TOM M. APOSTOL

What Is the Most Surprising

The first installment of this article described how one could conjecture the prime number theorem by examining a table of primes. The prime number theorem states that

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

Here $\pi(x)$ is the number of primes $\leq x$, and $\log x$ is the natural logarithm of x. Gauss and Legendre conjectured this theorem after studying tables of prime numbers, but neither made any progress toward obtaining a proof. The first positive step was made by Chebyshev in 1849 when he showed that if the ratio $\pi(x)(\log x)/x$ has a limit as $x \to \infty$, then this limit must equal 1.

The next significant step was made by G. F. B. Riemann in 1859. He attacked the problem with a new method, using a formula that Euler had discovered more than a century earlier in 1737 relating prime numbers and the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Every beginning calculus student learns about this series in studying convergence tests. The series converges for $s > 1$ and diverges for $s \leq 1$. Euler discovered that this series could also be expressed as an infinite product extended over all the primes as follows:

$$\zeta(s) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-s}}$$

also written more briefly as

$$\prod_p \frac{1}{1-p^{-s}},$$

where the product runs through all the primes p. Displaying the first three factors corresponding to the primes 2, 3, and 5 we have

$$\zeta(s) = \frac{1}{1-2^{-s}}\,\frac{1}{1-3^{-s}}\,\frac{1}{1-5^{-s}}\cdots$$

It is not hard to see why Euler's formula is true. The key is our old friend the geometric series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

which is also familiar to every freshman. The series converges if $|x| < 1$. If we expand each factor in the infinite product as a geometric series, taking $x = \frac{1}{p^s}$, we get

$$\frac{1}{1-2^{-s}}\frac{1}{1-3^{-s}}\frac{1}{1-5^{-s}} = \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\times \left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\cdots\right)\cdots.$$

When you multiply all these series together and arrange the terms according to increasing denominators you end up with $\sum_{n=1}^{\infty} \frac{1}{n^s}$ because of the fundamental theorem of arithmetic, which states that every integer greater than 1 can be factored in one and only one way as a product of prime powers, apart from the order of the factors. (There are some delicate questions of convergence here because we are multiplying together an infinite number of infinite series. Euler ignored these questions but the steps can be justified.)

Euler didn't do much with this formula, but Riemann realized that it had possibilities because the product on the right involves only primes. Riemann's main contribution was to replace s by a complex variable and to connect properties of the complex-valued function $\zeta(s)$ to the distribution of prime numbers. Because he did so much with the function $\zeta(s)$ it is now called the **Riemann zeta function.**

Riemann came close to proving that $\pi(x)(\log x)/x$ approaches 1 as a limit, but didn't succeed. In fact, in his lifetime not enough was known about the theory of functions of a complex variable to successfully carry out his ideas.

Thirty years later the necessary analytic tools were at hand, and in 1896 the French mathematician Jacques-Salomon Hadamard and the Belgian mathematician Charles-Jean de la Vallée Poussin independently and almost simultaneously succeeded in proving that

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1. \tag{1}$$

This remarkable accomplishment was one of the crowning achievements of a new branch of mathematics called **analytic number theory,** where methods of calculus and analysis are brought to bear on problems concerning the integers.

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Result in Mathematics? Part II

Some applications of the prime number theorem

In the first installment of this article we asked: **How does the nth prime p_n grow as a function of n?** The prime number theorem enables us to answer this question. Start with the prime number theorem as stated in (1) and take logarithms of both sides of the equation. We obtain

$$\lim_{x\to\infty}\{\log\pi(x)+\log\log x-\log x\}=0,$$

or

$$\lim_{x\to\infty}\left\{\log x\left(\frac{\log\pi(x)}{\log x}+\frac{\log\log x}{\log x}-1\right)\right\}=0.$$

Since $\log x\to\infty$ the last factor multiplying $\log x$ must tend to 0, so

$$\lim_{x\to\infty}\left(\frac{\log\pi(x)}{\log x}+\frac{\log\log x}{\log x}-1\right)=0.$$

The quotient $\frac{\log\log x}{\log x}$ tends to 0, hence

$$\lim_{x\to\infty}\frac{\log\pi(x)}{\log x}=1.$$

Multiply this relation by (1), cancel $\log x$, and we get

$$\lim_{x\to\infty}\frac{\pi(x)\log\pi(x)}{x}=1.$$

Now let $x=p_n$, so that $\pi(x)=n$. Then the last formula becomes

$$\lim_{n\to\infty}\frac{n\log n}{p_n}=1.$$

In other words, the nth prime p_n is asymptotically equal to $n\log n$ as $n\to\infty$. This is the answer to our original question. **For large n the nth prime grows like $n\log n$.** It can be shown that this also implies the prime number theorem, so it is logically equivalent to the prime number theorem.

As a consequence we see that the sum of the reciprocals of the primes $\sum\frac{1}{p_n}$ diverges because its terms are asymptotic to $1/(n\log n)$, and $\sum\frac{1}{n\log n}$ is a well known divergent series. However, you don't need the full power of the prime number theorem to prove this result, which was known to Euler. There are many direct proofs that the sum $\sum\frac{1}{p}$ diverges. The proof in the next section also displays the order of growth of its partial sums. This proof reveals the interplay between calculus and number theory, and gives you a glimpse into the methods of analytic number theory.

The partial sums of the series of reciprocals of the primes

Let S_n denote the sum of the reciprocals of the first n primes:

$$S_n=\sum_{k=1}^{n}\frac{1}{p_k}$$

We'll show that $S_n\to\infty$ as $n\to\infty$. The asymptotic value of S_n for large n is known to be $\log(\log n)$ (see Apostol, *Introduction to Analytic Number Theory,* Theorem 4.12.) By contrast, the asymptotic value of the nth partial sum of the harmonic series is $\log n$. This is easily deduced from the inequalities

$$\log(n+1)<\sum_{k=1}^{n}\frac{1}{k}<1+\log n,$$

which follow by comparing the area of a hyperbolic segment with inscribed and circumscribed rectangles.

Using only a few basic ideas from elementary calculus we can show that

$$S_n>\log(\log n)-1. \qquad (2)$$

Although this is not as strong as saying that S_n is asymptotic to $\log(\log n)$, it does show that $S_n\to\infty$ as $n\to\infty$. The proof requires a knowledge of the sum of the geometric series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n, \qquad (3)$$

valid for $|x|<1$, and the inequalities

$$\sum_{k=1}^{n}\frac{1}{k}>\log n, \qquad (4)$$

and

$$-\log(1-x)\le x+x^2 \quad \text{if } 0<x\le\tfrac{1}{2}. \qquad (5)$$

Inequality (5) follows easily from the power series expansion for $-\log(1-x)$, obtained by integrating (3), but it can also be deduced at once from the equation

$$-\log(1-x)=\int_0^x\frac{1}{1-t}dt.$$

Courtesy of The Huntington Library, San Marino, California.

Very little is known of the life of Euclid, who flourished around 300 B.C. and whose 13-volume *Elements* distills most of the mathematical wisdon of his day. Gauss deduced the fundamental theorem of arithmetic from Proposition 30 in Book 7. Proposition 20 in Book 9 of the *Elements* states that there are infinitely many primes. Many proofs of this theorem exist, but Euclid's original proof is the most elegant. He founded a school at Alexandria, in Egypt, and was a personal tutor to King Ptolemy I. When asked by Ptolemy if there was no shorter way to learn geometry than reading all 13 books, Euclid is said to have replied, "There is no royal road to geometry."

Just estimate the integrand. Simply note that if $0 < t \leq \frac{1}{2}$, then $0 < 2t \leq 1$ so $1 - 2t \geq 0$ and $t(1-2t) \geq 0$. Hence $1 + t - 2t^2 \geq 1$, or $(1-t)(1+2t) \geq 1$. This gives us the inequality

$$\frac{1}{1-t} \leq 1 + 2t.$$

Integrating this last inequality from 0 to x gives us inequality (5) if $0 < x \leq \frac{1}{2}$.

The proof also uses the fundamental theorem of arithmetic, which states that every positive integer can be factored uniquely (apart from order) as a product of prime powers, and the simple inequality $p_n > n$, which follows from the fact that not all integers are primes.

To prove inequality (2) we consider the finite product

$$P(n) = \prod_{k=1}^{n} \frac{1}{1 - \dfrac{1}{p_k}}.$$

This is like the Euler product for the Riemann zeta function, but it involves only a finite number of factors. Each factor in the product $P(n)$ can be expressed as a geometric series

$$\frac{1}{1 - \dfrac{1}{p_k}} = 1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots$$

and hence

$$P(n) = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right) \times \left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right) \cdots \left(1 + \frac{1}{p_n} + \frac{1}{p_n^2} + \cdots\right)$$

This is a product of a finite number of absolutely convergent series. When the series are multiplied together and the terms are rearranged according to increasing denominators we find

$$P(n) = \sum_{k \in T_n} \frac{1}{k},$$

where, because of the fundamental theorem of arithmetic,

$$T_n = \{k : \text{ all prime factors of } k \text{ are } \leq p_n\}.$$

Now T_n includes all positive integers $\leq p_n$ because if $k \leq p_n$ then every prime factor of k is $\leq p_n$. Hence

$$P(n) = \sum_{k \in T_n} \frac{1}{k} \geq \sum_{k=1}^{p_n} \frac{1}{k} > \log p_n > \log n.$$

We'll use this later. Now let's take the logarithm of the product defining $P(n)$. We get

$$\log P(n) = \sum_{k=1}^{n} -\log\left(1 - \frac{1}{p_k}\right)$$

At this stage we use the inequality in (5) with $x = 1/p_k$ and sum on k to obtain

$$\begin{aligned} \log P(n) &= \sum_{k=1}^{n} -\log\left(1 - \frac{1}{p_k}\right) \\ &\leq \sum_{k=1}^{n} \frac{1}{p_k} + \sum_{k=1}^{n} \frac{1}{p_k^2} < S_n + \sum_{k=2}^{n} \frac{1}{k^2} \\ &< S_n + \int_1^\infty \frac{1}{t^2} dt = S_n + 1. \end{aligned}$$

In other words, $\log P(n) < S_n + 1$, so

$$S_n > \log P(n) - 1.$$

But $P(n) > \log n$, so $S_n > \log(\log n) - 1$.

Elementary proof of the prime number theorem

The first proof of the prime number theorem given by Hadamard and de la Vallée Poussin was simplified by Landau and others in the early part of the 20th century, and new proofs were later discovered, all using sophisticated methods of real and complex analysis. In 1949 Atle Selberg and Paul Erdös discovered an elementary proof that makes no use of complex function theory. And in 1956 one of my PhD students, Basil Gordon, showed that you could also deduce the prime number theorem in an elementary way from Stirling's formula for $n!$. If you take the logarithm of Stirling's inequality you get the relation

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \log\sqrt{2\pi} + O\left(\frac{1}{n}\right)$$

This implies a weaker asymptotic formula:

$$\log n! = n \log n - n + o\left(\frac{n}{\log n}\right).$$

Gordon showed that this asymptotic formula for $\log n!$ implies the prime number theorem.

A probabilistic argument involving primes

In discussing the ubiquity of π we mentioned the following result:

The probability that two integers chosen at random have no prime factor in common is $6/\pi^2$.

Here's a heuristic argument that explains where this result comes from. Given an integer x and a prime p , divide x by p to get a quotient and a remainder x_p . If p divides x the remainder $x_p = 0$. If not, the remainder is one of the numbers $1, 2, \ldots, p-1$. Choose another integer y, and do the same to get a remainder y_p . There are p^2 possible pairs (x_p, y_p).

The pair $(x_p, y_p) = (0, 0)$ if and only if both x and y are divisible by p. So, the probability that both x and y are divisible by the same prime p is $1/p^2$. The complementary event is that at least one of x or y is not divisible by p, and the probability of the complementary event is $1 - 1/p^2$. This event depends on the prime p. It's reasonable to assume that for different primes these events are independent, so the probability that no prime divides both x and y is the product of all these probabilities:

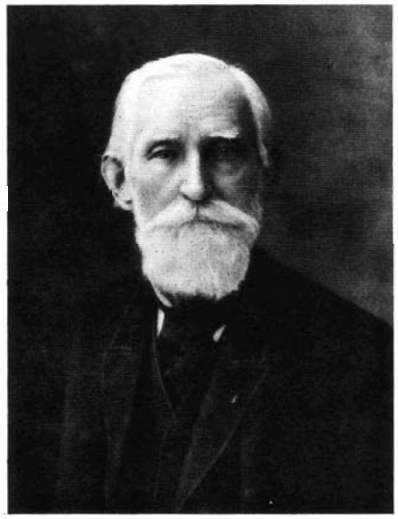
$$\prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

This formula reveals another surprise: the ubiquitous π has a connection with prime numbers!

Concluding Remarks

The prime number theorem is important not only because it makes an elegant and simple statement about primes and has many applications but also because much new mathematics was created in the attempts to find a proof. This is typical in number theory. Some problems, very simple to state, are often extremely difficult to solve, and mathematicians working on these problems often create new areas of mathematics of independent interest. Another example is the Fermat conjecture, which has received more publicity as an unsolved problem than any other result in mathematics. But Gauss himself considered the Fermat conjecture to be of only minor importance in mathematics and re-

Leonhard Euler (1703–1783) lost the use of his right eye to overwork when only 28. A cataract robbed him of his other eye at age 51, but his work continued undiminished with the assistance of his sons, an excellent memory, and a remarkable knack for mental computation.



Pafnuty Lvovich Chebyshev (1821–1894) was fascinated by mechanical toys as a boy. His quest to understand machinery led to an interest in geometry and ultimately to the rest of mathematics.

Charles-Jean de la Vallée Poussin (1866–1962) studied religion and engineering successively before turning to mathematics.

fused to work on it. The best thing that can be said about Fermat's last theorem is that it is largely responsible for the theory of algebraic numbers, which was created in an attempt to prove the Fermat conjecture.

Perhaps the most surprising result **about** mathematics is the profound impact that a few unsolved problems have had on the development of mathematics. The Fermat conjecture is just one of countless examples of mathematical problems that attract the intellectual curiosity of many individuals but resist efforts at solution. Repeated failure by eminent mathematicians to settle these problems by known procedures stimulates the invention of new methods, new approaches, and new ideas that, in time, become part of the mainstream of mathematics and even change the way mathematicians think about their subject. This is certainly true of the prime number theorem. Early attempts to prove the prime number theorem stimulated the development of the theory of functions of a complex variable, a branch of mathematics that is the life blood of mathematical analysis. Efforts to prove Fermat's last theorem led to the development of algebraic number theory, one of the most active areas of modern mathematical research, with ramifications far beyond the Fermat equation. One unexpected application of algebraic number theory is in designing security systems for computers.

In number theory alone there are hundreds of unsolved problems. New problems arise more rapidly than the old ones are solved, and many of the old ones have remained unsolved for centuries. Progress of our knowledge of numbers is advanced not only by what we already know about them, but also by realizing that there is much we do not know about them.

Addendum: Distribution of primes in arithmetic progressions

An arithmetic progression of integers with first term h and common difference k consists of all numbers of the form $kn+h$ as n runs through all the nonnegative integers $0, 1, 2, \ldots$. If h and k have a common prime factor p then each term of the progression is divisible by p and there can be no more than one prime in the progression. In 1803 Legendre considered whether there must be infinitely many primes in the progression if h and k have no common prime factor. For example, all odd numbers fall into two progressions, one containing numbers of the form $4n-1$, and the other containing numbers of the form $4n+1$, so at least one of these progression must contain infinitely many primes. In fact, both of these progressions contain infinitely many primes.

In a celebrated memoir published in 1837, Dirichlet showed that every arithmetic progression $kn+h$, where h and k have no prime factor in common, must contain infinitely many primes. Guided by Euler's proof of the infinitude of primes, Dirichlet used an ingenious argument to show that the sum of the reciprocals of all the primes in the progression $kn + h$ diverges. He did this by showing that the partial sums of this series containing the reciprocals of all primes $\leq x$ in the progression has the asymptotic value $\frac{1}{\varphi(k)} \log(\log x)$, where $\varphi(k)$ is the number of integers from 1 to k that have no prime factor in common with k. Since $\log(\log x) \to \infty$ as $x \to \infty$ this shows that the series diverges.

Dirichlet's proof of the infinitude of primes in arithmetical progressions was the first major triumph of analytic number theory. The ideas introduced in this proof laid the basis for areas of mathematical research that have had profound applications to both analytic and algebraic number theory. For example, there is a prime number theorem for arithmetic progressions first proved by de la Vallée Poussin. It states that the number of primes $\leq x$ in the progression $kn + h$ is asymptotic to $\frac{1}{\varphi(k)} \frac{x}{\log x}$.

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