in the field and passing through two points such that the time of motion between the two points is a minimum. His work led to 3-dimensional equations which must be simultaneously satisfied, similar to the 2-dimensional Euler's differential equation, Weierstrass's minimizing arc condition and Legendre's.

H. A. ROBINSON, *Secretary*

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# THE CONVERGENCE OF FOURIER SERIES[1]

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1. *Introduction.* Considerable parts of the theory of Fourier series have an interest, both for their mathematical content and by reason of the importance of their applications, for students whose experience of mathematics in general is only moderately advanced. The writer has had occasion more than once to give an introductory course on Fourier series and related topics for classes whose mathematical preparation was not assumed to extend beyond a first course in the calculus. The question has arisen each time how far it is possible to go beyond merely formal relationships, and to give such a class a genuine appreciation of some of the properties of convergence, even the most elementary of which are so characteristic of the type of series in question and have had so profound an influence on the course of modern mathematical development. This paper is an outline of the writer's most recent attempt in that direction. No part of the treatment is new, and most parts have been used long since for purposes of elementary exposition. The object of this account is merely to suggest one way of cutting the whole picture into pieces of convenient size, and arranging them in order so that at any stage the next piece is not far to seek.[2]

Specifically, the essential technicalities are brought within easy reach by the following devices:

(a) The main convergence proof is made to depend on nothing more abstruse than the fact that the general term of a convergent series approaches zero.

(b) Certain theorems relating to an arbitrary continuous function are made to depend on acceptance of the proposition that the graph of such a function can be approximated with any desired accuracy by a broken line. Elsewhere a reader unfamiliar with the precise definition of the word "continuous" may take it as self-explanatory, as far as an understanding of the main features of the argument is concerned.

(c) The phrase "uniformly convergent" is introduced as naturally descriptive of a type of convergence already characterized in quantitative terms.

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[1] Presented to the Mathematical Association of America at New Orleans, December 31, 1931.

[2] Among the numerous more or less extensive presentations of the theory of Fourier series the following may be specially mentioned in connection with the present paper: M. Bôcher, *Introduction to the Theory of Fourier's Series*, Annals of Mathematics, (2), vol. 7 (1905–06), pp. 81–152; H. Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906.

Such facts as the integrability of a continuous function, or of a function which is continuous except for a finite number of finite jumps, will be considered "obvious" on the basis of the interpretations that are customary in a first study of the calculus. An attempt is made throughout to give an exposition to which more critical study may have something to add, but from which it will have nothing to retract. In actual presentation to a class the less familiar ideas are naturally explained and illustrated at greater length than in the text.

2. *Formulas for the coefficients.* The Fourier series for a given function has the form

$$\begin{aligned} a_0/2 + a_1 \cos x + a_2 \cos 2x + \cdots \\ + b_1 \sin x + b_2 \sin 2x + \cdots , \end{aligned} \tag{1}$$

in which the coefficients are given by the formulas

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos kt\, dt, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \sin kt\, dt. \tag{2}$$

These are found by setting $f(t)$ equal to the series (1), written in terms of the variable $t$, multiplying by $\cos kt$ or $\sin kt$, and integrating term by term, with the use of the relations

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos jt \cos kt\, dt = 0, \qquad \int_{-\pi}^{\pi} \sin jt \sin kt\, dt = 0 && (j \neq k), \\ &\int_{-\pi}^{\pi} \sin jt \cos kt\, dt = 0 && (\text{for all } j \text{ and } k), \\ &\int_{-\pi}^{\pi} \cos^2 kt\, dt = \int_{-\pi}^{\pi} \sin^2 kt\, dt = \pi && (k \geqq 1). \end{aligned} \tag{3}$$

The representation of the constant term by $a_0/2$ rather than $a_0$ is an artifice to make the general formula for $a_k$ applicable without change when $k=0$.

If the calculation leading to the equations (2) were to be regarded as a "proof," some inquiry would be necessary as to the validity of the processes involved. In the discussion to be presented here the formulas (2), which are suggested by the calculation as at least presumptively important, and which have a meaning whenever $f(t)$ is an integrable function, will be taken outright as starting point. A series (1) will be written down with these coefficients, and it will be inquired whether the series does in fact converge and represent $f(x)$. It will be assumed throughout that $f(x)$ is periodic, with period $2\pi$.

3. *Order of magnitude of the coefficients in the case of certain continuous functions.* If $f(t)$ has a continuous derivative, integration by parts gives

$$\pi a_k = \left[\frac{1}{k} f(t) \sin kt\right]_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi} f'(t) \sin kt\, dt = -\frac{1}{k}\int_{-\pi}^{\pi} f'(t) \sin kt\, dt.$$

If there is a continuous second derivative,

$$\int_{-\pi}^{\pi} f'(t) \sin kt\, dt = \left[ -\frac{1}{k} f'(t) \cos kt \right]_{-\pi}^{\pi} + \frac{1}{k} \int_{-\pi}^{\pi} f''(t) \cos kt\, dt$$

$$= \frac{1}{k} \int_{-\pi}^{\pi} f''(t) \cos kt\, dt.$$

Let $M$ be the maximum of $|f''(t)|$. Then

$$|f''(t) \cos kt| \leqq M, \qquad \left| \int_{-\pi}^{\pi} f''(t) \cos kt\, dt \right| \leqq 2M\pi,$$

and $|a_k| \leqq 2M/k^2$. Similarly $|b_k| \leqq 2M/k^2$. It follows that

$$|a_k \cos kx + b_k \sin kx| \leqq 4M/k^2,$$

and as $1/k^2$ is the general term of a well known convergent series, the series (1) is also certainly convergent.

The same conclusion can be reached with a somewhat less restrictive hypothesis on $f(t)$, and this will be important for an application later. Let $f(t)$ still be continuous everywhere, and let it be supposed that any period interval can be divided into a finite number of subintervals throughout each of which $f(t)$ has continuous first and second derivatives, but that the derivatives may not be continuous in passing from one subinterval to the next. The graph of $f(t)$ over a period is then made up of a finite number of pieces, each having continuous curvature, but there may be corners (or, as an admissible alternative, abrupt changes of curvature without change of direction) at the points where two pieces come together. Let the successive points of division marking the subintervals of the period from $-\pi$ to $\pi$ be $x_1, x_2, \cdots, x_{p-1}$, and for uniformity of notation let $x_0 = -\pi$, $x_p = \pi$. The derivatives may have different values from the right and from the left at these points, but the function $f(t)$ itself has a determinate value at each of them. For each value of $i$ from 0 to $p-1$,

$$\int_{x_i}^{x_{i+1}} f(t) \cos kt\, dt = \left[ \frac{1}{k} f(t) \sin kt \right]_{x_i}^{x_{i+1}} - \frac{1}{k} \int_{x_i}^{x_{i+1}} f'(t) \sin kt\, dt.$$

When equations of this form are written for all $p$ subintervals and added, the terms $(1/k)f(x_i) \sin kx_i$ cancel, each occurring once with a plus and once with a minus sign, and

$$\int_{-\pi}^{\pi} f(t) \cos kt\, dt = -\frac{1}{k} \sum_{i=0}^{p-1} \int_{x_i}^{x_{i+1}} f'(t) \sin kt\, dt.$$

Another integration by parts gives

$$\int_{x_i}^{x_{i+1}} f'(t) \sin kt\, dt = \left[ -\frac{1}{k} f'(t) \cos kt \right]_{x_i}^{x_{i+1}} + \frac{1}{k} \int_{x_i}^{x_{i+1}} f''(t) \cos kt\, dt.$$

When these expressions are added for the various intervals the terms outside the signs of integration do not cancel, since $f'(x_i)$ at the left-hand end of one interval does not in general mean the same thing as $f'(x_i)$ at the right-hand end of the preceding interval. But under the hypotheses $f'(t)$ and $f''(t)$ remain finite everywhere, in spite of their discontinuities; if $M$ and $M_1$ are numbers such that $|f''(t)| \leqq M$, $|f'(t)| \leqq M_1$, for all values of $t$, then

$$|f'(x_{i+1}) \cos kx_{i+1} - f'(x_i) \cos kx_i| \leqq 2M_1$$

in each case, and

$$\left| \int_{x_i}^{x_{i+1}} f''(t) \cos kt\, dt \right| \leqq M(x_{i+1} - x_i).$$

Hence

$$|\pi a_k| = \left| \int_{-\pi}^{\pi} f(t) \cos kt\, dt \right| \leqq \frac{2pM_1}{k^2} + \frac{2\pi M}{k^2}.$$

A similar calculation applies to $b_k$. With a readily intelligible abbreviation of the hypothesis, the result may be stated as follows:

THEOREM I. *If $f(x)$ is a function which has a continuous second derivative except for a finite number of corners in a period, and if $a_k$, $b_k$ are the coefficients in its Fourier series, there is a number $C$, independent of $k$, such that*

$$|a_k| \leqq \frac{C}{k^2}, \qquad |b_k| \leqq \frac{C}{k^2}.$$

The convergence of the series is an immediate corollary. The special importance of the generalized hypothesis is that it applies in particular to a function whose graph over any period is made up of a finite number of straight line segments of finite slope joined end to end, or, as it may be described for brevity, *a function whose graph is a broken line.*

It must be recognized however that the series has not yet been proved to converge *to the value $f(x)$*. The preceding convergence proof would apply equally well, as far as it goes, to a series of cosines alone, without sine terms; but a cosine series is not adequate for the representation of an arbitrary periodic function. From the point of view of completeness of demonstration the question still remains whether the cosines and sines together are sufficient in all cases, or whether still other terms may sometimes be needed. An answer to this question will be found later, after some further preliminaries.

4. *Approach of the coefficients to zero in general* (*Riemann's Theorem*). Let $f(x)$ now be any function of period $2\pi$ which (with sufficient generality for the purposes of this paper) is continuous except for a finite number of finite jumps in a period. Let $a_k$, $b_k$ be its Fourier coefficients (2), and let $S_n(x)$ be the partial sum of the series (1) through terms of the $n$th order:

$$S_n(x) \equiv a_0/2 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx. \tag{4}$$

By the use of the relations (3) it is seen that

$$\int_{-\pi}^{\pi} S_n(t) \cos kt\, dt = \pi a_k = \int_{-\pi}^{\pi} f(t) \cos kt\, dt,$$

$$\int_{-\pi}^{\pi} S_n(t) \sin kt\, dt = \pi b_k = \int_{-\pi}^{\pi} f(t) \sin kt\, dt$$

for values of $k \leqq n$. If these equations, read from right to left, are multiplied by $a_0/2, a_1, \cdots, a_n, b_1, \cdots, b_n$ for the successive values of $k$ respectively and added, it is found that

$$\int_{-\pi}^{\pi} f(t)S_n(t)dt = \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right] = \int_{-\pi}^{\pi} [S_n(t)]^2 dt.$$

Hence

$$\begin{aligned}\int_{-\pi}^{\pi} [f(t) - S_n(t)]^2 dt &= \int_{-\pi}^{\pi} [f(t)]^2 dt - 2\int_{-\pi}^{\pi} f(t)S_n(t)dt + \int_{-\pi}^{\pi} [S_n(t)]^2 dt \\ &= \int_{-\pi}^{\pi} [f(t)]^2 dt - \int_{-\pi}^{\pi} [S_n(t)]^2 dt \\ &= \int_{-\pi}^{\pi} [f(t)]^2 dt - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2)\right].\end{aligned}$$

As the first member can not be negative, it must be that

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} [f(t)]^2 dt.$$

The fact that this is true for all values of $n$, while the last integral does not depend on $n$, means that the infinite series

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

is convergent. And since the general term of a convergent series approaches zero it must be that

$$\lim_{k\to\infty} a_k = 0, \quad \lim_{k\to\infty} b_k = 0.$$

It will be convenient to have this result stated for reference with another notation for the arbitrary function, and with the index $k$ replaced by $n$:

THEOREM II. *If $\phi(t)$ is any function of period $2\pi$ which is continuous except for a finite number of finite jumps in a period,*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \phi(t)\cos nt\,dt = \lim_{n\to\infty} \int_{-\pi}^{\pi} \phi(t) \sin nt\,dt = 0.$$

A simple consequence of this theorem, of no conspicuous interest in itself, will be required later as a lemma. It is clear that the hypothesis of periodicity is not essential, since the integrals to which the theorem relates do not involve values of the function outside the interval $(-\pi, \pi)$. If any function whatever is given over this interval, a periodic function can be constructed from it by suitable repetition of its values in successive intervals of length $2\pi$. If the original function as defined from $-\pi$ to $\pi$ approaches different limits at the two ends of this interval the corresponding periodic function will have a finite jump, to be sure, in passing from one interval to the next, but such a discontinuity is admissible under the hypothesis. If $\phi(x)$ is any function continuous from $-\pi$ to $\pi$ except for a finite number of finite jumps the same will be true of the functions $\phi(x) \cos (x/2)$ and $\phi(x) \sin (x/2)$, and the theorem can be applied to these functions, regardless of the fact that $\cos (x/2)$ and $\sin (x/2)$ do not of themselves have the period $2\pi$ when considered for unrestricted values of $x$. With a change of notation for the independent variable, application of the theorem to the combination

$$\phi(u) \sin (n + \tfrac{1}{2})u \equiv [\phi(u) \sin \tfrac{1}{2}u] \cos nu + [\phi(u) \cos \tfrac{1}{2}u] \sin nu$$

gives the

COROLLARY. *If $\phi(u)$ is any function which is continuous from $-\pi$ to $\pi$ except for a finite number of finite jumps,*

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \phi(u) \sin (n + \tfrac{1}{2})u\,du = 0.$$

5. *Integral formula for the partial sum of the series.* Further study of convergence depends on a trigonometric identity. Since

$$\begin{aligned}
&\sin \tfrac{1}{2}v[\tfrac{1}{2} + \cos v + \cos 2v + \cdots + \cos nv] \\
&\quad = \tfrac{1}{2} \sin \tfrac{1}{2}v + \sum_{k=1}^{n} \sin \tfrac{1}{2}v \cos kv \\
&\quad = \tfrac{1}{2} \sin \tfrac{1}{2}v + \tfrac{1}{2} \sum_{k=1}^{n} [\sin (k + \tfrac{1}{2})v - \sin (k - \tfrac{1}{2})v] \\
&\quad = \tfrac{1}{2} \sin (n + \tfrac{1}{2})v
\end{aligned}$$

it appears that

$$\tfrac{1}{2} + \cos v + \cdots + \cos nv = \frac{\sin (n + \tfrac{1}{2})v}{2 \sin \tfrac{1}{2}v}. \tag{5}$$

If the values of $a_k$, $b_k$ given by (2) are substituted explicitly in (4) and the resulting expression written out at length it is found that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} (\cos kt \cos kx + \sin kt \sin kx)\right]dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\tfrac{1}{2} + \sum_{k=1}^{n} \cos k(t - x)\right]dt,$$

which by the identity just obtained reduces to

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin (n + \frac{1}{2})(t - x)}{2 \sin \frac{1}{2}(t - x)}\,dt. \tag{6}$$

With the substitution $t-x=u$ this becomes

$$S_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x + u)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du.$$

The integrand has the period $2\pi$ when considered as a function of $u$. (Addition of $2\pi$ to $u$ reverses the signs of numerator and denominator in the fraction, but leaves the fraction as a whole unchanged.) It is a general fact that the integral of a periodic function over any interval whose length is a period is the same as the integral over any other interval of equal length. This is evident from the interpretation of the integral as the area under a curve, and is readily proved analytically with the aid of a suitable change of variable. In the present instance the integral from $-\pi-x$ to $\pi-x$ is the same as that from $-\pi$ to $\pi$, so that

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + u)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{7}$$

By integration of the identity (5) it is seen that

$$1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{8}$$

(This is in fact merely the form taken by the general expression (7) for the special case $f(x) \equiv 1$, since the Fourier series for any constant reduces to the constant itself.) If (8) is multiplied by $f(x)$, for any particular value of $x$, the factor $f(x)$, being independent of the variable of integration, may be placed under the integral sign:

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du.$$

Hence

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x + u) - f(x)]\,\frac{\sin (n + \frac{1}{2})u}{2 \sin \frac{1}{2}u}\,du. \tag{9}$$

*The problem of convergence of the series is thus reduced to the problem of showing that the last integral approaches zero*, under suitable hypotheses with regard to the function $f(x)$ under consideration.

6. *Convergence at a point of continuity.* Attention will be restricted for the present to the question of convergence at a point where $f(x)$ is continuous. *Let it be supposed that $f(x)$ is continuous everywhere, or continuous except for a finite number of finite jumps in a period, and that at the particular point where convergence is to be proved it is continuous and has a finite right-hand derivative and a finite left-hand derivative*, which may or may not be equal. This means that its graph is continuous at the point in question, and may be smooth or may have a corner there, with finite slopes from both sides.

Analytically the hypothesis implies that the difference quotient

$$\frac{f(x+u)-f(x)}{u}$$

approaches a definite limit as $u$ approaches zero through positive values, and approaches the same or a different limit as $u$ approaches zero through negative values. *Considered as a function of $u$, the quotient has at most a finite jump for* $u=0$. The same is true of the function $\phi(u)$ defined by the formula

$$\phi(u) = \frac{f(x+u)-f(x)}{2\sin\frac{1}{2}u} = \frac{f(x+u)-f(x)}{u}\cdot\frac{\frac{1}{2}u}{\sin\frac{1}{2}u},$$

since $(\frac{1}{2}u)/(\sin\frac{1}{2}u)$ has the limit 1. For any other value of $u$ between $-\pi$ and $\pi$ this $\phi(u)$, considered as a function of $u$ for a fixed value of $x$, is continuous if $f(x+u)$ is continuous, and has a finite jump if $f(x+u)$ has a finite jump. *The function $\phi(u)$ is continuous from $-\pi$ to $\pi$ except for a finite number of finite jumps.* For the value of $x$ in question, by (9)

$$S_n(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi}\phi(u)\sin(n+\tfrac{1}{2})u\,du.$$

Hence, by the Corollary of Theorem II, $S_n(x)-f(x)$ approaches zero as $n$ becomes infinite.[1] This proves

THEOREM III. *If $f(x)$ is continuous except for a finite number of finite jumps in a period, its Fourier series converges to the value $f(x)$ at every point where $f(x)$ is continuous and has a finite right-hand derivative and a finite left-hand derivative.*

7. *Uniform convergence.* Theorem III applies in particular to any function satisfying the hypotheses of Theorem I, and it is now established without question that under the conditions of Theorem I the series converges *to the value $f(x)$* for all values of $x$. To that extent Theorem I is superseded by Theorem III.

[1] For graphs illustrating the convergence in special cases see, e.g., Byerly, *Fourier's Series and Spherical Harmonics*, Boston, 1895, pp. 62–64.

The earlier theorem, however, gives important additional information with regard to the manner of convergence, under its own more restrictive hypotheses.

Let $f(x)$ be a function for which the conditions of Theorem I are satisfied By the conclusion of that theorem, together with the fact that the sum of the series is $f(x)$,

$$|f(x) - S_n(x)| = \left| \sum_{k=n+1}^{\infty} (a_k \cos kx + b_k \sin kx) \right| \leqq \sum_{k=n+1}^{\infty} \frac{2C}{k^2}.$$

It is clear that

$$\frac{1}{k^2} = \int_{k-1}^{k} \frac{du}{k^2} < \int_{k-1}^{k} \frac{du}{u^2},$$

since $u < k$ throughout the interior of the interval of integration, and hence

$$\sum_{k=n+1}^{\infty} \frac{1}{k^2} < \sum_{k=n+1}^{\infty} \int_{k-1}^{k} \frac{du}{u^2} = \int_{n}^{\infty} \frac{du}{u^2} = \frac{1}{n}.$$

So

$$|f(x) - S_n(x)| \leqq \frac{2C}{n},$$

for all values of $x$. The fact that the remainder does not exceed a quantity which is *independent of* $x$ and which approaches zero as $n$ becomes infinite is expressed by saying that the series is *uniformly convergent.* The conclusion thus noted may be recorded as

THEOREM IV. *If $f(x)$ is a function which has a continuous second derivative except for a finite number of corners in a period, its Fourier series converges uniformly to the value $f(x)$ for all values of $x$.*

8. *Weierstrass's Theorem for trigonometric approximation.* The preceding, like Theorem I, holds in particular for a broken-line function. Its application to such a function is not merely of interest in itself, but can be used to prove an important general theorem, known as Weierstrass's theorem, which says that any continuous function of period $2\pi$ can be approximately represented with any assigned degree of accuracy by a suitably constructed trigonometric sum. By a trigonometric sum is meant an expression of the form

$$\alpha_0/2 + \alpha_1 \cos x + \alpha_2 \cos 2x + \cdots + \alpha_n \cos nx$$
$$+ \beta_1 \sin x + \beta_2 \sin 2x + \cdots + \beta_n \sin nx,$$

with any constant coefficients $\alpha_k$, $\beta_k$.

Let $f(x)$ be any function which has the period $2\pi$ and is continuous for all values of $x$. It is clear that it can be approximated by a broken-line function as closely as may be desired. In terms of geometric representation this can be accomplished simply by marking points close enough together on the graph of

$f(x)$ and joining them in succession by line segments. Let $\epsilon$ be any positive number, arbitrarily small, and let $g(x)$ be a broken-line function constructed so that the difference $|f(x)-g(x)|$ is not merely less than $\epsilon$, but less than $\epsilon/2$, for all values of $x$. Let $T_n(x)$ be the partial sum of the Fourier series *for the function* $g(x)$, through the terms involving $\cos nx$ and $\sin nx$. Since the series converges uniformly to the value $g(x)$, it will be possible to take $n$ so large that $|g(x)-T_n(x)|<\epsilon/2$ for all values of $x$; if $C_0$ is the constant given by Theorem I, and entering into the proof of Theorem IV, as applied to $g(x)$, it is sufficient to take $n>4C_0/\epsilon$, so that $2C_0/n<\epsilon/2$. Then

$$|f(x) - T_n(x)| < \epsilon$$

for all values of $x$, and this is the essence of the conclusion to be proved. It may be stated as

THEOREM V (*Weierstrass's Theorem*). *If $f(x)$ is any continuous function of period $2\pi$, and if $\epsilon$ is any positive number, arbitrarily small, it is possible to construct a trigonometric sum $T_n(x)$ so that*

$$|f(x) - T_n(x)| < \epsilon$$

*for all values of $x$.*

If the Fourier series for $f(x)$ itself were known to converge uniformly to the right value there would of course be no need of bringing in the auxiliary function $g(x)$; but the Fourier series for a given function is *not* necessarily convergent if the function is merely assumed to be continuous. While continuous functions having divergent Fourier series are of complicated structure, and not likely to be encountered except when cited expressly for purposes of illustration, the fact that such functions exist gives significance to the general theorem of Weierstrass, which holds for all continuous functions without exception.

9. *Completeness of the series.* An important consequence for the theory of Fourier series, which will be stated first and then proved, is

THEOREM VI. *If $f(x)$ is a continuous function of period $2\pi$ whose Fourier coefficients are all zero, then $f(x)=0$ identically.*

The hypothesis means that

$$\int_{-\pi}^{\pi} f(x) \cos kx\, dx = \int_{-\pi}^{\pi} f(x) \sin kx\, dx = 0$$

for all integral values of $k$. It follows that

$$\int_{-\pi}^{\pi} f(x)T_n(x)dx = 0$$

if $T_n(x)$ is any trigonometric sum whatever. Let $M$ be the maximum of $|f(x)|$; the conclusion to be proved means of course that $M=0$, but that is not assumed

for the time being. Let $\epsilon$ be any positive quantity. Corresponding to the positive quantity $\epsilon/[2\pi(M+1)]$ let a trigonometric sum $T_n(x)$ be constructed according to Weierstrass's theorem so that

$$|f(x) - T_n(x)| \leqq \frac{\epsilon}{2\pi(M+1)}$$

for all values of $x$. Then

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = \int_{-\pi}^{\pi} f(x)[f(x) - T_n(x)]dx$$

$$\leqq \int_{-\pi}^{\pi} M \cdot \frac{\epsilon}{2\pi(M+1)} dx = \frac{M\epsilon}{M+1} < \epsilon.$$

Since this is true no matter how small $\epsilon$ is taken, it must be that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = 0,$$

from which it follows that $f(x) \equiv 0$, as the theorem asserts.

When the Fourier coefficients for a function are all zero, the corresponding series is of course convergent, and has zero for its sum. The theorem may be regarded as a statement that under these conditions, if the function is continuous, it is identical with the sum of the series. It is a special case of the general proposition, not yet proved in this paper, that any continuous function is equal to the sum of its Fourier series if the series is uniformly convergent. From the point of view of demonstration the special case is not trivial; on the contrary, it contains the essence of the general theorem, which can be deduced from it almost immediately with the aid of certain standard theorems on uniformly convergent series. These theorems, which will not be proved here, are to the effect that a uniformly convergent series of continuous functions represents a continuous function and can be integrated term by term. If they are assumed as known the reasoning proceeds as follows. Let $f(x)$ be the given continuous function, let $a_k$ and $b_k$ be its Fourier coefficients, and let $h(x)$ be the sum of the series, supposed uniformly convergent:

$$h(x) \equiv \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

By one of the theorems cited $h(x)$ also is continuous. The series will still be uniformly convergent, and so integrable term by term, if multiplied through by $\cos kx$ or $\sin kx$. When the integration is performed it is seen by means of (3) that

$$\int_{-\pi}^{\pi} h(x) \cos kx \, dx = \pi a_k, \qquad \int_{-\pi}^{\pi} h(x) \sin kx \, dx = \pi b_k.$$

Comparison of these formulas with (2), by means of which $a_k$ and $b_k$ are defined, shows that

$$\int_{-\pi}^{\pi} [f(x) - h(x)] \cos kx\, dx = \int_{-\pi}^{\pi} [f(x) - h(x)] \sin kx\, dx = 0$$

for all values of $k$, and application of Theorem VI to the difference $f(x)-h(x)$ gives $f(x)-h(x)\equiv 0$.

10. *Convergence at a point of discontinuity.* Throughout the discussion of convergence so far it has been assumed that the function is continuous at least at the point where convergence is to be proved, though in Theorem III it may have discontinuities elsewhere. By way of introduction to a treatment of convergence at a point of discontinuity let $F_0(x)$ be the particular function obtained by the following construction: it is defined by the formula $(\pi-x)/2$ for $0<x<2\pi$, is made periodic by repetition of these values in successive intervals of length $2\pi$, and at the points of discontinuity $x=2k\pi$, $k=0, \pm 1, \pm 2, \cdots$, it is expressly given the value zero. Its graph, except for isolated points, is thus made up of an infinite succession of straight line segments, each with slope $-\frac{1}{2}$, the right-hand end of each segment being $\pi$ units below the left-hand end of the next, while for the values of $x$ corresponding to breaks in the graph the value of the function is not the end value belonging to either of the segments concerned, but is half-way between them. Let $a_k$, $b_k$ be the Fourier coefficients of this function $F_0(x)$. Since the function satisfies the identity $F_0(-x)\equiv -F_0(x)$, the integral of $F_0(t)\cos kt$ from $-\pi$ to 0 cancels the integral from 0 to $\pi$, and $a_k=0$. It is readily found by explicit calculation that $b_k=1/k$. So $F_0(x)$ has the Fourier series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots . \tag{10}$$

It is known from Theorem III, without further inquiry, that the series must converge to $F_0(x)$ at all points where $F_0(x)$ is continuous.[1] And when $x$ is zero or any integral multiple of $2\pi$ the convergence of the series to the prescribed value is immediately apparent, since each term then reduces to zero separately. So $F_0(x)$ is represented by the series for *all* values of $x$. The value assigned to the function at the points of discontinuity of course has no influence on the determination of the coefficients; the essential observation is that the series (10) does converge at these points, and the value to which it converges is the one designated.

Now let $f(x)$ be any function of period $2\pi$ which is continuous except for a finite number of finite jumps in a period, which actually does have a finite jump for $x=0$, and which has a derivative from the right and a derivative from the

[1] In particular, setting $x=\pi/2$ gives

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

in agreement with the result obtained by setting $x=1$ in the power series for arc tan $x$. So the present reasoning incidentally gives a proof of the validity of the power series for $x=1$.

left there; the last requirement is naturally interpreted to mean that the function has a derivative from the right if defined for $x=0$ by the limit approached from the right, and a left-hand derivative if defined by the limit from the left, or in other words that each segment of the graph has a determinate finite slope where the break occurs. Let the limits approached by $f(x)$ from the right and from the left be denoted by $f(0+)$ and $f(0-)$, and let $D=f(0+)-f(0-)$. Let $f_0(x)=f(x)-(D/\pi)F_0(x)$. This function approaches the limit $[f(0+)+f(0-)]/2$ as $x$ approaches zero from the right, and has the same limit for approach from the left; moreover it has a right-hand and a left-hand derivative for $x=0$, by the corresponding hypothesis on $f(x)$. So its Fourier series converges for $x=0$ to the value $[f(0+)+f(0-)]/2$, by direct application of Theorem III. As the Fourier series for $f(x)$ can be obtained by adding the series for $f_0(x)$ and the series for $(D/\pi)F_0(x)$, and as the latter converges to the value zero, it is seen that the series for $f(x)$ converges to the mean value $[f(0+)+f(0-)]/2$.

Similar reasoning is applicable in the case of a finite jump for any other value of $x$. The question is merely that of convergence at the point where the discontinuity occurs; convergence at points of continuity is already taken care of by Theorem III. If carried through in detail, the proof would involve something amounting to explicit verification of the fact that the Fourier series for $F_0(x-c)$ as a function of $x$ is

$$\sum_{k=1}^{\infty}\left(\frac{\cos kc}{k}\sin kx-\frac{\sin kc}{k}\cos kx\right)=\sum_{k=1}^{\infty}\frac{\sin k(x-c)}{k},$$

if $c$ is any constant. One way of accomplishing this is as follows: As the Fourier coefficients for $F_0(x)$ are known to be 0 and $1/k$,

$$\frac{\sin kx}{k}=\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(t)\cos k(t-x)dt.$$

Let $F_0(x-c)$ be denoted by $F(x)$, and let $A_k$, $B_k$ be its Fourier coefficients. Then

$$\begin{aligned}A_k\cos kx+B_k\sin kx&=\frac{1}{\pi}\int_{-\pi}^{\pi}F(u)\cos k(u-x)du\\&=\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(u-c)\cos k(u-x)du.\end{aligned}$$

By the substitution $u-c=t$ the last expression becomes

$$\frac{1}{\pi}\int_{-\pi-c}^{\pi-c}F_0(t)\cos k[t-(x-c)]dt,$$

and as integration over any period interval gives the same result this reduces to

$$\frac{1}{\pi}\int_{-\pi}^{\pi}F_0(t)\cos k[t-(x-c)]dt=\frac{\sin k(x-c)}{k}.$$

The conclusion with regard to convergence at a point of discontinuity may be summarized in

THEOREM VII. *If $f(x)$ is continuous except for a finite number of finite jumps in a period, if it has a finite jump for $x=x_0$, the limits approached from the right and from the left being $f(x_0+)$ and $f(x_0-)$, and if it has a derivative from the right and a derivative from the left at this point, its Fourier series converges for $x=x_0$ to the value $[f(x_0+)+f(x_0-)]/2$.*

11. *Least-square property.* An important characteristic of the Fourier series is the least-square property, according to which the integral

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 dx,$$

where $S_n(x)$ is the partial sum of the series, has a smaller value than that which is obtained if $S_n(x)$ is replaced by any other trigonometric sum of the $n$th order. This property however is not needed for the present discussion of convergence, and it is not necessary to repeat a proof of it here.

12. *Summation by the first arithmetic mean.* A matter which does have an intimate connection with the preceding work is the *summation* of Fourier series by the method of the arithmetic mean. The mean in question is the quantity

$$\sigma_n(x) \equiv \frac{S_0(x) + S_1(x) + \cdots + S_{n-1}(x)}{n}.$$

Although, as has been stated, there exist continuous functions for which the sums $S_n(x)$ do not give a convergent approximation, Fejér[1] proved the striking theorem that *$\sigma_n(x)$ always converges uniformly to the value $f(x)$ if $f(x)$ is continuous.* A proof[2] will be given here which is in part different in arrangement from that of Fejér.

A preliminary observation is that the arithmetic mean converges uniformly in the case of any broken-line function of the sort previously considered. Let $g(x)$ be any such function, let $T_n(x)$ be the partial sum of its Fourier series, and let

$$\tau_n(x) \equiv (1/n)[T_0(x) + T_1(x) + \cdots + T_{n-1}(x)].$$

Let $\epsilon$ be any positive quantity, and in accordance with the uniform convergence of the series let $p$ be a number so large that $|g(x) - T_k(x)| < \epsilon/2$ everywhere for $k=p$ and for all larger values of $k$. For $n>p$,

[1] L. Fejér, *Untersuchungen über Fouriersche Reihen*, Mathematische Annalen, vol. 58 (1904), pp. 51–69.

[2] For the method cf. A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Dissertation, Göttingen, 1909, reprinted in Mathematische Annalen, vol. 69 (1910), pp. 331–371.

$$g(x) - \tau_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} [g(x) - T_k(x)]$$
$$= \frac{1}{n} \sum_{k=0}^{p-1} [g(x) - T_k(x)] + \frac{1}{n} \sum_{k=p}^{n-1} [g(x) - T_k(x)].$$

Let the sums from 0 to $p-1$ and from $p$ to $n-1$ be denoted respectively by $\Sigma_1$ and $\Sigma_2$, so that $g(x)-\tau_n(x)=(1/n)(\Sigma_1+\Sigma_2)$. Then

$$|g(x) - \tau_n(x)| \leqq \frac{1}{n}|\Sigma_1| + \frac{1}{n}|\Sigma_2|.$$

In $\Sigma_2$ each difference $g(x)-T_k(x)$ is less than $\epsilon/2$ in absolute value, and as the number of terms in the summation is not greater than $n$ it is certain that $|\Sigma_2| < n\epsilon/2$ and $(1/n)|\Sigma_2| < \epsilon/2$. The sum $\Sigma_1$ does not depend on $n$; if $G$ is the maximum of its absolute value, $(1/n)|\Sigma_1| < \epsilon/2$ as soon as $n > 2G/\epsilon$. When the last condition is satisfied,

$$|g(x) - \tau_n(x)| < \epsilon$$

for all values of $x$. When any positive $\epsilon$ is chosen, no matter how small, the inequality is satisfied for all values of $n$ from a certain point on, and $\tau_n(x)$ thus converges uniformly toward $g(x)$.

It is readily seen (though this is not necessary for present purposes) that the method of proof is of more general applicability, and that if any series whatever is convergent the corresponding means will converge to the same value. The significance of the process of "summation" lies in the fact that the means will sometimes converge when the original series does not.

From the identity (6), written with $k$ in place of $n$, it is seen that

$$\sigma_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} S_k(x) = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \left[ \sum_{k=0}^{n-1} \frac{\sin (k + \frac{1}{2})(t - x)}{2 \sin \frac{1}{2}(t - x)} \right] dt.$$

The relations

$$\sin \tfrac{1}{2}v \sum_{k=0}^{n-1} \sin (k + \tfrac{1}{2})v = \tfrac{1}{2} \sum_{k=0}^{n-1} [\cos kv - \cos (k + 1)v]$$
$$= \tfrac{1}{2}(1 - \cos nv) = \sin^2 \tfrac{1}{2}nv$$

give

$$\frac{1}{\sin \frac{1}{2}v} \sum_{k=0}^{n-1} \sin (k + \tfrac{1}{2})v = \frac{\sin^2 \frac{1}{2}nv}{\sin^2 \frac{1}{2}v}$$

and hence

$$\sigma_n(x) = \frac{1}{n\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin^2 \frac{1}{2}n(t - x)}{2 \sin^2 \frac{1}{2}(t - x)} dt.$$

For the study of convergence an important difference between $\sigma_n(x)$ and $S_n(x)$ is that the trigonometric factor in the last integral is always positive or zero, while the corresponding factor in (6) is of variable sign. This is the underlying reason for the greater simplicity of some of the properties of $\sigma_n(x)$. (It is not implied that the present formulas are in any sense to be regarded as superseding the earlier ones; the original sums $S_n(x)$ continue to be of more fundamental significance and have important advantages of their own, and in particular are likely to converge more rapidly when they do converge.)

In the case of a function $f(x)$ which is identically 1 each $S_k(x)$ reduce to 1, and $\sigma_n(x)$ consequently is identically 1 also, for any value of $n$, so that

$$1 \equiv \frac{1}{n\pi}\int_{-\pi}^{\pi} \frac{\sin^2 \frac{1}{2}n(t-x)}{2\sin^2 \frac{1}{2}(t-x)}\,dt. \tag{11}$$

An inference from (11) is that for any $f(x)$, if $M$ is a number such that $|f(x)| \leqq M$ for all values of $x$,

$$|\sigma_n(x)| \leqq \frac{1}{n\pi}\int_{-\pi}^{\pi} M\,\frac{\sin^2 \frac{1}{2}n(t-x)}{2\sin^2 \frac{1}{2}(t-x)}\,dt = M. \tag{12}$$

It is possible now to proceed to the proof of the main convergence theorem for $\sigma_n(x)$. Let $f(x)$ be an arbitrary continuous function of period $2\pi$, and $\sigma_n(x)$ the arithmetic mean of the first $n$ partial sums of its Fourier series. Let $\epsilon$ be any positive quantity. Let $g(x)$ be a broken-line function constructed so that

$$|f(x) - g(x)| < \epsilon/3$$

for all values of $x$, and let $r(x) = f(x) - g(x)$. Let $\tau_n(x)$ and $\rho_n(x)$ be the arithmetic means pertaining to $g(x)$ and $r(x)$ respectively. Then

$$\sigma_n(x) = \tau_n(x) + \rho_n(x), \qquad \sigma_n(x) - f(x) = [\tau_n(x) - g(x)] + [\rho_n(x) - r(x)],$$

and

$$|\sigma_n(x) - f(x)| \leqq |\tau_n(x) - g(x)| + |r(x)| + |\rho_n(x)|.$$

The definition of $r(x)$ makes

$$|r(x)| < \epsilon/3.$$

By application of (12) to the function $r(x)$, whose maximum absolute value is less than $\epsilon/3$, it is seen that

$$|\rho_n(x)| < \epsilon/3$$

for all $x$ and all $n$. As it has been shown that $\tau_n(x)$ converges uniformly toward $g(x)$,

$$|\tau_n(x) - g(x)| < \epsilon/3$$

for all values of $n$ from a certain point on. For such values of $n$ it follows that

$$|\sigma_n(x) - f(x)| < \epsilon,$$

and this relation, satisfied for all values of $x$, expresses the property of uniform convergence. This completes the proof of

THEOREM VIII. *If $f(x)$ is any continuous function of period $2\pi$, and $\sigma_n(x)$ the arithmetic mean of the partial sums $S_0(x), S_1(x), \cdots, S_{n-1}(x)$ of its Fourier series, $\sigma_n(x)$ converges uniformly toward $f(x)$ for all values of $x$.*

13. *Weierstrass's Theorem for polynomial approximation.* Inasmuch as $\sigma_n(x)$ is a trigonometric sum, Theorem V is incidentally obtained again as a corollary of Theorem VIII, though with the order of presentation followed here the first proof is simpler.

In conclusion it may be noted that the better known theorem of Weierstrass which relates to polynomial approximation is also immediately obtainable from the work that has been done. The earlier form of proof will be preferred again.

Let $f(x)$ be a function of $x$ which is continuous for $-1 \leqq x \leqq 1$. Let $x = \cos\theta$, and let $f(x) = f(\cos\theta) = \phi(\theta)$. This is a function of period $2\pi$ which is defined and continuous for all values of $\theta$, and which furthermore satisfies the relation $\phi(-\theta) \equiv \phi(\theta)$. Let $\epsilon$ be any positive quantity, and let $g(\theta)$ be defined for $0 \leqq \theta \leqq \pi$ as a broken-line function such that $|\phi(\theta) - g(\theta)| < \epsilon/2$ throughout the interval. Then if $g(\theta)$ is defined for $-\pi \leqq \theta \leqq 0$ by the relation $g(-\theta) = g(\theta)$, and for values of $\theta$ outside the interval $(-\pi, \pi)$ by the requirement that it shall have the period $2\pi$, it is a function to which Theorem IV is applicable, and $|\phi(\theta) - g(\theta)| < \epsilon/2$ for all values of $\theta$. Let $T_n(\theta)$ be the partial sum of the Fourier series for $g(\theta)$, and let $n$ be taken so large that by virtue of the uniform convergence $|g(\theta) - T_n(\theta)| < \epsilon/2$ everywhere, and hence $|\phi(\theta) - T_n(\theta)| < \epsilon$. Since $g(-\theta) \equiv g(\theta)$ each sine coefficient $b_k$ in $T_n(\theta)$ is zero; the cosine of $k\theta$ is expressible as a polynomial of the $k$th degree in $\cos\theta$ for each value of $k$; and hence $T_n(\theta)$ is a polynomial in $\cos\theta$, which may be denoted by $P_n(\cos\theta)$ or $P_n(x)$. As $\phi(\theta) \equiv f(x)$, an approximating polynomial has been constructed for $f(x)$ so that

$$|f(x) - P_n(x)| < \epsilon$$

for $-1 \leqq x \leqq 1$.

This result can be extended to any interval $(a, b)$ by the change of variable $y = (2x - a - b)/(b - a)$. Any continuous function of $x$ for $a \leqq x \leqq b$ is a continuous function of $y$ for $-1 \leqq y \leqq 1$; an approximating polynomial in terms of $y$ can be found for this function; and any polynomial in $y$ is a polynomial of the same degree in $x$. The general conclusion can be stated as

THEOREM IX (*Weierstrass's Theorem for polynomial approximation*). *If $f(x)$ is any continuous function for $a \leqq x \leqq b$, and if $\epsilon$ is any positive number, arbitrarily small, it is possible to construct a polynomial $P_n(x)$ so that*

$$|f(x) - P_n(x)| < \epsilon$$

*for $a \leqq x \leqq b$.*

The discussion of convergence of Fourier series given in the earlier sections

can be carried over in part to the case of Legendre series and to more general developments in series of polynomials. This is done, together with an extension to less elementary parts of the theory, in a paper in the Annals of Mathematics.[1]

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# THE POSTULATIONAL METHOD IN MATHEMATICS[2]

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It is an interesting coincidence that the beginning of the Century of Progress which is now being celebrated in Chicago coincides almost exactly with the beginning of the postulational method in mathematics.

The postulational method in mathematics may be said to have had its origin in the non-Euclidean geometries of Lobachevsky and Bolyai which were published almost exactly one hundred years ago.

A systematic development of postulational technique began with Peano and his school in Italy in 1889, and Hilbert's "Foundation of Geometry" attracted wide attention to the method ten years later. In the United States important papers were published early in the twentieth century by O. Veblen, E. H. Moore, L. E. Dickson, B. A. Bernstein, H. M. Sheffer, and many others, in this field. After a short period of semi-quiescence, renewed interest in the postulational method has been recently shown by numerous books and papers, both on the mathematical side and on the philosophical side. The semi-popular accounts of the method in E. T. Bell's "The Queen of the Sciences," in the Century of Progress Series, and in C. J. Keyser's little book on "Thinking about Thinking," indicate the widespread popular interest in the subject. A rapidly increasing number of technical papers have been based on E. H. Moore's notion of "complete independence;" and the debate on the foundations of pure logic, which has centered for over twenty years around the monumental "Principia Mathematica" of Whitehead and Russell, is being carried on with renewed vigor in very recent years. Moreover in many widely separated fields of science the methods of attack on fundamental scientific problems are coming more and more to resemble the postulational method used in mathematics. P. W. Bridgman's "operational" theory in physics, although built up (as V. Lenzen has shown) from an entirely different point of view, may be mentioned as of particular interest in this connection.

Progress is so rapid that it is doubtless too early to attempt any assessment of the permanent value of the postulational method. All that I shall undertake to do in this brief paper is to examine one or two typical examples of simple sets

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[1] D. Jackson, *Series of orthogonal polynomials*, Annals of Mathematics, vol. 34 (1933), pp. 527–545.

[2] A paper read before the Mathematical Association of America at the joint meeting with Section A of the A.A.A.S. held in Chicago, June 20, 1933.