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The Ultimate Flat Tire

The Flattest Wheel

How flat can a tire be and still roll? Can something as straight as a straight line be used as a wheel? Sure, if, as explained below, one uses some care in defining its center. The insight is due to G. B. Robison in 1960 [1]; he also realized that by suitably truncating a doubly infinite straight line one could form a square wheel, which would indeed roll on a properly shaped road.

To set the stage we must be precise about what "roll" means. A round circle rolls on a straight line in the sense that the center of the wheel stays horizontal. So for a non-circular wheel, we will say that it rolls on a curvy road if the center of the wheel moves in a horizontal line as the wheel moves without slipping along the road. Here I explain why a square rolls on a sequence of inverted catenaries. (Recall that a catenary is the curve made by a flexible chain allowed to hang with both ends held at the same height; its equation is simply $y = \cosh x = (e^x + e^{-x})/2$.)

As the Wheel Turns

Suppose the road is given as a function $y = f(x)$ and the wheel is described in polar coordinates, $r = r(\theta)$. An example is given in Figure 1, where $f(x)$ is $\cos(x) - \sqrt{17}$ and

$$r(\theta) = \cos\left[2\arctan\left[\frac{1}{4}\left(\sqrt{17}-1\right)\tan[2\theta]\right]\right] - \sqrt{17}.$$

As the wheel rolls, the distance from its center to the road must match the depth of the road: this means that the two dashed lines have equal length. And the road surface and wheel circumference must match, so the two thick curves must have equal arc length. These conditions will allow us to relate the polar equation $r = r(\theta)$ of the wheel to the Cartesian equation $y = f(x)$ of the road on which it rolls smoothly. The graph on the right shows the important function $\theta(x)$, which tells us the polar angle of the straight-down radius when the wheel has rolled enough so that its center is above the point x on the x-axis. That is, the segment joining the center of the wheel to the point on the wheel touching the road at x made the angle θ with the positive x-axis before the wheel started rolling. We are using standard polar coordinates, so $\theta(0) = -\pi/2$.

Now, the condition arising from the equality of the dashed lines leads to the radius condition

$$r[\theta(x)] = -f(x). \tag{1}$$

The negative sign is included because r should be positive but $f(x)$, which defines the road, will be negative. Note that the initial condition becomes $r[\theta(0)] = -f(0)$, or $r(-\pi/2) = -f(0)$.

Next we match the arc lengths. The road length is given by the familiar formula

$$\int_0^x \sqrt{1+[f'(t)]^2}\,dt,$$

while the wheel circumference is the slightly less familiar

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The author taking a ride. Photos by Deanna Haunsperger.

$$\int_{-\frac{\pi}{2}}^{\theta(x)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta .$$

Equating these integrals, differentiating both sides with respect to x, and squaring yields:

$$1+[f'(x)]^2 = \left(\frac{d\theta}{dx}\right)^2 \left[r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2 \right]. \tag{2}$$

But the radius condition (1) can be differentiated with respect to x to yield $(dr/d\theta)(d\theta/dx) = -f'(x)$. Substituting into (2) yields:

$$1+[f'(x)]^2 = \left(\frac{d\theta}{dx}\right)^2 r(\theta)^2 + [f'(x)]^2 .$$

which simplifies to $d\theta/dx = 1/r(\theta)$. This is what we want: a differential equation, quite simple as it turns out, that relates the rolling function $\theta(x)$ and the shape of the wheel $r(\theta)$. The variables separate into $r(\theta)d\theta = dx$ so integration and the initial conditions can be used to get x in terms of θ as follows:

$$x = \int_{-\frac{\pi}{2}}^{\theta} r(\theta)\, d\theta \tag{3}$$

If we can solve this for $\theta(x)$, we will know the shape of the road, since, for any x, $f(x) = -r[\theta(x)]$. (It is an interesting exercise to show that if we match tangent slopes instead of matching arc lengths we get the same fundamental relationship (3), in a way that avoids the arc length integrals.)

Why a Catenary?

Now consider the straight-line wheel with polar equation $r = -\csc\theta$, $-\pi < \theta < 0$. We will take the origin as center of the wheel and perform the analysis described above.

To get the rolling relationship we evaluate the integral of (3):

$$x = \int -\frac{1}{\sin\theta}\, d\theta . \tag{4}$$

Standard calculus techniques, using the facts that the θ-domain is $(-\pi, 0)$ and the initial condition is $x(-\pi/2) = 0$, tell us that $x = -\log(-\tan(\theta/2))$.

This inverts to $\theta(x) = 2\arctan(-e^{-x})$. It follows that the road we seek is the graph of $y = r[\theta(x)] = -\csc[2\arctan(-e^{-x})]$. This simplifies to just $y = -(e^x + -e^{-x})/2$, which is the familiar

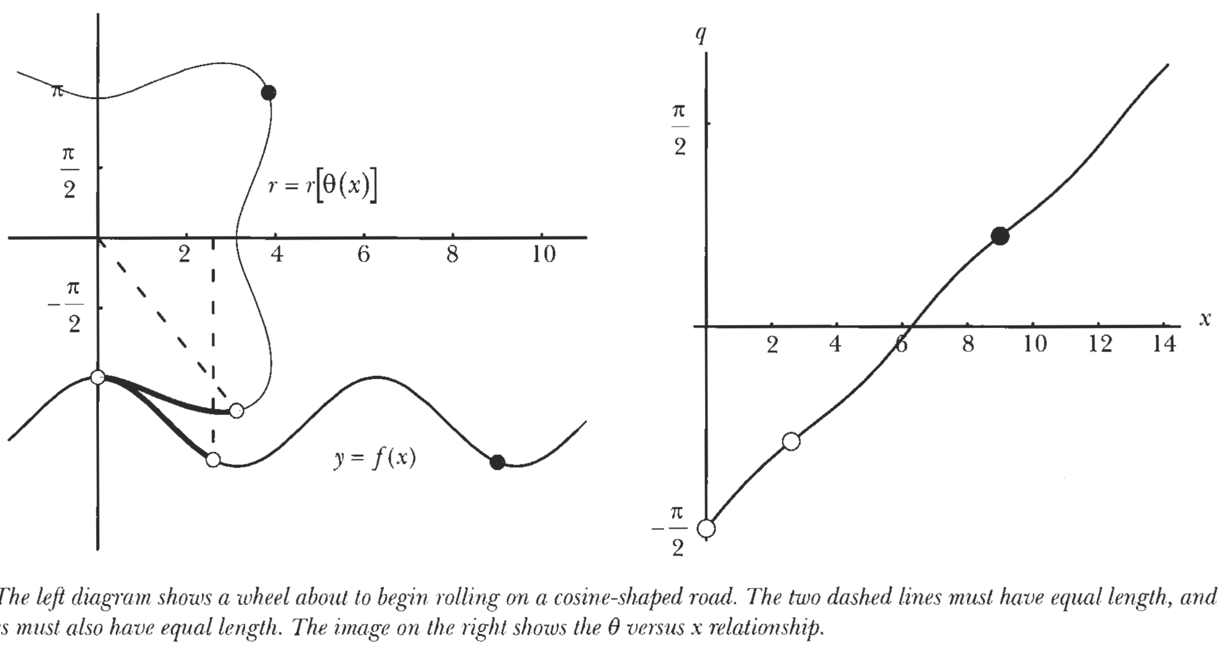

Figure 1. *The left diagram shows a wheel about to begin rolling on a cosine-shaped road. The two dashed lines must have equal length, and the two thick curves must also have equal length. The image on the right shows the θ versus x relationship.*

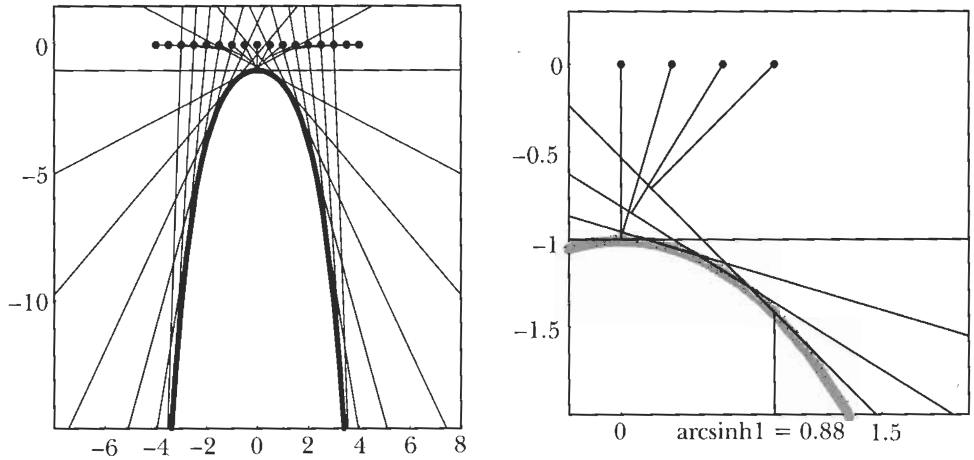

Figure 2. *A wheel that is a straight line rolls on a catenary so that the hub, the point that started at* (0,0), *stays horizontal. The close-up view shows that when the center is at* arcsinh(1), *the line makes a* 45° *angle with the horizontal.*

catenary arch, $-\cosh x$. Thus our straight line will roll on a catenary, as shown in Figure 2.

A Square Wheel

Now that we understand why the straight line rolls on the catenary it is easy to see how to handle a square wheel. Just truncate the catenary at the point at which the rolling straight line will make a 45° angle with the horizontal. Then when an identical truncated catenary is placed beside the first, the cusp will have a 90° angle; when a second straight line is placed perpendicular to the first one, we will have a rolling right angle. Do the same for the other angles of the square and, presto, a square wheel. Figure 3 shows how the wheel rolls; note that the locus of one of the corners occasionally goes backward, reminiscent of the classic puzzle about the locus of the point at the bottom of the flanged wheel of a train.

Many people, on seeing the rolling square, wonder if a rolling pentagon or hexagon is possible. Indeed, essentially the same argument shows that any regular n-gon will roll on a catenary road for any $n \geq 3$. The triangular case is actually impossible in practice, the vertex of the triangle crashes into the next bump before it can settle into the cusp. It also crashes into the previous bump as it climbs out of a cusp. So, in theory you could just lay the road as you need it and rip it up as you pass over it, but that would be slow going.

Is the Ride Smooth?

There is a subtle difference between a square wheel and a round one. For a normal bike, x is proportional to θ, where x is the distance traveled and θ is the angle pedaled. If you pedal more, you travel farther, and the correspondence is linear: pedal twice as much and you travel twice as far. This is almost true of the square wheel, but not quite. Figure 4 shows this relationship for the square wheel, with a straight line shown for comparison. The discrepancy is so small that it cannot be felt by a rider. (To see more clearly the difference from linearity, examine the Maclaurin series of $\theta(x)$, which is $2\arctan(-e^{-x})$.)

A Working Model

Inspired by various models I had seen (a small one at San Francisco's Exploratorium and a larger one built by the Center of Science and Industry in Columbus, Ohio), I asked

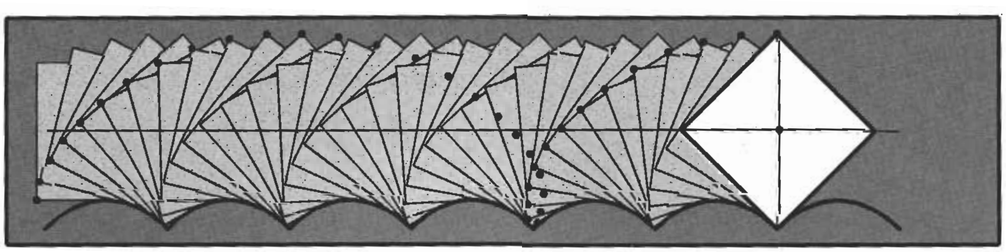

Figure 3. *When a square rolls on a sequence of appropriately truncated catenaries, the ride is smooth in the sense that the hub of the wheels experiences no up or down motion. The dots show the locus of a corner.*

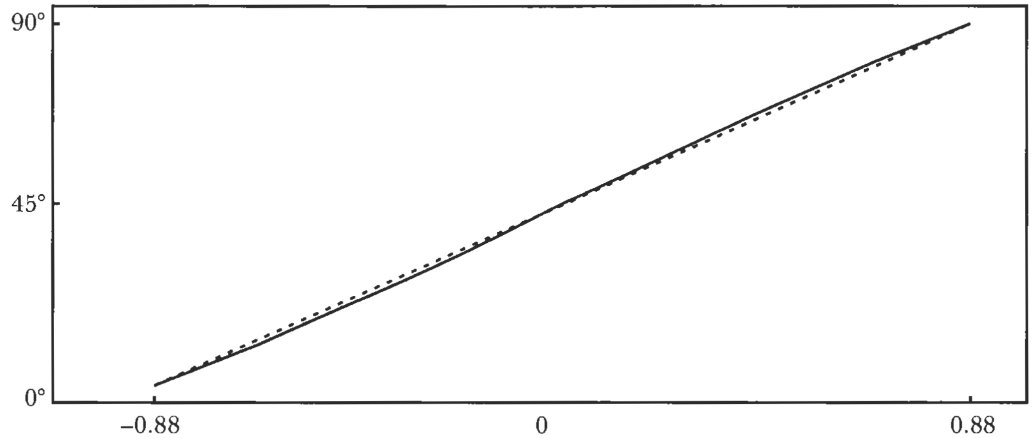

Figure 4. *The θ vs. x relationship for a square wheel. The graph is very close to a straight line (dashed) so that even though horizontal speed is not a linear function of pedaling speed, the difference is very small, and not detectable on a full-size square-wheel bicycle.*

Loren Kellen, a neighbor who knows carpentry and bicycles, if he could build a working model of a square-wheel bike. He was enthusiastic and six months later it was done; the full-size model (the road is 23 feet long) is on permanent display in the science center at Macalester and is open for public riding. We decided on a three-wheel design for stability. Friction is a big concern: there must be enough friction between the tire and the road to prevent slippage, or "creep" as it is called by professionals in catenary road-building. Also the bike frame had to be sawn in two and rewelded so that the frame would fit the road, whose spacing is in turn decided by the size of the square wheels. I thought steering would be a problem, but in fact one can steer the thing provided one does so at the top of each arch!

More Surprises

Thanks to the power of modern software, an investigation such as this often leads to new insights. Leon Hall and I, after seeing the Exploratorium model, made an extensive investigation into the shapes of various road-wheel pairs. One surprising discovery is related to the age-old definition of a cycloid as the locus of a point on a round wheel rolling on a straight road. We found that the locus of a point of a limaçon as it rolls smoothly on a trochoid (itself a type of cycloid) is also an exact cycloid! Thus the cycloid we know and love can be viewed as one of a matched pair (see Figure 5). See [2] for more such relationships. Here is a final puzzle, due to Robison: Find the unique road-wheel pair for which the road and wheel have *identical shapes*. ■

References

1. G. B. Robison, Rockers and rollers, *Mathematics Magazine*, 33 (1960) 139–144.

2. L. Hall and S. Wagon, Roads and wheels, *Mathematics Magazine*, 65 (1992) 283–301.

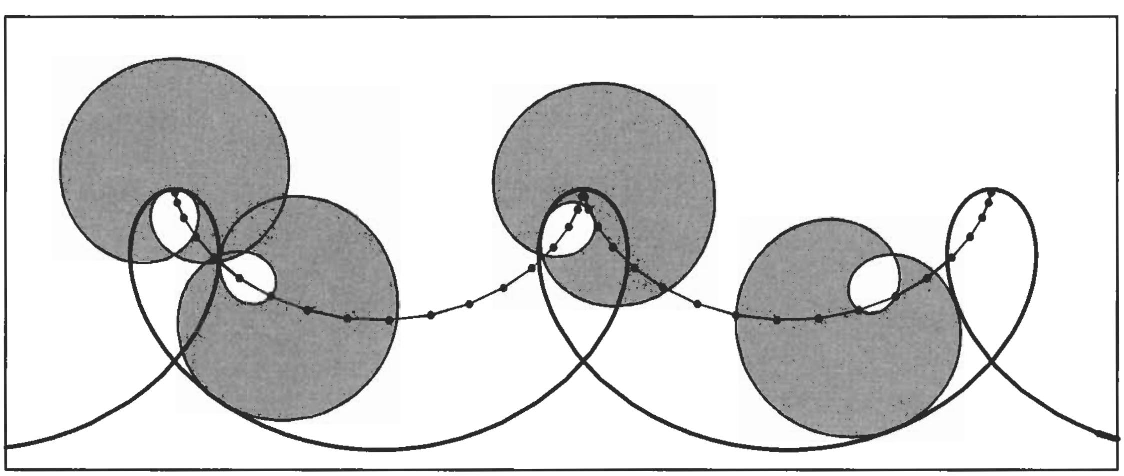

Figure 5. *When a limaçon rolls on a trochoid, the locus of one of the wheel's points is an exact cycloid.*