LUDWIG BIEBERBACH'S CONJECTURE AND ITS PROOF BY LOUIS DE BRANGES

J. KOREVAAR
Mathematics Institute, University of Amsterdam, Roetersstraat 15, 1018 WB Amsterdam, The Netherlands

Summary. 1984 has been an exciting year for complex analysis. It even brought strong rumors that the Riemann hypothesis had been proved, but so far, the rumor has not been confirmed. However, we know for sure that the difficult Bieberbach conjecture has been settled this year. As many of you know, this famous conjecture of 1916 concerns the class S of normalized injective holomorphic functions. That class consists of the 1-1 holomorphic functions from the unit disc U into the complex plane $\mathbb{C}$ with a power series of the form

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad |z| < 1.$$

The conjecture asserts that $|a_n| \leqslant n$ for every f in S and every n. Louis de Branges has proved this conjecture as well as some stronger conjectures for the class S.

Each of the following items has played an essential role in the proof:

(i) Löwner's partial differential equation for so-called Löwner chains $\{f_t(z)\}$ of injective holomorphic functions from U to $\mathbb{C}$.

(ii) The observations of Lebedev and I. M. Milin, especially their inspired conjecture for the so-called logarithmic coefficients of f in S, that is, the coefficients in the expansion $\Sigma_1^\infty c_k z^k$ for a branch of $\log\{f(z)/z\}$.

(iii) De Branges' striking breakthrough, namely, the creation of a functional associated with the Lebedev-Milin conjecture which varies monotonically along Löwner chains.

(iv) De Branges' introduction and solution of a system of differential equations which he devised to make the functional manageable.

(v) A positivity result for hypergeometric functions which is a tool in establishing the monotonicity of the functional.

Of the above, (i) dates back to 1923, while (ii) and (v) are relatively recent. The Lebedev-Milin observations date from the years 1965–1970 and became well known in the West only around 1977. The hypergeometric functions result occurs in work of Askey and Gasper of 1976.

1. Historical introduction. Our starting point is the well-known *conformal mapping theorem* formulated by Riemann. Let D be an arbitrary simply connected domain in the complex plane $\mathbb{C}$ which is not the whole plane. Then there exists a conformal (or 1-1 holomorphic) map $w = f(z)$ from the unit disc U: $\{|z| < 1\}$ onto D. One may arbitrarily prescribe the image $f(0)$ (in D) of the origin, as well as the angle $\arg f'(0)$ through which directions at the origin are rotated. However, such data determine the map uniquely. The first complete proofs of the theorem were given around 1900: by Hilbert, who put the Dirichlet principle on a rigorous basis, and by Osgood, who constructed and used Green's functions for D.

Questions on the fine structure of conformal maps became a popular topic in German mathematics around 1910 (Koebe, Carathéodory and others). Let us normalize our injective holomorphic maps f from U to $\mathbb{C}$ by requiring $f(0) = 0$, $f'(0) = 1$. Then we obtain the class S of normalized 1-1 holomorphic functions ("schlicht" or univalent functions)

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \text{ on } U. \tag{1}$$

The author has worked in complex analysis and approximation theory, including Tauberian theorems and Müntz-type approximation. Born and educated in the Netherlands, he became an admirer of Hardy-Littlewood-Pólya-Szegö-Wiener. He has spent 25 years in the U.S., mostly at the University of Wisconsin (Madison) and the University of California (San Diego). Since 1974 he has been a professor at the University of Amsterdam and a member of the Netherlands Academy of Sciences; he holds an honorary degree from the University of Gothenburg. Mathematics is a family interest (the father-in-law story applies), as are music, languages and mountain hiking.

EXAMPLES. The formula

$$w = \frac{1+z}{1-z} = 1 + 2z + \cdots + 2z^n + \cdots$$

defines a conformal map of U onto the right half-plane H: $\{\operatorname{Re} w > 0\}$. Squaring, we obtain a conformal map

$$w = \left(\frac{1+z}{1-z}\right)^2$$

of U onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$. Normalization gives the important *Koebe function*

$$K_0(z) = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\} = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots + nz^n + \cdots \tag{2a}$$

which maps U onto $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$. This function and its rotations

$$K_\theta(z) = e^{-i\theta} K_0(e^{i\theta} z) = \frac{z}{(1 - e^{i\theta} z)^2} \tag{2b}$$

(which we will also call Koebe functions) provide the solution to many extremal problems for the class S.

Some extremal problems. In 1916 Bieberbach [**2**] proved that $|a_2| \leqslant 2$ for every f in S, with equality only for the Koebe functions (2). In a footnote he remarked that perhaps quite generally

$$|a_n| \leqslant n \quad \text{for } f \in S. \tag{3}$$

This footnote became the famous Bieberbach conjecture which remained unproven until 1985, although a great deal of work was expended on it. The a_2-result can be used to show that the image $f(U)$ contains the disc $|w| < 1/4$ in the w-plane for every f in S. Moreover, if $f(U)$ contains no larger disc about 0, then f is a Koebe function. Related results are Koebe's distortion theorems, of which we mention

$$|f'(z)| \leqslant \frac{1+|z|}{(1-|z|)^3}, \quad \frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2}. \tag{4}$$

These inequalities hold for all f in S, with strict inequality for all $z \neq 0$ unless f is a Koebe function.

General references for this section and the next are the books by Goluzin [**10**], Pommerenke [**14**], Duren [**6**] and Goodman [**11**].

2. More on the Bieberbach conjecture. Using the partial differential equation named after him, see Section 4(iii), Löwner proved in 1923 that $|a_3| \leqslant 3$ for every f in S [**12**]. Later, Schiffer and others developed a number of variational methods for injective holomorphic functions. In the years 1955–1972 those techniques yielded rather laborious proofs for the special cases $n = 4$, 6 and 5 of the Bieberbach conjecture. From time to time, proofs for other special cases have been announced, but they have not been substantiated.

Turning to general n, the upper bound for $|f(z)|$ in the distortion relations (4) and Cauchy's inequality for the coefficients of a power series readily show that $|a_n| < en^2$. In 1925 Littlewood found the correct order of the upper bound for $|a_n|$ as $n \to \infty$:

$$|a_n| < en \quad \text{for all } f \in S.$$

The best result of this kind until this year was that of FitzGerald (1972), including a slight improvement by his student Horowitz (1978):

$$|a_n| < 1.07n.$$

There is also a beautiful regularity theorem of Hayman (1953): the limit

$$\lim_{n\to\infty} \frac{|a_n|}{n}$$

exists for every f in S, and is smaller than 1 unless f is a Koebe function. This result remains of interest even now that the Bieberbach conjecture has been proved.

Odd functions. For f in S, it is often useful to look at the related function

$$f_1(z) = \sqrt{f(z^2)} = b_1 z + b_3 z^3 + \cdots + b_{2n-1} z^{2n-1} + \cdots, \quad b_1 = 1. \tag{5}$$

This is an odd function in S, and every odd function f_1 in S can be represented as such a square root transform. By (1) and (5)

$$a_n = b_1 b_{2n-1} + \cdots + b_{2n-1} b_1. \tag{6}$$

If f is the Koebe function K_0 then

$$f_1(z) = \frac{z}{1-z^2} = z + z^3 + \cdots + z^{2n-1} + \cdots.$$

In 1932 Littlewood and Paley proved that there is a constant $C \leqslant 14$ such that for all odd functions in S,

$$|b_{2n-1}| \leqslant C, \quad n = 1, 2, \ldots.$$

In a footnote they remarked: "No doubt the true bound is given by $C = 1$." Observe that by (6), the truth of this conjecture would imply the Bieberbach conjecture!

Recently V. I. Milin proved that one may take $C = 1.14$. Before that, Hayman had shown that

$$\lim_{n\to\infty} |b_{2n-1}|$$

exists for every odd function f_1 in S, and is smaller than 1 unless f_1 is the root transform of a Koebe function. These results would seem to support the Littlewood-Paley conjecture. However, the latter had been disproved already in 1933 by Fekete and Szegö: there exist odd functions in S for which $|b_5| > 1$. No wonder that some experts doubted the Bieberbach conjecture as well!

However, there are always people around with the intuition to come up with a good conjecture. It was observed by Robertson that by (6) and the Cauchy-Schwarz inequality, the Bieberbach conjecture would already follow from the inequality

$$\sum_1^n |b_{2k-1}|^2 \leqslant n. \tag{7}$$

(7) became known as the *Robertson conjecture* for odd functions in S (1936).

Logarithmic coefficients. Since 1940, one has increasingly used certain logarithmic transforms of injective holomorphic functions. The associated Grunsky and Goluzin inequalities have been successfully applied to various extremal problems. More recently, Lebedev and I. M. Milin have focused on the expansion

$$\log \frac{f(z)}{z} = \sum_1^\infty c_k z^k, \qquad |z| < 1 \tag{8}$$

for f in S. (Note that $f(z)/z$ is holomorphic and zero-free on U; one takes the branch of the logarithm which vanishes at the origin.) For the Koebe function K_0 one has $c_k = 2/k$. For the case of image domains $f(U)$ that are star-shaped relative to the origin, one has $|c_k| \leqslant 2/k$ and this inequality readily implies the Bieberbach conjecture (3) for such "starlike" functions f.

The latter result goes back to Nevanlinna (1920). For starlike f, a geometric argument at the boundary shows that

$$\operatorname{Re} z\frac{f'(z)}{f(z)} = \operatorname{Re}\left(1 + \sum_1^\infty k c_k z^k\right) > 0,$$

so that by a well-known inequality of Carathéodory for functions with positive real part, $k|c_k| \leqslant 2$, cf. Section 4(vi).

The inequality $|c_k| \leqslant 2/k$ or

$$k|c_k|^2 - \frac{4}{k} \leqslant 0$$

is not true for every f in S, but Lebedev and Milin conjectured that the latter inequality is true in the following average sense:

$$\Omega_n \overset{\text{def}}{=} \sum_{p=1}^{n-1} \sum_{k=1}^{p} \left(k|c_k|^2 - \frac{4}{k}\right) = \sum_{k=1}^{n-1} \left(k|c_k|^2 - \frac{4}{k}\right)(n-k) \leqslant 0 \tag{9}$$

for $n = 2,3,\ldots$ and all $f \in S$. This amazing conjecture occurs in Milin's book of 1971; the book became available in English only in 1977 [**13**]. The L-M conjecture implies the Robertson conjecture and hence also the Bieberbach conjecture.

EXAMPLES. The L-M conjecture is easy to prove for $n = 2$ and 3, cf. Section 4(vii). For $n = 2$ it asserts that $|c_1|^2 \leqslant 4$. Since

$$\begin{aligned} 1 + a_2 z + a_3 z^2 + \cdots &= \frac{f(z)}{z} = \exp(c_1 z + c_2 z^2 + \cdots) \\ &= 1 + c_1 z + \left(\frac{1}{2}c_1^2 + c_2\right) z^2 + \cdots, \end{aligned} \tag{10}$$

Bieberbach's inequality is an immediate consequence:

$$|a_2| = |c_1| \leqslant 2.$$

For $n = 3$ the L-M conjecture is equivalent to $|c_1|^2 + |c_2|^2 \leqslant 5$. Löwner's inequality is an easy corollary:

$$|a_3| = \left|\frac{1}{2}c_1^2 + c_2\right| \leqslant \frac{1}{2}|c_1|^2 + |c_2| \leqslant \frac{5}{2} - \frac{1}{2}|c_2|^2 + |c_2| \leqslant 3$$

(by calculus!).

Turning to the general case, there is a useful inequality of Lebedev and Milin for the coefficients of

$$\sum_0^\infty \beta_k z^k = \exp\left(\sum_1^\infty \gamma_k z^k\right).$$

It asserts that

$$\sum_0^{n-1} |\beta_k|^2 \leqslant n \exp\left\{\frac{1}{n}\sum_{p=1}^{n-1} \sum_{k=1}^{p} \left(k|\gamma_k|^2 - \frac{1}{k}\right)\right\}, \quad n = 1,2,\ldots,$$

see [**6**]. Applying this inequality to the identity

$$\begin{aligned} b_1 + b_3 z + \cdots + b_{2n-1} z^{n-1} + \cdots &= \frac{f_1(z^{\frac{1}{2}})}{z^{\frac{1}{2}}} \\ &= \left\{\frac{f(z)}{z}\right\}^{\frac{1}{2}} = \exp\left\{\frac{1}{2}\log\frac{f(z)}{z}\right\} = \exp\left(\frac{1}{2}\sum_1^\infty c_k z^k\right), \end{aligned}$$

cf. (5) and (8), one obtains

$$|b_1|^2 + \cdots + |b_{2n-1}|^2 \leqslant n \exp(\Omega_n/4n). \tag{11}$$

Thus if the L-M conjecture (9) holds for $f \in S$ and a certain n, then the Robertson conjecture (7) holds for the corresponding f_1 and the same n, so that also the Bieberbach conjecture (3) must be true for the same n, cf. (6). Moreover, if $\Omega_n < 0$ for some n, then one has strict inequality in (7) and hence also in (3).

De Branges has proved the L-M conjecture and thereby also the Robertson and Bieberbach conjectures [**3**], [**4**], [**5**].

3. De Branges' theorem [4]. *Let f be an arbitrary function in S, let the power-series coefficients a_n be defined by* (1) *and the logarithmic coefficients c_k by* (8). *Then the conjectured* L-M *inequality* (9) *and hence the conjectured Bieberbach inequality* (3) *are true for every $n \geqslant 1$. Equality holds in* (3) *and hence in* (9) *for a certain $n \geqslant 2$ if and only if f is a Koebe function* (2).

4. The proof. I will present de Branges' ideas in as simple a way as I can. The following arrangement is based on de Branges' lecture in Amsterdam (July 10) and an early write-up in Russian which he had at that time. It also shows the influence of his later manuscript [**4**] and of a manuscript by FitzGerald and Pommerenke [**8**] based on de Branges' work. The proof will be spread over a number of steps.

(i) *We may take D nice.* For the proof of the L-M conjecture (9), it may be assumed that f maps U onto a domain D bounded by an analytic Jordan curve. Indeed, for any given f in S and $0 < \rho < 1$ we may define

$$f^*(z) = \frac{1}{\rho} f(\rho z) = z + a_2 \rho z^2 + \cdots + a_n \rho^{n-1} z^n + \cdots.$$

The function f^* maps U onto the set $(1/\rho) f(\rho U)$. The latter domain is bounded by the analytic Jordan curve given by $(1/\rho)$ times the image of the circle $|z| = \rho$ under f.

Since

$$\log \frac{f^*(z)}{z} = \log \frac{f(\rho z)}{\rho z} = \sum_1^\infty c_k \rho^k z^k,$$

the logarithmic coefficients c_k^* for f^* are equal to $c_k \rho^k$. Hence if (9) has been proved for the coefficients c_k^*, it follows for the coefficients c_k by letting ρ tend to 1.

(ii) *Löwner chains.* Given $D = f(U)$ as in (i), it is easy to construct a nice continuously increasing family of simply connected domains D_t, $0 \leqslant t < \infty$, such that

$$D_0 = D, \quad D_s \subsetneqq D_t \text{ if } s < t \quad \text{and} \quad D_t \to \mathbb{C} \text{ as } t \to \infty. \tag{12a}$$

One can actually do this for *every* simply connected domain D, cf. [**14**].

We define

$$f_t(z) = f(z, t), \quad 0 \leqslant t < \infty,$$

as the 1-1 conformal map of U onto D_t such that

$$f_t(0) = 0, \quad f_t'(0) > 0.$$

Then $\omega(t) = f_t'(0)$ will be a strictly increasing continuous function such that $\omega(0) = 1$ and $\omega(t) \to \infty$ as $t \to \infty$. Introducing a new parameter u by setting $\omega(t) = e^u$, if necessary, one may assume from the beginning that $\omega(t) = e^t$. The corresponding family of injective holomorphic functions

$$f_t(z) = f(z, t) = e^t\left(z + a_2(t) z^2 + \cdots\right), \quad 0 \leqslant t < \infty; \quad f_0(z) = f(z) \tag{12b}$$

(which depend continuously on t) is called a Löwner chain starting at $f(z)$.

A little more effort shows that *every* $f \in S$ is the starting point of a Löwner chain, cf. **[14]** p. 159.

(iii) *Heuristic derivation of the Löwner differential equation*. The functions $f(z, t)$ of a Löwner chain satisfy the partial differential equation of Löwner **[12]**:

$$\frac{\partial f}{\partial t} = z\frac{\partial f}{\partial z}p(z, t), \tag{13a}$$

where

$$p(z, t) \text{ is analytic in } z, \quad \operatorname{Re} p(z, t) > 0, \quad p(0, t) = 1. \tag{13b}$$

Geometrically, equation (13) represents an outward flow in the plane. Indeed, the vector z gives the direction of the outward normal to the circle C_r: $|z| = r$. Thus $z(\partial f/\partial z)$ gives the direction of the outward normal to the curve $f(C_r)$ at the point $f(z, t)$. By (13), the velocity vector $\partial f/\partial t$ should make an angle with the normal there less than $\frac{1}{2}\pi$.

We now indicate how (13) comes about. Let $0 \leqslant s < t$ and define

$$\varphi(z) = \varphi(z, s, t) = f_t^{-1} \circ f_s(z) = e^{s-t}z + \cdots .$$

This φ is a holomorphic map from U into U, but not onto U, such that 0 is carried to 0. Hence by the Schwarz lemma,

$$|\varphi(z, s, t)| < |z| = |\varphi(z, s, s)|$$

for all $z \neq 0$ ("inward flow on the unit disc"). Let us assume that $\partial\varphi/\partial t$ exists (and is analytic in z). Then the angle between the vector $\partial\varphi/\partial t$ for $t = s$ and the vector $-z$ must be bounded by $\frac{1}{2}\pi$. It follows that

$$\left.\frac{\partial\varphi}{\partial t}\right|_{t=s} = -zp(z, s) \text{ with } \operatorname{Re} p(z, s) > 0 \tag{14}$$

and $p(z, s)$ analytic in z, $p(0, s) = 1$.

From the definition of φ,

$$f_t \circ \varphi(z, s, t) = f_s(z).$$

Differentiating with respect to t and then setting $t = s$, we obtain

$$\frac{\partial f_t}{\partial t} + \frac{\partial f_t}{\partial z}\frac{\partial\varphi}{\partial t} = 0 \quad \text{for } t = s. \tag{15}$$

Combination of (15) and (14) gives Löwner's equation for $t = s$.

The assumption that φ is a nice function of its arguments is no real restriction, since we may assume that our domains D_t depend analytically on t. However, the Löwner differential equation holds for arbitrary Löwner chains, cf. Pommerenke **[14]** Chapter 6. A crucial observation (which makes use of the distortion formula (4)) is that $f(z, t)$ is Lipschitzian in t; equation (13a) will hold for almost all t. Conversely, every partial differential equation (13) determines a Löwner chain of conformal maps.

(iv) *Logarithmic coefficients for* $f(z, t)/e^t$. It is natural to introduce the expansions

$$\log\frac{f(z, t)}{e^t z} = \sum_1^\infty c_k(t) z^k. \tag{16}$$

Since $f(z, t)/e^t$ is in S, cf. (12b), we know from Section 2 that there exist constants A_k, for example $A_k = ek^2$, such that $|a_k(t)| \leqslant A_k$ for all t. Hence by recursion, cf. equation (10), there will be constants C_k such that

$$|c_k(t)| \leqslant C_k \quad \text{for all } t. \tag{17}$$

We may differentiate relation (16) with respect to t and with respect to z. We substitute the results in the Löwner equation (13a), setting

$$p(z,t) = 1 + 2\sum_1^{\infty} d_k(t) z^k. \tag{18}$$

Equating the coefficients of like powers of z, we thus obtain the system of differential equations

$$c_k'(t) = 2d_k(t) + kc_k(t) + 2\sum_1^{k-1} jc_j(t) d_{k-j}(t), \quad k = 1,2,\dots. \tag{19}$$

(v) *The auxiliary functional* Ω. We now take *n fixed*. With an eye to the L-M conjecture (9) and following de Branges' ideas, cf. also [**8**], we introduce the auxiliary functional

$$\Omega(t) = \Omega_n(t) = \sum_1^{n-1}\left\{k|c_k(t)|^2 - \frac{4}{k}\right\}\sigma_k(t), \tag{20}$$

where the weight functions $\sigma_k(t)$ are to be chosen in a suitable manner. What properties besides some smoothness do we want the $\sigma_k(t)$ to have?

It is desired that the relation $\Omega(0) \leqslant 0$ be the conjectured L-M inequality (9). Noting that $c_k(0) = c_k$, we therefore impose the initial conditions

$$\sigma_k(0) = n - k, \quad k = 1,\dots,n-1. \tag{21}$$

Clearly the inequality $\Omega(0) \leqslant 0$ would follow if $\Omega(t)$ were a non-decreasing function of t which vanishes at $t = +\infty$, that is, if

$$\Omega'(t) \geqslant 0 \quad \text{for } 0 \leqslant t < \infty, \tag{22}$$

while $\Omega(t) \to 0$ as $t \to \infty$. Because of the boundedness of every $c_k(t)$, cf. (17), the last condition will be satisfied if

$$\lim_{t\to\infty} \sigma_k(t) = 0, \quad k = 1,\dots,n-1. \tag{23}$$

Do there really exist functions $\sigma_k(t)$ satisfying (21) and (23) such that at the same time $\Omega'(t) \geqslant 0$?

(vi) *Differential equation conditions on the* σ_k *in order to make* Ω' *manageable*. We calculate $\Omega'(t)$ using the differential equations (19) for the $c_k(t)$. The resulting expression is quite complicated. However, after some experimentation it is seen to simplify if we impose de Branges' conditions

$$\sigma_k - \sigma_{k+1} = -\left(\frac{\sigma_k'}{k} + \frac{\sigma_{k+1}'}{k+1}\right), \quad k = 1,\dots,n-1; \quad \sigma_n \equiv 0, \tag{24}$$

where the variable t has been suppressed. The result of the calculation may then be written in the form

$$\Omega' = -\sum_1^{n-1} Q_k(c,d)\sigma_k', \tag{25}$$

where the Q_k are nonnegative functions of the $c_k(t)$ and the $d_k(t)$.

Since it is of importance for the case of equality in the L-M conjecture, we indicate the precise form of $Q_k(c,d)$. Using the Herglotz representation for holomorphic functions on the unit disc with positive real part, we have

$$p(z,t) = \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu_t(\theta),$$

where μ_t is a positive Borel measure of total mass equal to $p(0, t) = 1$. Thus by (18), the coefficients of $p(z, t)$ have the form

$$d_k(t) = \int_{-\pi}^{\pi} e^{-ik\theta}\, d\mu_t(\theta).$$

Introducing the sums

$$s_k = \sum_{j=1}^{k} jc_j(t)\, e^{ij\theta}, \quad s_0 = 0 \tag{26}$$

we may write $kc_k(t) = (s_k - s_{k-1})e^{-ik\theta}$ and by (19)

$$c_k' = \int_{-\pi}^{\pi} (2 + s_{k-1} + s_k)\, e^{-ik\theta}\, d\mu_t(\theta).$$

Ω' may now be written as an integral relative to μ_t; the integrand is a sum involving the s_k, σ_k and σ_k'. After a summation by parts and from the differential equation (24), the result is (25) with

$$Q_k(c, d) = \frac{1}{k}\int_{-\pi}^{\pi} |2 + s_{k-1} + s_k|^2 d\mu_t. \tag{27}$$

(vii) *Explicit form of the* σ_k'. By the formula for Ω' (25), we would have its positivity (22), if we could *guarantee* that

$$\sigma_k' \leqslant 0, \quad k = 1, \ldots, n-1. \tag{28}$$

Observe, however, that first σ_{n-1} and next $\sigma_{n-2}, \ldots, \sigma_1$ are *completely determined* by the system of differential equations (24) and the initial conditions (21)! Could it be true that for the solutions, the additional conditions (23) and (28) are miraculously satisfied?

EXAMPLES (cf. the examples in Section 2). For $n = 2$, one has $\sigma_2 \equiv 0$ and hence $\sigma_1 = e^{-t}$. Thus $\sigma_1' \leqslant 0$. It follows that the L-M inequality holds for $n = 2$ and thus also the Bieberbach inequality $|a_2| \leqslant 2$.

For $n = 3$, one has $\sigma_3 \equiv 0$ and next

$$\sigma_2 = e^{-2t}, \quad \sigma_1 = 4e^{-t} - 2e^{-2t}.$$

Again $\sigma_k' \leqslant 0$, thus proving the L-M inequality for $n = 3$ and hence Löwner's inequality $|a_3| \leqslant 3$.

Of course, de Branges went on. For general n he found a solution of his system of differential equations and initial conditions which may be written as

$$\sigma_k(t) = k \sum_{\nu=0}^{n-k-1} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-1-\nu}}{(k+\nu)\nu!(n-k-1-\nu)!} e^{-\nu t - kt},$$

$k = 1, \ldots, n-1$. Here

$$(a)_\nu = a(a+1)\cdots(a+\nu-1) \quad \text{for } \nu \geqslant 1, (a)_0 = 1.$$

It is clear that the functions $\sigma_k(t)$ will vanish at infinity (condition (23)). However, what about the negativity condition $\sigma_k' \leqslant 0$ in (28)? In other words, could it be true that the sums

$$-\frac{\sigma_k'}{k} e^{kt} = \sum_{\nu=0}^{n-k-1} (-1)^\nu \frac{(2k+\nu+1)_\nu (2k+2\nu+2)_{n-k-1-\nu}}{\nu!(n-k-1-\nu)!} e^{-\nu t} \tag{29}$$

are nonnegative for $k = 1, \ldots, n-1$ and all $n \geqslant 2$?

(viii) *Completion of the proof of the L-M conjecture*. For relatively small n, de Branges could verify immediately that the sums (29) are positive on $(0, \infty)$. But what about larger values of n? At this stage de Branges went to his numerical colleague Gautschi at Purdue University for help. He told Gautschi that he had a way of proving the Bieberbach conjecture, but needed to establish certain inequalities involving hypergeometric functions. Would Gautschi be willing to check as many of these inequalities as possible on the computer? Gautschi wrote a suitable program with a feeling that he might soon hit a value of n for which the consistent positivity of expressions related to (29) would come to an end. Much to his surprise, however, he discovered that the crucial expressions were positive for all values of n which he tried: $2 \leqslant n \leqslant 30$. Thus at this time, assuming that the theoretical work was correct, de Branges and the computer had verified the Bieberbach conjecture for all n up to 30!

How to continue? Gautschi had the idea to call Askey at the University of Wisconsin, the world's expert on special functions. At first Askey was incredulous that the supposed positivity of sums such as those in (29) would prove the Bieberbach conjecture. However, he realized very soon that those sums were essentially generalized hypergeometric functions of a very special kind which are known to be positive:

$$-\frac{\sigma_k'}{k}e^{kt} = \binom{n+k}{n-k-1}\,{}_3F_2\left(\begin{matrix}-n+k+1,\,k+\frac{1}{2},\,n+k+1\\ k+\frac{3}{2},\,2k+1\end{matrix}\,\middle|\, e^{-t}\right). \tag{30}$$

Here

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\,\middle|\,x\right) = \sum_{\nu=0}^{\infty}\frac{(a)_\nu(b)_\nu(c)_\nu}{(d)_\nu(e)_\nu}\frac{x^\nu}{\nu!};$$

for the special values of a through e in (30), the positivity of ${}_3F_2$ followed from a joint result of Askey and Gasper [**1**].

Thus de Branges' proof of the L-M conjecture was complete, thanks to a known result on special functions.

(ix) *The case of equality.* We now start with an arbitrary function $f \in S$ and an associated Löwner chain. It is easy to see that equality holds in the L-M inequality (9) for f and given $n \geqslant 2$ if and only if $\Omega' \equiv 0$. Since $\sigma_k' < 0$ on $(0, \infty)$ for $1 \leqslant k \leqslant n-1$, the latter condition requires that $Q_k(c, d) \equiv 0$ for those values of k, cf. (25). In particular, the condition $Q_1(c, d) \equiv 0$ is necessary for equality. By the representation (27) with positive μ_t this condition implies

$$2 + s_1 = 2 + c_1(t)e^{i\theta} = 0 \text{ a.e. } [\mu_t].$$

Thus the absolutely continuous part of μ_t must be zero, and in fact, μ_t must be a point mass 1 at some point θ_t. It follows that $|c_1(t)| \equiv 2$ and in particular $|c_1| = 2$, hence $|a_2| = 2$ so that f must be a Koebe function. For a Koebe function, one indeed has $\Omega(0) = 0$ for every n.

For the case of equality in the "Bieberbach inequality" (3), one may now use the remark at the end of Section 2.

5. Final remarks. De Branges was born in Paris in 1932. He studied in the U.S. and has been at Purdue University since 1963. In his mathematical career, he has tackled a number of difficult problems. Early in 1984 he completed a manuscript of 385 pages for a new edition of his book "Square summable power series". This manuscript culminated in a proof of the Bieberbach conjecture. With the manuscript, de Branges departed for Leningrad in April 1984 for a scheduled exchange visit. As he tells it, he was disappointed that the U.S. mathematicians to whom he had sent his manuscript had not yet been able to verify his long proof. In Leningrad, de Branges presented his work to the members of the seminars in functional analysis and geometric function theory. In a large number of sessions, the proof was verified and some inessential errors corrected. Finally, through hard work under de Branges' direction, a relatively short proof of the Lebedev-Milin conjecture was distilled from the original manuscript.

Upon his return from Leningrad, de Branges lectured on his proof at a number of universities, among them the Free University at Amsterdam. An early report on the proof in Russian was widely circulated. It reached FitzGerald and Pommerenke at La Jolla in July. They restated the proof in their own words, as mathematicians do when they try to understand new material. They also treated the case of equality in the Bieberbach conjecture [**8**], as would others. In the mean time, de Branges produced a more sophisticated write-up of his proof which includes the case of equality in a very natural way [**4**]. Comments on the exciting events have been written up by FitzGerald [**7**] and by Gautschi [**9**], among others. (Added in proof: see also Pommerenke [**15**].)

References

1. R. Askey and G. Gasper, Positive Jacobi polynomial sums II, Amer. J. Math., 98 (1976) 709–737.

2. L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, S.-B. Preuss. Akad. Wiss., 1916, pp. 940–955.

3. L. de Branges, A proof of the Bieberbach conjecture, preprint E-5-84, Steklov Math. Inst., Leningrad Branch, 1984, pp. 1–21.

4. ______, A proof of the Bieberbach conjecture, Acta Math., 154 (1985) 137–152.

5. ______, Square summable power series, 2nd ed., to appear.

6. P. L. Duren, Univalent Functions, Springer, Berlin, 1983.

7. C. H. FitzGerald, The Bieberbach conjecture: retrospective. Notices, Amer. Math. Soc., 32 (1985) 2–6.

8. C. H. FitzGerald and C. Pommerenke, The de Branges theorem on univalent functions, Trans. Amer. Math. Soc., 290 (1985) 683–690.

9. W. Gautschi, Reminiscences of my involvement in de Branges' proof of the Bieberbach conjecture, to appear.

10. G. M. Goluzin, Geometric theory of functions of a complex variable. English transl., Amer. Math. Soc., Providence, R.I., 1969.

11. A. W. Goodman, Univalent Functions, vols. I, II, Mariner, Tampa, FL, 1983.

12. K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I, Math. Ann., 89 (1923) 103–121.

13. I. M. Milin, Univalent functions and orthonormal systems. English trans., Amer. Math. Soc., Providence, RI, 1977.

14. C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

15. ______, The Bieberbach conjecture, Math. Intelligencer, 7 (1985) 23–25, 32.

NON-SEXIST SOLUTION OF THE MÉNAGE PROBLEM

KENNETH P. BOGART

Department of Mathematics, Dartmouth College, Hanover, NH 03755

PETER G. DOYLE

AT & T Bell Labs, Murray Hill, NJ 07974

1. The ménage problem. The *ménage problem* (problème des ménages) asks for the number M_n of ways of seating n man-woman couples at a circular table, with men and women

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Kenneth Bogart received his B.S. in Mathematics from Marietta College in 1965 and his Ph.D. in Mathematics from CalTech in 1968. He has been teaching at Dartmouth College, where he is a professor of Mathematics and Computer Science, ever since. His research interests are in the applications of algebra to combinatorics, and the applications of combinatorics, especially to Computer Science. Much of his research has involved the theory or applications of ordered sets. His textbook, *Introductory Combinatorics*, was published in 1983 by Pitman Publishing Company. He is currently writing a textbook on discrete mathematics.

Peter Doyle got his Ph.D. in mathematics from Dartmouth College in 1982. While he was at Dartmouth, he and Laurie Snell wrote a book called *Random Walks and Electric Networks* which was published by the MAA in 1984 in the Carus Mathematical Monographs series. After getting his degree he worked as a systems programmer for a year, and spent a year at the Institute for Mathematics and its Applications in Minneapolis. He now works in the Mathematical Sciences Research Center at AT & T Bell Labs in Murray Hill, NJ.