$(a^2+b^2)^{1/2}M<1$ is sufficient. But $(a^2+b^2)^{1/2}=|dx/ds|=\kappa$, the curvature of C. Hence a sufficient condition is

$$M < \inf_s \rho(s).$$

where $\rho(s)$ is the radius of curvature.

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THE SIMPLE CONTINUED FRACTION EXPANSION OF e

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1. Introduction. Students, and others, have asked where they can find a readable account of the simple continued expansion of e, namely, the beautiful result due to Euler [1] that

$$\text{(1)}\qquad \begin{aligned} e &= 2+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots \\ &= \langle 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, \cdots\rangle. \end{aligned}$$

Simple continued fractions have the form

$$\text{(2)}\qquad a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\cfrac{1}{a_4+\cdots}}},$$

where a_1 is usually a positive or negative integer (but could be zero), and where the terms $a_2, a_3, a_4, \cdots$ are positive integers. It is convenient to write (2) in the form

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\,\cdots,$$

with the "+" signs after the first one lowered to indicate the "step-down" process in forming a continued fraction; or simply to represent it by the symbol

$\langle a_1, a_2, a_3, \cdots \rangle$. To understand the continued fraction part of this expository paper one need only read, say, a few pages of Chapter 7 in Niven and Zuckerman [4]; noting, in particular, that the convergents of the continued fraction (2), namely,

$$c_1 = \frac{a_1}{1} = \frac{p_1}{q_1}, \; c_2 = a_1 + \frac{1}{a_2} = \frac{a_1a_2 + 1}{a_2} = \frac{p_2}{q_2}, \cdots, \tag{3}$$

can be calculated, successively, from the equations

$$\begin{aligned} p_i &= a_ip_{i-1} + p_{i-2}, \\ q_i &= a_iq_{i-1} + q_{i-2}, \end{aligned} \tag{4}$$

for $i = 1, 2, 3, 4, \cdots$, provided the *undefined* terms which occur are assigned the values $p_{-1} = 0$, $p_0 = 1$, $q_{-1} = 1$, $q_0 = 0$.

Euler derived the expansion (1) by converting the infinite series expansion of e into a continued fraction, an effective method when it succeeds. One does not always end up with a *simple* continued fraction. See Wall [6, p. 17].

Since Euler's time, mathematicians such as Lambert, Gauss, Liouville, Hurwitz, Stieltjes, to mention only a few, established continued fractions as a field worthy of independent study, with applications to many branches of mathematics. It comes as no surprise, then, that the expansion (1), and similar ones, are special cases of later developments of the subject.

However, for historical reasons, we present Hermite's derivation of (1). The main ideas are contained in his famous paper [2] in which he gave the first proof that e is a transcendental number. Hermite needed approximations to e and its integral powers, and, as a matter of fact, those he used were not convergents to their respective continued fractions. What follows, then, is not a mere translation of what Hermite wrote, but, rather, a re-working of his ideas, with changes and additions to make a self-contained exposition starting with the integral (5), given below, and ending with (1).

2. Hermite's Method. Hermite starts with an integral

$$\int e^{-rx} f(x)\, dx, \tag{5}$$

where $r \neq 0$ is an arbitrary constant, and where $f(x)$ is a polynomial of degree $n = 2m$. Repeated integration by parts transforms (5) into the form

$$\begin{aligned} \int e^{-rx} f(x) dx &= -\frac{1}{r} e^{-rx} f(x) + \frac{1}{r} \int f^{(1)}(x) e^{-rx} dx \\ &= -\frac{1}{r} e^{-rx} f(x) - \frac{1}{r^2} e^{-rx} f^{(1)}(x) + \frac{1}{r^2} \int f^{(2)}(x) e^{-rx} dx \end{aligned} \tag{6}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= -e^{-rx}\left(\frac{1}{r}f(x) + \frac{1}{r^2}f^{(1)}(x) + \cdots + \frac{1}{r^{2m+1}}f^{(2m)}(x)\right)$$
$$= -e^{-rx}\Phi(x).$$

Hence,

$$\int_0^1 e^{-rx}f(x)dx = -e^{-rx}\Phi(x)\Big]_0^1 = \Phi(0) - e^{-r}\Phi(1). \tag{7}$$

In (7) let $f(x) = x^m(x-1)^m$, then

$$f^{(j)}(0) = f^{(j)}(1) = 0, \qquad j = 0, 1, 2, \cdots, m-1,$$

consequently, the expression $\Phi(x)$ in (6) reduces for $x=0$ and $x=1$, respectively, to

$$\Phi(0) = \sum_{j=m}^{2m} \frac{f^{(j)}(0)}{r^{j+1}}, \qquad \Phi(1) = \sum_{j=m}^{2m} \frac{f^{(j)}(1)}{r^{j+1}}. \tag{8}$$

On the other hand, Taylor's expansion about $x=0$ shows that

$$f(x) = \sum_{j=m}^{2m} \frac{f^{(j)}(0)}{j!} x^j = \sum_{j=m}^{2m} \alpha_j x^j, \tag{9}$$

where $f^{(j)}(0) = j!\alpha_j$. Since $f(x)$ is a polynomial with integral coefficients, the α_j's are integers. Similarly, writing $f(x) = [(x-1)+1]^m(x-1)^m$, Taylor's expansion about $x=1$, shows that

$$f(x) = \sum_{j=m}^{2m} \frac{f^{(j)}(1)}{j!}(x-1)^j = \sum_{j=m}^{2m} \beta_j(x-1)^j, \tag{10}$$

where $f^{(j)}(1) = j!\beta_j$, the β_j's being integers. Hence, the equations (8) take the form

$$\begin{aligned} \Phi(0) &= \sum_{j=m}^{2m} \frac{j!\alpha_j}{r^{j+1}} = \frac{m!M_m(r)}{r^{2m+1}}, \\ \Phi(1) &= \sum_{j=m}^{2m} \frac{j!\beta_j}{r^{j+1}} = \frac{m!N_m(r)}{r^{2m+1}}, \end{aligned} \tag{11}$$

where $M_m(r)$ and $N_m(r)$ are polynomials of degree m in r with integral coefficients. Using (11), with $f(x) = x^m(x-1)^m$, we rewrite (7) in the form

$$e^rM_m(r) - N_m(r) = \frac{r^{2m+1}e^r}{m!}J_m = V_me^r, \tag{12}$$

where

$$J_m = \int_0^1 e^{-rx}x^m(x-1)^m dx, \qquad V_m = \frac{r^{2m+1}}{m!}J_m. \tag{13}$$

Setting $m=0$, $f(x)=1$, $\Phi(x)=r^{-1}$, and so $\Phi(0)=r^{-1}$, $M_0(r)=1$, $N_0(r)=1$. Similarly, for $m=1$, $f(x)=x^2-x$, and so $M_1(r)=2-r$, $N_1(r)=2+r$.

The crucial part of Hermite's development hinges on a relationship between J_m, J_{m-1}, and J_{m-2}, $m\geqq 2$, of the form $J_m+aJ_{m-1}+bJ_{m-2}=0$. To this end, integration by parts shows that

$$J_m = \frac{m}{r}\int_0^1 e^{-rx}x^{m-1}(x-1)^m dx + \frac{m}{r}\int_0^1 e^{-rx}x^m(x-1)^{m-1}dx. \tag{14}$$

In the first integral above replace $(x-1)^m$ by $(x-1)^{m-1}(x-1)$ to obtain

$$\int_0^1 e^{-rx}x^m(x-1)^{m-1}dx = \frac{r}{2m}J_m + \frac{1}{2}J_{m-1}, \tag{15}$$

and with m replaced by $m-1$,

$$\int_0^1 e^{-rx}x^{m-1}(x-1)^{m-2} = \frac{r}{2(m-1)}J_{m-1} + \frac{1}{2}J_{m-2}. \tag{16}$$

A second integration by parts shows that

$$\int_0^1 e^{-rx}x^m(x-1)^{m-1}dx = \frac{m}{r}J_{m-1} + \frac{m-1}{r}\int_0^1 e^{-rx}x^m(x-1)^{m-2}dx. \tag{17}$$

On the other hand, since $(x-1)^{m-1}=x(x-1)^{m-2}-(x-1)^{m-2}$, it follows that

$$J_{m-1} = \int_0^1 e^{-rx}x^m(x-1)^{m-2}dx - \int_0^1 e^{-rx}x^{m-1}(x-1)^{m-2}dx. \tag{18}$$

Using (16), and then solving (18) for the first integral on the right, we get

$$\int_0^1 e^{-rx}x^m(x-1)^{m-2}dx = J_{m-1} + \frac{r}{2(m-1)}J_{m-1} + \frac{1}{2}J_{m-2}. \tag{19}$$

Now, substitute (19) into (17), and follow this by equating the right side of (15) with the new form of (17). The result, after simplifications, is

$$r^2J_m - 2m(2m-1)J_{m-1} - m(m-1)J_{m-2} = 0. \tag{20}$$

Using the relationship between V_m and J_m given in (13), we can replace (20) by

$$V_m - (4m-2)V_{m-1} - r^2V_{m-2} = 0. \tag{21}$$

Next, calculate V_m, V_{m-1}, V_{m-2} from (12) and substitute into (21) to obtain

$$\begin{aligned} &e^r[M_m(r) - (4m-2)M_{m-1}(r) - r^2M_{m-2}(r)] \\ &\qquad - [N_m(r) - (4m-2)N_{m-1}(r) - r^2N_{m-2}(r)] = 0, \end{aligned} \tag{22}$$

and since e^r is irrational this demands that

$$\begin{aligned} M_m(r) - (4m-2)M_{m-1}(r) - r^2M_{m-2}(r) &= 0, \\ N_m(r) - (4m-2)N_{m-1}(r) - r^2N_{m-2}(r) &= 0. \end{aligned} \tag{23}$$

From (11) we know that $M_m(r)$ and $N_m(r)$ are polynomials of degree m in r. Hence, if we replace r by $2/k$, where k is a positive integer, then these polynomials can be written in the form

$$M_m(2/k) = S_m/k^m, \qquad N_m(2/k) = R_m/k^m, \tag{24}$$

where S_m and R_m are integers. Next, substitute the values of M_m and N_m from (24) into (12), use (13) to get

$$e^{2/k}S_m - R_m = \frac{2^{2m+1}e^{2/k}}{k^{m+1}m!}\int_0^1 e^{-2x/k}x^m(x-1)^m dx.$$

Since $|x(x-1)| \leqq 1/4$ if $0 \leqq x \leqq 1$, and since $\int_0^1 e^{-2x/k}dx < k/2$, it follows easily that

$$\left| e^{2/k}S_m - R_m \right| < \frac{e^{2/k}}{k^m m!}\cdot \tag{25}$$

Again, using (24) the relations (23) can be replaced by

$$\begin{aligned} S_m &= (4m-2)kS_{m-1} + 4S_{m-2}, \\ R_m &= (4m-2)kR_{m-1} + 4R_{m-2}. \end{aligned} \tag{26}$$

If, for convenience, we let

$$\begin{aligned} S_m + R_m &= 2^{m+1}\, T_m, \\ S_m - R_m &= -\, 2^{m+1}\, Z_m, \end{aligned} \tag{27}$$

then the recurrence relations (26) can be replaced by

$$\begin{aligned} T_m &= (2m-1)kT_{m-1} + T_{m-2}, \\ Z_m &= (2m-1)kZ_{m-1} + Z_{m-2}, \end{aligned} \tag{28}$$

where, in particular,

$$\begin{aligned} &T_0 = 1,\ T_1 = k,\ T_2 = 3k^2 + 1,\ T_3 = 15k^3 + 6k, \cdots, \\ &Z_0 = 0,\ Z_1 = 1,\ Z_2 = 3k,\ Z_3 = 15k^2 + 1, \cdots. \end{aligned}$$

Referring to equations (4), this shows that T_0/Z_0, T_1/Z_1, T_2/Z_2, $\cdots$, are the convergents to the simple continued fraction $\langle k, 3k, 5k, 7k, \cdots \rangle$. Moreover, from (27) we have

$$S_m = 2^m(T_m - Z_m),\ R_m = 2^m(T_m + Z_m),$$

which transforms the inequality (25) into the form

$$\left| (e^{2/k} - 1)T_m - (e^{2/k} + 1)Z_m \right| < \frac{e^{k/2}}{(2k)^m m!},$$

and after dividing both sides by $Z_m(e^{2/k}-1)$, and noting by the recursion formula (28) that Z_m increases as m increases, we see that

$$\left|\frac{e^{2/k}+1}{e^{2/k}-1}-\frac{T_m}{Z_m}\right|<\frac{e^{2/k}}{(2k)^m m!Z_m(e^{2/k}-1)}\to 0$$

as $m\to\infty$. We are justified, then, in writing as a simple continued fraction

$$\frac{e^{2/k}+1}{e^{2/k}-1}=k+\frac{1}{3k+}\frac{1}{5k+}\frac{1}{7k+}\cdots=\langle k,3k,5k,7k,\ldots\rangle. \tag{29}$$

Subtracting 1 from both sides of (29) gives

$$\frac{2}{e^{2/k}-1}=(k-1)+\frac{1}{3k+}\frac{1}{5k+}\frac{1}{7k+}\cdots. \tag{30}$$

This can give excellent approximations to e. For example, if $k=2$, then from (30) we get

$$\frac{e-1}{2}=\frac{1}{1+\frac{1}{6+}\frac{1}{10+}\cdots}. \tag{31}$$

Using the 7th convergent to (31), we have $(e-1)/2\approx 342762/398959$, which shows that $e\approx 2.718281828458\cdots$, in error by only one unit in the last place.

From (31) we see that

$$e=1+\frac{2}{1+\frac{1}{6+}\frac{1}{10+}\frac{1}{14+}\cdots}, \tag{32}$$

and it is quite easy to transform this into a simple continued fraction by the use of two transformations. Write

$$\frac{2}{a+\cfrac{1}{b+\cfrac{1}{c+\cdots}}}=\frac{2}{a+\cfrac{1}{b+\cfrac{1}{y}}}.$$

Then it follows easily that if a is even,

$$\frac{2}{a+\cfrac{1}{b+\cfrac{1}{y}}}=\frac{1}{\cfrac{a}{2}+\cfrac{1}{2b+\cfrac{2}{y}}},$$

and if a is odd, that

$$\frac{2}{a+\cfrac{1}{b+\cfrac{1}{y}}} = \frac{1}{\cfrac{(a-1)}{2}+\cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{(b-1)+\cfrac{1}{y}}}}}.$$

Thus, the continued fraction (32) transforms easily into Euler's result:

$$(33) \qquad e = 2 + \frac{1}{1\,+}\,\frac{1}{2\,+}\,\frac{1}{1\,+}\,\frac{1}{1\,+}\,\frac{1}{4\,+}\cdots = \langle 2, \overline{1, 2n, 1}\rangle_{n=1}^{\infty}$$

where the "bar" indicated the periodic part of the fraction.

A similar discussion would show that

$$(34) \qquad e^2 = \langle 7, \overline{3n-1, 1, 1, 3n, 6(2n+1)}\rangle_{n=1}^{\infty} \quad ;$$

and that for $k \geqq 3$ and odd

$$(35) \qquad e^{2/k} = \langle \overline{1, \tfrac{1}{2}[(6n+1)k-1], 6k(2n+1), \tfrac{1}{2}[(6n+5)k-1], 1}\rangle_{n=1}^{\infty} \quad .$$

3. Conclusion. The continued fraction (29) was published by Lambert in 1761. For historical data on continued fractions see Perron [5]. For the technique for proving e transcendental, see Chapter 2 and 9 in Niven [3].

What gave Hermite the idea to start with the integral (5)? A partial answer can be found by thumbing through his previous papers to see his great skill in handling various difficult integrals involving transcendental functions. One soon senses that he must already have had in mind all the basic techniques needed to prove that e is transcendental.

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