightrightarrows\frac{d}{dx}\,F(x)$$

(c) $$\frac{d}{dx} f(x) \rightrightarrows -ix\,F(x)$$

(d) $$F(x) \rightrightarrows f(-x)$$

(e) If $F(x)=G(x)$, then $f(x)=g(x)$

(f) $$e^{i\tau x}f(x) \rightrightarrows F(x+\tau)$$

(g) $$f(x+\tau) \rightrightarrows e^{-i\tau x}F(x)$$

(h) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-u)g(u)du \rightrightarrows F(x)G(x)$$

(i) $$f(x)g(x) \rightrightarrows \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x-u)G(u)du.$$

8. *Uniqueness.* From property (d) there follows immediately another important property of the Fourier transformation, namely:

(e) *It is a one-to-one transformation.* This means that to each one single function $f(x)$ there corresponds only one transform $F(x)$; and conversely that each one transform $F(x)$ is the transform of only one function $f(x)$. A function cannot have two transforms (as we see from the definition, which is not multiple valued); and a single transform cannot belong to two distinct functions. The latter fact is deeper, and says that if $F(x)=G(x)$, then $f(x)=g(x)$; or in terms of integrals, if

$$\int_{-\infty}^{\infty} e^{isx}f(s)ds = \int_{-\infty}^{\infty} e^{isx}g(s)ds \quad \text{for all real } x,$$

then $$f(x)=g(x) \quad \text{for all* real } x.$$

This theorem holds for all classes of functions for which the property (d) holds, as the following argument shows. For if $f(x) \rightrightarrows F(x)$ and $g(x) \rightrightarrows G(x)$ then $F(x) \rightrightarrows f(-x)$ and $G(x) \rightrightarrows g(-x)$; so if $F(x) \equiv G(x)$, then $f(-x) \equiv g(-x)$, and $f(x)$ and $g(x)$ are identical.

This uniqueness property has actually been used before we formally stated it. Thus, in section 5 we drew the conclusion (5.4) by noting that the right numbers of (5.2) and (5.3) are equal and assuming

*Actually the conclusion is true for all x except a set of Lebesgue measure zero. Thus, if $f(x)$ and $g(x)$ were equal with the exception of one single value of x where they differed, the integrals would still be equal for all x. But sets of Lebesgue measure zero are negligible for all of the calculations in which we are interested, and we therefore consider $f(x)$ and $g(x)$ as being equivalent.

that that implied that the left members were equal. This amounts to assuming that (e) holds.

9. *Translation properties and difference equations.* Another type of equation that Fourier transforms help to solve is the type known as difference equations, in which different values of the independent variable occur in the same function. Thus,

$$f(x)+f(x+1)+f(x+2)=e^x$$

is called a difference equation; while (1.5) is called a difference-differential equation because the derivatives of $f(x)$ occurs as well as $f(x)$ and $f(x+1)$. The reason Fourier Transforms can be applied in certain difference equations is that the Fourier transformation has the following translation properties:

(f) *It replaces a multiplication by* $\exp(i\tau x)$ by a translation of τ units to the left, and

(g) *Similarly, it replaces a translation of* τ *units to the left by multiplication by* $\exp(-i\tau x)$.

If we write these statements out symbolically they read

$$e^{i\tau x}f(x) \rightrightarrows F(x+\tau)$$

$$f(x+\tau) \rightrightarrows e^{-i\tau x}F(x),$$

and in terms of integrals they read

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x+\tau)}f(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}[e^{i\tau s}f(s)]ds$$

and $$e^{-i\tau x}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s)ds = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s+\tau)ds.$$

Written in this form, the truth of the first statement is self-evident and the second is seen to be correct as soon as we put the factor $\exp(-i\tau x)$ under the integral sign in the first member and replace s by $s'-\tau$ in the second member. The statement (g) may also be obtained from (f) by the use of (d).

Now to solve (1.5), let $f(x) \rightrightarrows F(x)$, and transform the equation

$$\frac{d}{dx}f(x)+f(x)+f(x+1) = \frac{1}{x^2+1}$$

by applying the formulas listed in section 7. Thus we have

$$\frac{d}{dx}f(x) \rightrightarrows -ix\,F(x)$$

$$f(x+1) \rightrightarrows e^{-ix}F(x)$$

and

$$\frac{d}{dx}f(x)+f(x)+f(x+1) \rightrightarrows -ixF(x)+F(x)+e^{-ix}F(x).$$

But by (6.5),

$$\frac{1}{x^2+1} \rightrightarrows \sqrt{\frac{\pi}{2}}\,e^{-|x|}\quad,$$

so the transformed equation is

$$(-ix+1+e^{-ix})F(x)=\sqrt{\frac{\pi}{2}}\,e^{-|x|}\,.$$

Solving for $F(x)$, we find

$$F(x) = \sqrt{\frac{\pi}{2}}\;\frac{e^{-|x|}}{(-ix+1+e^{-ix})}\;;$$

and since by (d)

$$F(x) \rightrightarrows f(-x)\quad,$$

it follows that

$$f(-x)=\tfrac{1}{2}\int_{-\infty}^{\infty} e^{isx}\,\frac{e^{-|s|}}{(-is+1+e^{-is})}\,ds;$$

so

$$f(x)=\tfrac{1}{2}\int_{0}^{\infty}\frac{e^{-isx}e^{-s}ds}{1+e^{-is}-is}+\tfrac{1}{2}\int_{-\infty}^{0}\frac{e^{-isx}e^{s}ds}{1+e^{-is}-is}\,.$$

We next substitute $-s$ for s in the second integral and write the complex exponentials in terms of trigonometric functions:

$$f(x)=\tfrac{1}{2}\int_{0}^{\infty}\frac{(\cos sx-i\sin sx)e^{-s}ds}{1+\cos s-i\sin s-is}$$

$$+\tfrac{1}{2}\int_{0}^{\infty}\frac{(\cos sx+i\sin sx)e^{-s}ds}{1+\cos s+i\sin s+is}\,.$$

Finally, we put the two fractions together under one integral sign, adding the fractions in the usual way by first reducing to common denominator. All the imaginary terms now drop out, and we obtain the final answer

$$f(x)=\int_0^\infty \frac{[\cos sx+\cos(sx-s)+s\sin sx]}{2+2\cos s+2s\sin s+s^2}\,e^{-s}ds\ .$$

Of course we need to check this answer; for we have obtained it by purely formal manipulations based on very shaky logical foundations; and we have not attempted to justify each step by seeing that the functions are of the type to which the formulas apply. However, questionable methods of arriving at an answer do not invalidate the correctness of an answer if it actually satisfies the required equation; and it is easy to verify that the integral we have just obtained does converge and does satisfy the required equation. We of course do not claim that this is the only solution.

10. *Certain integral equations.* The integral equations that Fourier transforms help us to solve are those in which the unknown function occurs in what is called a *convolution* or *faltung* or *resultant*. The convolution $h(x)$ of two functions $f(x)$ and $g(x)$ may* be defined to be

$$h(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-s)g(s)ds\ ;$$

so that a convolution is a doubly infinite integral of the product of the two functions, the variables $x-s$ and s being substituted in the functions in place of x. Of course any other letter would do in place of s, and it does not matter in which function we substitute the s. For if $x-s=t$, we have

$$\int_{-\infty}^{\infty} f(x-s)g(s)ds=\int_{-\infty}^{\infty} f(t)g(x-t)dt\ ;$$

since the change of sign in the differential nullifies the change of sign due to the necessary interchange of limits after substitution.

Integral equations involving convolutions frequently arise in physics and other branches of applied mathematics, and it is therefore important to know how to solve them. The reason that we can solve

*The constant $1/\sqrt{(2\pi)}$ is only included in the definition for convenience. The term *convolution* does not necessarily include this constant.

them is that the Fourier transformation changes convolutions into ordinary products. The transformation has properties (h) and (i):

(h) *It replaces convolutions by products.*

(i) *It replaces products by convolutions.* Written symbolically, these statements are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-u)g(u)du \rightrightarrows F(x)G(x)$$

and

$$f(x)g(x) \rightrightarrows \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x-u)G(u)du\ ;$$

and written out in detail, they are

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\left[\int_{-\infty}^{\infty} f(s-u)g(u)du\right] ds$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{itx}f(t)dt\right]\cdot\left[\int_{-\infty}^{\infty} e^{iux}g(u)du\right]$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}f(s)g(s)ds$$

$$=\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\left\{\left[\int_{-\infty}^{\infty} e^{is(x-u)}f(s)ds\right]\cdot\left[\int_{-\infty}^{\infty} e^{itu}g(t)dt\right]\right\}du.$$

The first of these statements has a simple formal proof based on interchanging order of integration and replacing $s-u$ by t:

$$\int_{-\infty}^{\infty} e^{isx}\int_{-\infty}^{\infty} f(s-u)g(u)duds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{isx}f(s-u)g(u)duds$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ix(s-u)}f(s-u)e^{ixu}g(u)dsdu.$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixt}f(t)e^{ixu}g(u)dtdu$$

$$=\int_{-\infty}^{\infty} e^{ixu}g(u)\left[\int_{-\infty}^{\infty} e^{ixt}f(t)dt\right]du$$

$$=\left[\int_{-\infty}^{\infty} e^{ixt}f(t)dt\right]\cdot\left[\int_{-\infty}^{\infty} e^{ixu}g(u)du\right].$$

The second statement may be obtained by combining the first statement with (d).

It is now possible to solve the first problem mentioned in the introduction:

$$\int_{-\infty}^{\infty} f(x-t)f(t)dt+2f(x)=g(x)\ ;$$

for we let, $f(x)\rightrightarrows F(x)$ and $g(x)\rightrightarrows G(x)$,

and have $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-t)f(t)dt\rightrightarrows F(x)\cdot F(x)$$

so that $$\int_{-\infty}^{\infty} f(x-t)f(t)dt+2f(x)\rightrightarrows\sqrt{2\pi}[F(x)]^2+2F(x)$$

and $$\sqrt{2\pi}[F(x)]^2+2F(x)=G(x).$$

But this transformed equation is an ordinary quadratic equation in $F(x)$, and we can solve it by the quadratic formula, obtaining

$$F(x)=\frac{-2\pm\sqrt{4+4\sqrt{2\pi}\,G(x)}}{2\sqrt{2\pi}}$$

$$=\frac{1}{\sqrt{2\pi}}\left(-1\pm\sqrt{1+\sqrt{2\pi}\,G(x)}\right)$$

$$=\frac{1}{\sqrt{2\pi}}\left[-1\pm\sqrt{1+\int_{-\infty}^{\infty} e^{isx}g(s)ds}\right].$$

Transforming back by the inverse Fourier transformation (6.1), we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left[-1\pm\sqrt{1+\int_{-\infty}^{\infty} e^{ist}g(s)ds}\right]dt.$$

Though this is formally two solutions, the formal statement is very deceptive in this regard. It may, as a matter of fact, represent infinitely many different solutions, because we may choose the signs one way for some values of t and the other way for other values of t. However, there is one important case in which there cannot be more than one solution. This is the case in which $g(x)$ is of class L_1 and its Fourier transform is nowhere equal to $-1/\sqrt{(2\pi)}$ and we are seeking for solutions $f(x)$ of class L_1. In this case it can be shown that there is not more than one solution of class L_1. Moreover there is a simple rule for determining whether there is one or no solution. This consists of tracing out the values taken on by the Fourier transform of $g(x)$ as x varies continuously from $-\infty$ to $+\infty$. These values will trace a continuous curve in the complex plane, beginning and ending at zero. If this curve winds an even number of times around the point $-1/\sqrt{(2\pi)}$, there will be a solution of class L_1; but if it winds an odd number of times around $-1/\sqrt{(2\pi)}$, there will be no solution of class L_1. The reason is roughly that the square root must assume the value $+1$ at both $-\infty$ and $+\infty$ if the outside integral is to converge; and this can only happen if the expression under the radical winds an even number of times around the origin. In particular, if

$$g(x)=\frac{4x^2+10}{\pi(x^4+5x^2+4)} ,$$

we obtain

$$f(x)=\frac{1}{\pi(x^2+1)}$$

as the unique solution of class L_1

11. *A linear integral equation.* While dealing with integral equations it seems worth while to take up the more ordinary case of the linear integral equation. We shall take such a case for our last illustrative example. Let us therefore consider the equation given in (1.4), namely

$$\rho(x)+\int_0^\infty \rho(x-t)e^{-t}dt=\frac{1}{x^2+1} .$$

The integral in this equation does not appear to be a convolution because it is only taken from 0 to $+\infty$ instead of from $-\infty$ to $+\infty$. However, we can replace the lower limit by $-\infty$ if we replace $\exp(-t)$

by a function $g(t)$ which equals $\exp(-t)$ whenever t is positive but equals zero when t is negative. For then we have

$$\int_{-\infty}^{\infty} \rho(x-t)g(t)dt = \int_{-\infty}^{0} \rho(x-t)g(t)dt + \int_{0}^{\infty} \rho(x-t)g(t)dt$$

$$= \int_{-\infty}^{0} \rho(x-t)\cdot 0\cdot dt + \int_{0}^{\infty} \rho(x-t)e^{-t}dt = \int_{0}^{\infty} \rho(x-t)e^{-t}dt \,;$$

and the equation becomes

$$\rho(x)+\int_{-\infty}^{\infty} \rho(x-t)g(t)dt = \frac{1}{x^2+1} \,.$$

Now if $\rho(x)\rightleftarrows\Phi(x)$ and $g(x)\rightleftarrows G(x)$, it follows that

$$\rho(x)+\int_{-\infty}^{\infty} \rho(x-t)g(t)dt \rightleftarrows \Phi(x)+\sqrt{2\pi}\Phi(x)G(x) \,;$$

and since
$$\frac{1}{x^2+1} \rightleftarrows \sqrt{\frac{\pi}{2}}\, e^{-|x|} \,,$$

we obtain the transformed equation

$$\Phi(x)[1+\sqrt{2\pi}G(x)] = \sqrt{\frac{\pi}{2}}\, e^{-|x|} \,.$$

But
$$G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}g(s)ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{isx}\cdot o\cdot ds + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{isx}e^{-s}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{s(ix-1)}ds = \frac{1}{\sqrt{2\pi}} \quad \frac{1}{1-ix}$$

and hence
$$\Phi(x)\left[1+\frac{1}{1-ix}\right] = \sqrt{\frac{\pi}{2}}\, e^{-|x|}$$

and
$$\Phi(x) = \sqrt{\frac{\pi}{2}}\left(\frac{1-ix}{2-ix}\right) e^{-|x|} \,.$$

Transforming back, we have

$$\rho(x)=\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-isx}\,\frac{(1-is)}{(2-is)}\,e^{-|s|}ds\ ;$$

and after the usual simplification, we obtain the answer given in the introduction. Substitution shows that this is correct.

12. *Integrals allied to the Fourier transform.* Let us consider a function $f(x)$ defined on the interval from 0 to $+\infty$; and from this function let us construct four functions, all of which are defined on the whole interval from $-\infty$ to $+\infty$, as follows:

$$f_c(x)=\begin{cases} f(x) & \text{when } x \text{ is positive}\\ f(-x) & \text{" " " negative}\end{cases}$$

$$f_s(x)=\begin{cases} -if(x) & \text{" " " positive}\\ if(-x) & \text{" " " negative}\end{cases}$$

$$f_l(x)=\begin{cases} \sqrt{2\pi}\,f(x) & \text{" " " positive}\\ 0 & \text{" " " negative}\end{cases}$$

$$f_m(x)=\sqrt{2\pi}\,f(e^x)\quad \text{for all real } x.$$

Then if $F_c(x)$, $F_e(x)$, $F_s(x)$, $F_m(x)$ are the Fourier transforms of $f_c(x)$, $f_s(x)$, $f_e(x)$, $f_m(x)$, we obtain by formal substitution in the definition (2.1) and formal simplification the integrals

$$F_c(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(s)\cos sx\,ds$$

$$F_s(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(s)\sin sx\,ds$$

$$F_l(ix)=\int_0^{\infty} f(s)e^{-sx}ds$$

$$F_m(ix)=\int_0^{\infty} f(t)t^{x-1}dt\ .$$

These four important integrals are known as the Fourier cosine transform, the Fourier sine transform, the Laplace transform, and the Mellin transform respectively of $f(x)$. They have properties somewhat similar to those of the Fourier transforms, yet differing from them in many important points. For a discussion of these properties, the reader should consult Titchmarsh's *Introduction to the Theory of Fourier Integrals.*

A different type of modification of the Fourier transform is obtained when we deal with functions of a complex variable and use some path of integration other than the real axis. Thus the modified Fourier transform is

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_C e^{isx} f(s) ds \tag{12.1}$$

where the contour C is chosen so that as many as possible of the formal properties of Fourier transforms still hold. In particular, C should either be a closed curve or a curve which goes to infinity in some direction at both ends. This is necessary to preserve property (c); for if there are finite end points, the values at these end points will have to be substituted in the UV term of the integration by parts, and an extra term will crop up and spoil property (c). If the contour is infinite, it must of course be chosen so that $f(s)$ approaches zero as we approach infinity.

The contour integral (12.1) satisfies properties (a), (b), and (c); and since these were the only properties used in section 4 in solving (1.2), this integral could have been used there instead of the integral going from $-\infty$ to $+\infty$. But the only place in which the integral itself was used was in the last step, where we obtain $Y(x)$ from $y(x)$. Thus the only difference that the use of (12.1) could make would be that the final integral would be taken over C instead from $-\infty$ to $+\infty$. But such an answer would have just as much formal justification as the one we actually obtained, and this leads us to wonder whether every contour would produce the same answer or would at least produce a solution to the problem. As a matter of fact, every contour along which the integral converges properly does give us a solution to the equation; but not necessarily the same solution; and this enables us to complete the solution of the problem. You remember that we obtained only part of the general solution and mentioned that a method would later be given by which we could obtain the rest. That method is merely to obtain different parts of the solution by varying the contour used, and then to add these parts together to form the general solution. Since the equation with which we are dealing is linear and has its right hand member zero, the sum of two solutions is again a solution, and this process is valid.

In the present case, our solution is to be a multiple of

$$\int_C e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds,$$

where C is to be chosen so as to approach infinity in some other way than positively and negatively along the real axis. We wish the integrand to approach zero, and this means that the real part of the exponent

$$i(sx-(s^3/3))-(s^2/2)$$

must approach $-\infty$. Now for numerically large s, the numerically largest term is $-is^3/3$; and this will be real if s is pure imaginary. If $s=it$, then $-is^3/3=-t^3/3$, and it approaches $-\infty$ as s approaches infinity. Thus we can let one end of C go out along the upper part of the imaginary axis; and of course the other end can go out in either of the directions used before. Let us therefore take C as a contour starting at $i\infty$ and coming down the imaginary axis to zero, and then turning right and going out along the real axis to $+\infty$. Using this contour, we have

$$\begin{aligned}
&-Bi\int_C e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds \\
&= -Bi\int_\infty^0 e^{i(itx+(it^3/3))}e^{t^2/2}i\,dt - Bi\int_0^\infty e^{i(sx-(s^3/3))}e^{-(s^2/2)}ds \\
&= -B\int_0^\infty e^{-(t^3/3)+(t^2/2)-tx}dt \\
&\quad -Bi\int_0^\infty [\cos(sx-(s^3/3))+i\sin(sx-(s^3/3))]e^{-(s^2/2)}ds.
\end{aligned}$$

We may as well drop the cosine term of this solution, as it is just like the part already obtained and is therefore itself a solution and may be included with the first part by a change of the constant A. We therefore have as the second part of our solution

$$B\int_0^\infty [e^{-(s^2/2)}\sin(sx-(s^3/3))-e^{-(s^3/3)+(s^2/2)-sx}]\,ds;$$

and the problem is completely solved. The convergence of this second integral and the fact that it is a solution of the differential equation can be directly verified; and it can also be shown that neither part of the solution is identically zero or a constant multiple of the other part.

13. *Conclusion and warning.* One frequently hears the statement that a little knowledge of medicine is a dangerous thing. A similar statement might well be made in regard to certain branches of mathematics; particularly Fourier transforms. A mere formal knowledge of Fourier transforms will lead the manipulator to all sort of false conclusions. The situation here is much worse than in elementary calculus where lack of mathematical rigor and optimistically formal use of limit theorems *may* lead to false conclusions, but usually do not. Here purely formal work is sure to lead one into difficulties, and rather soon at that. You see, in this work there is really nothing that can be called "formally correct", because the formal rules are not even self consistent unless we put strong restrictions on the functions; and when we begin to state these restrictions there is no half way about it. We have to go all the way and do exact, rigorous mathematics.

As we stated in the introduction, our purpose here is merely to give the reader a general idea of the way a Fourier transform ordinarily behaves when suitably restricted. We hope that the reader may become interested in Fourier transforms as he sees the sort of thing that can be done with them, and that he may be willing to take time to learn the details of their exact behavior after he has had this little non-technical glance at the way they act when the machinery is well oiled with sufficiently powerful hypotheses. If the reader would like to gain a real understanding of the subject, he should study one of the standard works such as Titchmarsh's *Theory of Fourier Integrals,* Carslaw's *Fourier Series and Integrals,* Bochner's *Vorlesungen über Fouriersche Integrale,* Wiener's *The Fourier Integral,* and Paley and Wiener's *Fourier Transforms in the Complex Domain.*