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Weird Dice

Imagine you are in the middle of a game of monopoly and someone substitutes a pair of dice labeled with 1, 2, 2, 3, 3, 4 and 1, 3, 4, 5, 6, 8 for the standard dice. Does this change the game in any way?

The first thing that comes to mind is that the probabilities for the various sums might be different. Suprisingly, this is not the case. Figure 1 reveals that the frequencies of the possible sums are exactly the same for either pair of dice (1 way to roll a sum of 2; 2 ways to roll a sum of 3, etc.)

Because of this one might conclude that the change in dice does not effect the game. But recall that one of the rules of monopoly is that if a player is in jail (not visiting) and he/she rolls doubles then the player advances the number of spaces shown on the dice. With the standard dice each of the sums 2, 4, 6, 8, 10 and 12 has a 1/36 chance of occurring as the result of a double. With the weird dice the sums 2 and 8 have a 1/36 chance of occurring; the sums 4, 10 and 12 have a 0 chance; and the sum 6 has a 1/18 chance.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

	1	2	2	3	3	4
1	2	3	3	4	4	5
3	4	5	5	6	6	7
4	5	6	6	7	7	8
5	6	7	7	8	8	9
6	7	8	8	9	9	10
8	9	10	10	11	11	12

Consequently, a person with hotels on the orange property St. James Place (6 spaces past jail) is better off with the weird dice! Moreover, a person that owns the maroon property Virginia (4 spaces past jail) is worst off. In fact, this small change in the probabilities for certain doubles using weird dice causes St. James Place to move up from the 10th most frequently landed space to the 6th most landed on space whereas Virginia drops from 24th place to 27th place in the rankings.

These weird dice raise two interesting mathematical questions: How were the labels derived and are there other weird labels consisting of positive integers? It is possible to answer these questions with a simple analysis. To do so we begin by finding a way to model summing the faces of a pair of dice.

First, let us ask ourselves how we may obtain a sum of 6, say, with an ordinary pair of dice. Well, there are five possibilities for the two faces: (5,1), (4,2), (3,3), (2,4) and (1,5). Next, we consider the product of the two polynomials created by using the ordinary dice labels as exponents:

$$(x^6+x^5+x^4+x^3+x^2+x)$$
$$\times(x^6+x^5+x^4+x^3+x^2+x).$$

Observe that using the distributive property we pick up the term x^6 in this produce in precisely the following ways: $x^5 \cdot x^1$, $x^4 \cdot x^2$, $x^3 \cdot x^3$, $x^2 \cdot x^4$, $x^1 \cdot x^5$. Notice that the correspondence between pairs of labels whose sums are 6 and pairs of terms whose products are x^6. This correspondence is one-to-one, and it is valid for all sums and all dice—including our weird dice and any other dice that yield the desired probabilities. So, let a_1, a_2, a_3, a_4, a_5, a_6 and b_1, b_2, b_3, b_4, b_5, b_6 be any two lists of positive integer labels for a pair of cubes with the property that the probability of rolling any particular sum with these dice is the same as the probability of rolling that sum with ordinary dice labeled 1 through 6. Using our observation about products of polynomials, this means that

$$(x^6+x^5+x^4+x^3+x^2+x)$$
$$\times(x^6+x^5+x^4+x^3+x^2+x)$$

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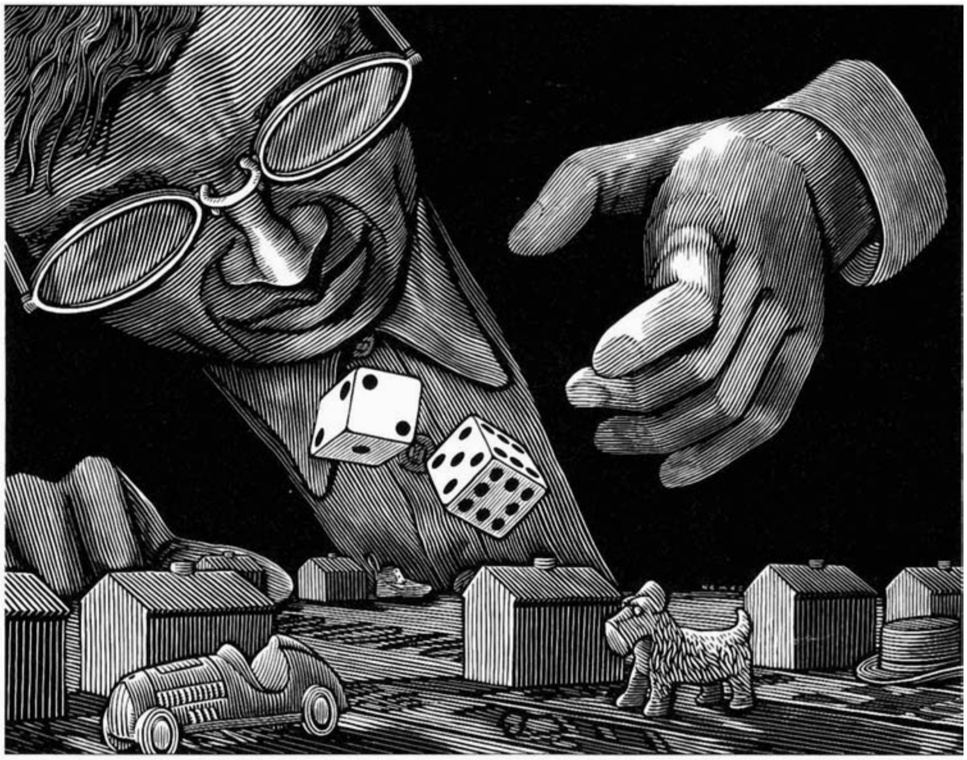

Illustration by Greg Nemec

$$= (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})$$
$$\times (x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}).$$

Now all we have to do is solve this equation for the a's and b's. How can we solve one equation in 8 unknowns? You didn't learn this in high school but you did spend much of your efforts factoring. So, let's factor the left-hand side of the equation. The polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x$ factors uniquely into irreducibles as

$$x(x+1)(x^2+x+1)(x^2-x+1)$$

so that the left-hand side of the equation has the irreducible factorization

$$x^2(x+1)^2(x^2+x+1)^2(x^2-x+1)^2.$$

This means that these factors are the only possible irreducible factors of $P(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$. Thus, $P(x)$ has the form

$$x^q(x+1)^r(x^2+x+1)^s(x^2-x+1)^t,$$

where $0 \leq q, r, s, t \leq 2$.

To further restrict the possibilities for q, r, s, and t we evaluate $P(1)$ in both forms for $P(x)$. That is,

$$P(1) = 1^{a_1} + 1^{a_2} + \cdots + 1^{a_6} = 6$$

and

$$P(1) = 1^q 2^r 3^s 1^t.$$

Clearly, this means that $r = 1$ and $s = 1$. What about q? Evaluating $P(0)$ in two ways just as we did for $P(1)$ shows that $q \neq 0$. On the other hand, if $q = 2$, the smallest possible sum one could roll with the corresponding labels for dice would be 3 since q is the smallest of the a's and the smallest permissible b is 1. Because the sum of 2 is possible with ordinary dice this violates our assumption about the dice having equal probabilities. Thus, $q = 2$ is not permissible. We have now reduced our list of possibilities for q, r, s, and t to $q = 1$, $r = 1$, $s = 1$, and $t = 0, 1, 2$. Let's consider each of these possibilities for t in turn.

When $t = 0$,

$$P(x) = x^4 + x^3 + x^3 + x^2 + x^2 + x,$$

so the die labels are 4, 3, 3, 2, 2, 1—one of our weird dice.

When $t = 1$,

$$P(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x,$$

so the die labels are 6, 5, 4, 3, 2, 1—an ordinary die.

When $t = 2$,

$$P(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x,$$

so the die labels are 8, 6, 5, 4, 3, 1—the other weird die.

Thus we have *derived* the weird dice labels and *proved* that they are the *only* other pair of dice that have this property.

We invite the reader to investigate the analogous questions for the tetrahedron, octahedron, dodecahedron, and icosahedron dice. For the answers see [1] and [2].

References

1. Broline, Duane. (1979). "Renumbering the faces of dice," *Mathematics Magazine,* vol. 52, 312–315.

2. Gallian, Joseph A. and David Rusin. (1979). "Cyclotomic polynomials and nonstandard dice," *Discrete Mathematics,* vol. 27, 245–259.