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1996—A Triple Anniversary

People tend to observe anniversaries. This is true whether the observance is of happy events—like weddings—or of unhappy ones—like stock market crashes. Somehow, the very act of remembering seems irresistible.

The scientific community, too, commemorates its milestones. Earlier this year, for instance, the University of Pennsylvania held a conference to recognize the fiftieth anniversary of the development of ENIAC, the world's first electronic computer. As part of those festivities, Chess Grandmaster Garry Kasparov matched wits with the IBM "Deep Blue" computer in an intellectual clash between carbon and silicon. I take no little pride in reporting that Kasparov cleaned Deep Blue's (internal) clock.

Because we are creatures of the base-10 numeration system, certain anniversaries assume particular importance. That is why Penn went ape over the fiftieth—as opposed to, say, the 37th or 43rd. We tend to celebrate events remote from us in multiples of ten. And we remember those separated from us by centuries ($10^2 = 100$) as being especially significant.

Thus, for mathematicians, 1996 is a year rich in anniversaries. Not only is it the centennial of the first proof of the prime number theorem, and not only is it the bicentennial of the discovery of the geometric constructibility of the regular 17-gon, but it is also the tricentennial of the publication of the first calculus textbook. We have not one, not two, but *three* reasons to celebrate the current year.

The Prime Number Theorem (1896)

Primes, of course, are the multiplicative "atoms" from which all whole numbers can be built. For two and a half millennia, mathematicians have studied their fascinating but surprisingly elusive properties. It was Euclid in 300 B.C. who proved that no finite collection of primes can include them all—in other words, that primes are infinitely abundant. His argument, appearing as Proposition 20 of Book IX of the *Elements*, is regarded as a logical *tour de force*, one of the most elegant, most beautiful proofs in all of mathematics.

One reason for its fame is that Euclid established the infinitude of primes without providing any explicit formula or pattern for these numbers. Indeed, the distribution of primes seems quite haphazard. Consider the first three dozen of them:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151

Anyone see a pattern here?

Well, all but the first are odd, but that's not terribly profound. Slightly more perceptive is that, after the first two, each is either one more or one less than a multiple of 6, but again this is of marginal value. Sometimes adjacent primes are close together, like 41 and 43; at other times they seem fairly far apart, like 113 and 127. All in all, primes appear to be distributed pretty much at random.

But they're not. To see why, we first define the arithmetic function $\pi(n)$ to be the number of primes less than or equal to n. For example, $\pi(9) = 4$ because there are four primes less than or equal to 9—namely, 2, 3, 5, and 7. Likewise, $\pi(10) = 4$, $\pi(100) = 25$, and $\pi(151) = 36$.

Instead of counting primes less than or equal to n, we can look at the *proportion* of primes among the numbers less than or equal to n. That is, we consider $\pi(n)/n$. Clearly $\pi(9)/9 = 0.4444$, $\pi(10)/10 = 0.4000$, $\pi(100)/100 = 0.2500$, and $\pi(151)/151 = 0.2384$. This can be extended to much larger numbers, yielding such proportions as $\pi(10^7)/10^7 = 0.06645790$ or $\pi(10^{10})/10^{10} = 0.04550525$.

Hidden amid these fragmentary results is a subtle asymptotic pattern known as "the prime number theorem." Like so many profound results from number theory, it was suspected long before it was conclusively proved.

One who perceived order amid the chaos was young Carl Friedrich Gauss (1777–1855). As a pastime in his mid-teens (no kidding!), Gauss compared the frequency of primes and the entries in a table of logarithms. "I soon recognized," he wrote, "that behind all of its fluctua-

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tions, this frequency is on the average inversely proportional to the logarithm."[1] The 15-year-old Gauss confided to his diary the cryptic statement:

$$\text{prime numbers below } a\ (=\infty)\ \frac{a}{\mathrm{l}(a)}$$

Replacing $\mathrm{l}(a)$ with the modern "$\ln(a)$" and using the function $\pi(a)$ as defined above, we can translate Gauss's jottings into:

$$\text{for large } n,\ \pi(n) \approx n/\ln(n)$$

$$\text{or, equivalently, } \pi(n)/n \approx 1/\ln(n).$$

For example, compare $\pi(10^{10})/10^{10} = 0.04550525$ with $1/\ln(10^{10}) = 0.04342945$. That's darn close.

With one final adjustment, the prime number theorem in modern form is stated as:

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln(n)} = 1.$$

The young Gauss had recognized this pattern by employing both a phenomenal insight and a tremendous amount of perseverance (after all, these computations were done at a time when "digital technology" meant counting on your fingers). His discovery provides dramatic evidence—as if any were needed—of the vast difference between guesswork and Gausswork.

Although inferring the theorem, Gauss gave no proof. All he really provided was a promising conjecture. *Proving* the prime number theorem would occupy mathematicians throughout the nineteenth century. Although frustrating, this quest gave birth to the important field known as analytic number theory.

It is to Peter Gustav Lejeune-Dirichlet (1805–1859) that analytic number theorists usually trace their beginnings. In 1837 Dirichlet proved the long-suspected fact that any suitable arithmetic progression must include infinitely many primes. More precisely, if we begin with relatively prime whole numbers a and b and examine the progression

$$a,\ a+b,\ a+2b,\ a+3b, \ldots, a+nb, \ldots,$$

then there must be infinitely many primes among them. Note that if $a = 1$ and $b = 1$, the progression is simply the set of all whole numbers, so that Euclid's result on the infinitude of primes becomes a corollary of Dirichlet's stronger theorem.

What made his proof so remarkable was that it employed the analytic tools of convergence and divergence to answer a question about whole numbers. There is something unexpected about applying analysis—the science of the continuous—to something as discrete (i.e., non-continuous) as the positive integers. This surprising and wonderful interconnection is the essence of analytic number theory.

But the prime number theorem remained unproved. In the early 1850s Pafnuti Lvovich Tchebycheff (1821–1894) showed that, *if*

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln(n)}$$

exists, then it must fall somewhere between 0.92129 and 1.10555. Unfortunately he never could establish the existence of this limit and thus could not draw the desired conclusion from all his efforts. Georg Friedrich Bernhard Riemann (1826–1866) next took up the challenge and in 1859 brilliantly advanced the frontiers of analytic number theory by examining what we now call the Riemann zeta function. Alas, his pursuit of the prime number theorem was also unsuccessful.

(By the way, it seems to me that analytic number theory—boasting such innovators as Peter Gustav Lejeune-Dirichlet, Georg Friedrich Bernhard Riemann, and Pafnuti Lvovich Tchebycheff—favors those with mellifluous, multisyllabic names. A syllabically-challenged person like "Cher" just wouldn't stand a chance.)

The prime number theorem resisted the efforts of mathematicians until the century had nearly run its course. At last, in 1896—exactly one hundred years ago—two individuals independently furnished the long-sought proof. One was the aptly named Charles-Jean-Gustave-Nicholas de la Vallée-Poussin (1866–1962). The other was Jacques Hadamard (1865–1963), whose name

Ch. J. de la Vallée Poussin

J. Hadamard

seems insufficiently flamboyant for this crowd. Their simultaneous discoveries, coupled with their nearly identical lifespans, have led some to doubt that these were really two different people. Rest assured, they were.

Vallée-Poussin and Hadamard finally laid the proposition to rest. A century ago, thanks to their genius and the power of analytic number theory, the prime number conjecture became the prime number *theorem.*

Construction of the Regular Heptadecagon (1796)

In the previous section we met the 15-year-old Gauss counting primes. Here we celebrate the bicentennial of his breathtaking announcement that a regular heptadecagon (hepta = 7 and deca = 10) can be constructed using only the Euclidean tools of compass and straightedge. This he discovered at the relatively advanced age of 18.

The problem was an ancient one: which regular polygons can be drawn with compass and straightedge? Euclid, who addressed the subject in Book IV of the *Elements,* knew how to construct regular triangles, squares, pentagons, hexagons, octagons, and pentadecagons (15-gons). Euclid also knew (although he somehow neglected to mention it) that polygons formed by repeatedly doubling the number of sides of any of these were likewise constructible. Thus, one could do a regular $2 \times 8 = 16$-gon or a regular $2 \times 16 = 32$-gon, and so on.

What Euclid left unsaid was whether these were the *only* regular polygons that were geometrically constructible. Over the centuries no one had found any others, so there was reason to believe that Euclid had netted them all in 300 B.C. It thus came as a shock when, in 1796, the young and as yet unknown Carl Friedrich Gauss informed the mathematical world that Euclid had missed some. In Gauss's words, "It seems to me then to be all the more remarkable that besides the usual polygons there is a collection of others which are constructible geometrically, e.g., the 17-gon."[2]

Needless to say, his argument cannot be squeezed into a few paragraphs; if it were that simple, someone would have stumbled upon it during the previous 21 centuries. Yet the difficulty lies more in the intricate interconnections of its steps than in the complexity of any step in particular. Let me outline the basic structure of his proof.

First, it was known since the time of Descartes in the early seventeenth century that, beginning with a unit length, one can construct any magnitude whose length is expressible in terms of integers and finitely many applications of the operations $+$, $-$, $\times$, $\div$, and $\sqrt{}$. Such expressions are called "quadratic surds."

Of course, it is obvious how, starting with a segment of length 1, we can construct a segment of length 2, or 3, or 4. Less clear is how to construct one of (say) length $\sqrt{5}$. If you haven't seen it before, here it is:

Along a straight line, mark off segment AB of length 5 and BC of unit length (as shown in Figure 1). Bisect AC at O—a familiar compass and straightedge construction—and using O as center and $OA = OC$ as radius, draw a semicircle. From B, erect a perpendicular to AC meeting the semicircle at D, another simple construction.

Now cash in on some elementary geometry. Triangle ADC is right because $\angle ADC$ is inscribed in a semicircle. Thus DB, the altitude to the hypotenuse, splits $\triangle ADC$ into two similar triangles, namely $\triangle ABD$ and

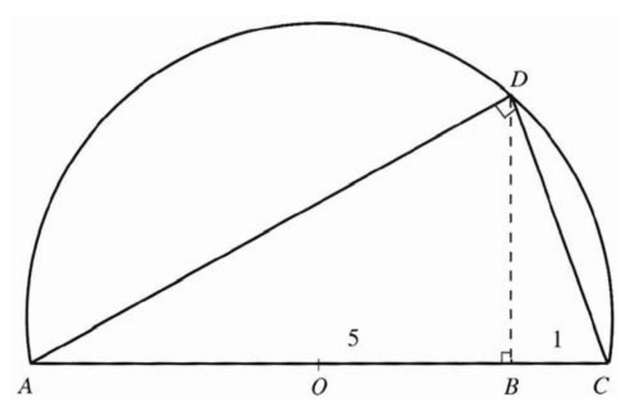


Figure 1

$\triangle DBC$. By similarity,

$$\frac{AB}{BD} = \frac{BD}{BC}$$

so that

$$(BD)^2 = AB \times BC = 5 \times 1 = 5.$$

Hence BD has length $\sqrt{5}$. And here's the bottom line (pun intended): the length $\sqrt{5}$ was constructed *with compass and straightedge.*

As noted, one can also construct products and quotients of previously constructed lengths, although the interested reader will have to discover how for herself. (O.K., if you don't want to do it yourself, look at [3]).

So—to repeat—by nesting these sorts of constructions, it is possible to draw any segment whose length is a quadratic surd. For instance, it is possible to construct a segment of length

$$\frac{2+\sqrt{3+\sqrt{5-\sqrt{7}}}}{4-\sqrt{1+\sqrt{6}}},$$

although I wouldn't particularly want to.

Next, Gauss recognized that if he could construct $\cos(2\pi/n)$, then he could easily construct a regular n-gon. That is, inside a unit circle, copy the constructed length $\cos(2\pi/n)$ as segment OC in Figure 2. From C construct a perpendicular upward, meeting the circle at B. If $\theta = \angle BOC$, then clearly

$$\begin{aligned} \cos\theta &= \frac{OC}{OB} \\ &= \frac{\cos(2\pi/n)}{1} \\ &= \cos(2\pi/n), \end{aligned}$$

and so the central angle $\theta = 2\pi/n$. Copying the chord AB n times around the unit circle, we must return exactly to A and in the process shall have constructed a regular n-gon.

Summarizing to this point: we know that a regular 17-gon is constructible if $\cos(2\pi/17)$ is and, further, that

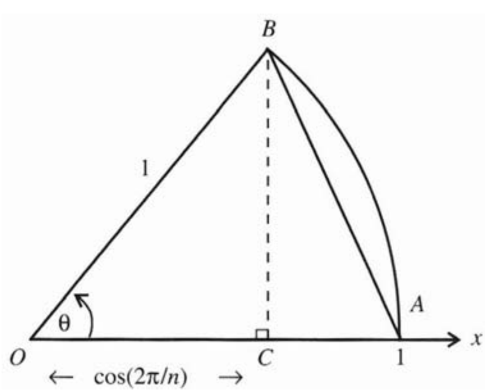


Figure 2

$\cos(2\pi/17)$ is constructible if it is a quadratic surd. The remaining obstacle, then, was to establish the quite unexpected fact that $\cos(2\pi/17)$ really is such a surd. This is what Gauss did.

In the process, he displayed his extraordinary genius by veering off into the world of complex numbers. At first this seems bizarre. Geometric constructions, after all, occur in the real world; there's nothing imaginary about them.

But note that $\cos(2\pi/17)$ appears as the real part of the complex number $z = \cos(2\pi/17) + i\sin(2\pi/17)$, which is itself one of the seventeenth roots of unity. This suggests a bridge into the imaginary realm, a bridge Gauss crossed with spectacular success. Once on the far side, he proved that $\cos 2\pi/17$ is indeed a quadratic surd. To be specific,

$$\begin{aligned} \cos(2\pi/17) = &-\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}} \\ &+\frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}} \end{aligned}$$

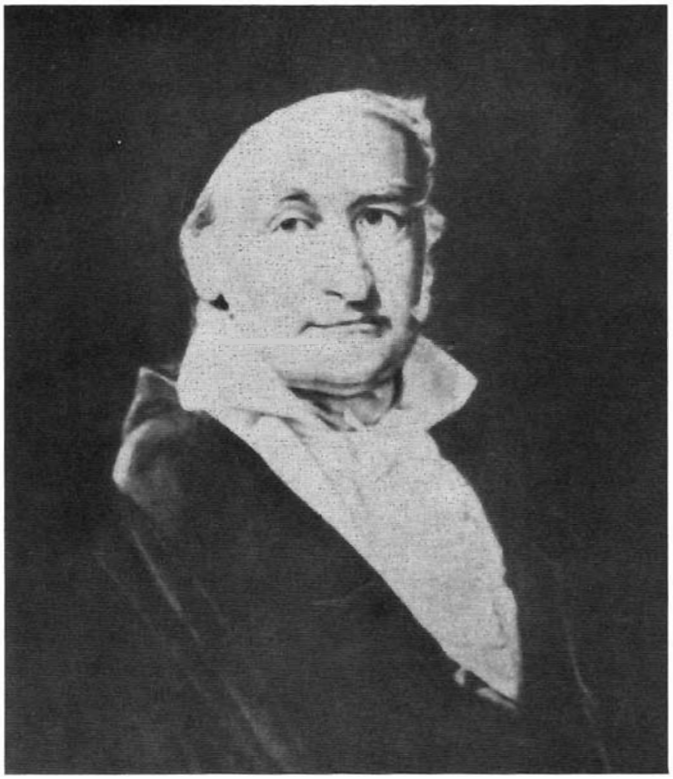
Although this may strike some readers as being a quadratic *ab*surd, it is perfectly correct.

Because this complicated expression is built from integers that are added, subtracted, multiplied, divided, and square-rooted, it is constructible. Hence $\cos(2\pi/17)$ is constructible, and it follows from the criterion above that the regular 17-gon is constructible as well. So goes the proof.

Wow! Like investigating primes with the techniques of analysis, the construction of geometric polygons using the properties of complex numbers links two seemingly unrelated subjects. Once again we see that unexpected mathematical interconnections can yield the most remarkable of theorems.

A striking feature of this result is its Janus-like quality. In addressing the constructibility of regular polygons, Gauss looked backward to the Greeks; in introducing complex variables, he looked forward to a subject whose importance would explode in the coming century. Of course, the *practical* significance of all this was nil. It didn't help anyone balance a ledger or powder a wig. In truth, it was quite useless. But if theorems can ever be breathtaking in their boldness and sweep, this was one.

And Gauss's proof served another purpose. It showed that Euclidean geometry runs deeper than is usually imagined. From my own student days, I remember thinking that Euclid's geometry was simply the careful demonstration of self-evident facts. It was fun to prove that the base angles of an isosceles triangle were congruent, but surely anyone would have guessed this by drawing a few pictures. The proofs may have required insight, but the theorems as stated drew only yawns.



C. F. Gauss

Not so here. Prior to Gauss, no one—*no one*—anticipated that a regular 17-gon could be constructed within the constraints of Euclid's innocent-looking system. The construction of the regular heptadecagon exactly two hundred years ago forcefully reminded mathematicians that Euclidean geometry still held some big surprises.

The First Calculus Text (1696)

If our first two anniversaries commemorate specific theorems, the third recognizes a broader achievement: the publication in 1696 of the first textbook on calculus. Its author was Guillaume François Antoine de l'Hospital (1661–1704), and the story of how he came to write it demands a brief digression.

L'Hospital was a French marquis, a minor nobleman who was an ardent, if unschooled, mathematician. Late in 1691, l'Hospital was introduced to the young Johann Bernoulli (1667–1748), then fresh from his triumphant discovery of the catenary curve. In making that discovery, Bernoulli had artfully employed the techniques of calculus as first described in the 1684 and 1686 papers of Gottfried Wilhelm Leibniz, its creator. Bernoulli later recalled, "I knew right away that he [l'Hospital] was a good geometer ... but that he knew nothing at all of the differential calculus, of which he scarcely knew the name, and still less had he heard talk of the integral calculus, which was only just being born."[4]

L'Hospital wanted to learn. And, as a marquis, he had deep pockets. He thus hired Johann Bernoulli to teach him this new and powerful subject. For the better part of a year, Bernoulli worked to bring l'Hospital up to speed. In the process, he provided (or, more precisely, "sold") l'Hospital his own discoveries in calculus. Essentially, l'Hospital bought the rights to Bernoulli's theorems.

By all accounts, l'Hospital made excellent progress and soon felt himself ready to write a book on the subject. It appeared three hundred years ago, in 1696. Titled *Analyse des infiniment petits* (*Analysis of the infinitely small*), it was largely the recycled memoirs of Johann Bernoulli. L'Hospital acknowledged his debt to Leibniz and, especially, to Bernoulli and forthrightly observed that "I frankly return to them whatever they please to claim as their own."[5]

Later in life Johann grumbled that l'Hospital had garnered undeserved credit for this text. Initially skeptics dismissed Bernoulli's protestations as sour grapes. Johann Bernoulli was, after all, a contentious, argumentative egotist—the sort of person who gives arrogance a bad name. Actually, as documents subsequently revealed, Bernoulli was quite justified in claiming much of the book as his own.

And so it was that, in 1696, the world saw its first calculus text. *Analyse des infiniment petit* treated differential calculus only. It was modest in size, unlike today's 14-pound monsters that require a wheelbarrow to carry to class. There were no color illustrations nor any calculator exercises indicated in the problem sets. In fact, there were no problem sets.

Yet it *was* a calculus book.

L'Hospital began with a few postulates and definitions. For instance:

Postulate I: Grant that...a quantity which is increased or decreased only by an infinitely small quantity may be considered as remaining the same.

Definition II: The infinitely small part whereby a variable quantity is continually increased or decreased is called the differential of that quantity.[6]

To such statements, the modern reader is likely to respond, "Huh?"

"Infinitely small quantity?" "Continually increased?" What do these mean? Clearly the precision and rigor of modern analysis lay far, far in the future.

Early in the book, l'Hospital noted that the differential of a constant is zero. That is, if a is constant, then $da = 0$ (where the symbol "d" for differential was borrowed from Leibniz's original 1684 paper). A bit later, l'Hospital gave the following proof of the product rule:

To find the differential of the product xy, we increase x by an infinitely small part dx to get $x + dx$. Similarly, y is increased by an infinitely small part to $y + dy$. Thus the product xy will be increased to $(x + dx)(y + dy) = xy + x\,dy + y\,dx + dx\,dy$, and so the differential of xy will be the difference

$$d(xy) = [xy + x\,dy + y\,dx + dx\,dy] - xy$$
$$= x\,dy + y\,dx + dx\,dy.$$

Then l'Hospital noted that, "because $dx\,dy$ is infinitely small with respect to the other terms,"[7] we simply can throw it away. And so—Bingo! —we have the product rule $d(xy) = x\,dy + y\,dx$.

L'Hospital then dispatched the quotient rule in short order:

To find $d(\frac{x}{y})$, introduce the auxiliary variable $v = \frac{x}{y}$, so that $x = vy$. Apply the just-proved product rule to this last expression to get

$$dx = d(vy) = v\,dy + y\,dv,$$

and therefore

$$dv = \frac{dx - v\,dy}{y}$$

Multiplying numerator and denominator by y yields

$$dv = \frac{y\,dx - vy\,dy}{y^2},$$

and we now merely recall that $v = \frac{x}{y}$ and $x = vy$ to deduce the famous quotient rule:

$$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}.$$

Such was calculus in 1696.

We must not conclude without mentioning the result that appeared in Section IX of the *Analyse*. There l'Hospital gave the famous rule for finding

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

when both $f(x)$ and $g(x)$ tend to 0 as x approaches a (although he didn't state it in this modern form). In his words,

> if the differential of the numerator be found, and that be divided by the differential of the denominator, after having made $x = a \ldots$, we shall have the value ... sought.[8]

Modern textbooks state this result more concisely as

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

It should come as no surprise that l'Hospital's Rule was actually discovered by Johann Bernoulli. It is one of those nuggets that l'Hospital had promised to return to its rightful owner. Unfortunately for Johann, the return was never made. Everyone today knows "l'Hospital's Rule" as a wonderful application of differential calculus. Not everyone knows that it should be called "Bernoulli's Rule."

After stating the result that would guarantee him immortality, l'Hospital provided his first example. It was a doozy. He asked for the limit as x approaches a of

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{a^2x}}{a - \sqrt[4]{ax^3}}$$

This, I repeat, was his *first* example!

My, how textbooks have changed. For the sake of comparison, I checked out some of the popular texts of today:

Stewart's first example of l'Hospital's Rule is the tame

$$\lim_{x \to 0} \frac{2^x - 1}{x}.$$

Finney, Thomas, Demana, and Waits (the authors, not the law firm) start with the easy

$$\lim_{x \to 0} \frac{3x - \sin x}{x}$$

And Anton begins with the positively wimpy

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}.$$

These problems are mere lightweights compared to l'Hospital's snarl of symbols. If you dare, try it for yourself—although be mindful of that old adage, "A chain rule is only as strong as its weakest link."

With this challenge, we end our anniversary celebration of three mathematical milestones—the prime number theorem, the regular heptadecagon, and the first calculus text. Readers are now free to break out the champagne, throw confetti, and cheer wildly!

But don't celebrate overlong. After all, one big question remains to be answered: a century from now, what mathematical milestone will our ancestors be celebrating from 1996?

Get working!

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