RAVI VAKIL

The Youngest Tenured Professor

Noam Elkies has been playing around with arithmetic and number puzzles for as long as he can remember. He was born in New York in 1966, and in early childhood he was intrigued by the association of numbers to fingers and hence to keys on a piano (so Frère Jacques would begin 1231|1231|345|345). This fascination would foreshadow his adult fascination with both mathematics and music. Thanks to special camps and enriched education in Israel between 1970 and 1978, his ability in mathematics was encouraged from an early age. He happened across a Hebrew translation of Euclid's *Elements* on his parents' bookshelves, and fell in love with mathematics from reading it.

When Noam returned to New York, he enrolled in Stuyvesant High School, one of the best mathematical schools in the US. He plundered the public libraries in search of more mathematics. Martin Gardner's books were a particular source of inspiration. Indeed, one of Gardner's problems later became a prize-winning project for Noam at the 1982 *Westinghouse Science Talent Search,* a national science fair.

Noam also competed in the series of competitions run by the Mathematical Association of America. The pinnacle of this series is the *USA Mathematical Olympiad,* which Noam won in his final two years of high school. Based on that performance, he was invited to compete for the US at the *International Mathematical Olympiad,* where he won Gold Medals in each of his two years.

Math or Music?

While at Stuyvesant, Noam also attended the Juilliard School of Music. Noam considers mathematics and music similar aesthetic pursuits; abstraction and a certain playfulness are essential to both. In fact, Noam considered a career in music as late as his final year of high school; he still gives regular recitals, sometimes of his own compositions.

After Stuyvesant and Juilliard, Noam studied mathematics and music at Columbia University, graduating after three years when he was only 18. In each of those three years, he won top honors in the North American *Putnam Mathematical Competition.*

In 1985, Noam entered the PhD program at Harvard. In the summer after his first year, he stunned his professors by proving a conjecture in number theory that had defied mathematicians for over 25 years. This important result became his PhD thesis, and the following year, after only two years of study, he received his doctorate in mathematics.

In 1987, another long-standing problem succumbed to Noam's insight. It had been known for centuries that it is not possible for one cube to be written as the sum of two cubes. But sometimes a cube can be written as the sum of *three*

Photo by Laura Wulf. Courtesy of Harvard University News Office.

Mathematician Elkies at the piano.

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in Harvard History

cubes; for example, $3^3+4^3+5^3=6^3$. Leonhard Euler (1707–1783) conjectured that for $n > 2$ it is possible for n nth powers to add to another nth power, but impossible for $n-1$ to do so.

In 1966, L.J. Lander and T.R. Parkin showed that Euler's conjecture is false: $27^5+84^5+110^5+133^5=144^5$. But it had been unknown since Euler's day whether there are three fourth powers summing to a fourth. Using elegant and sophisticated tools of number theory, Noam hunted down a counterexample:

$$2682440^4+15365639^4+18796760^4 = 20615673^4.$$

He then proved that, in some sense, there are *lots* of counterexamples, and showed how to produce an infinite number of them. (The counterexamples are very large; the numbers in Noam's second counterexample have almost seventy digits each!)

Noam's main concern about completing a PhD early was that he would be forced to leave Harvard's hothouse academic environment too soon. Such fears would prove groundless. He was offered a Junior Fellowship at Harvard from 1987–90, and then an Associate Professorship from 1990–93. Finally, in 1993, at the age of 26, he became the youngest person ever to be granted tenure at Harvard University.

Noam's favorite problems include the famous *Riemann Hypothesis*, a long-standing conjecture in number theory that has tremendous ramifications in the field. "All number theorists are inherently fascinated by it," says Noam. But despite the concentrated effort being expended on this one problem, Noam notes that "it might well be solved by accident."

The *ABC* Conjecture

Another problem Noam has tackled is the *ABC Conjecture* at the crossroads between number theory and geometry. The *ABC* Conjecture implies many important results, including Fermat's Last Theorem. Noam proved another important implication in 1991.

The *ABC* Conjecture (due to Masser and Oesterlé) is reasonably simple to understand. Imagine that there were a solution to Fermat's Last Theorem: $x^n+y^n=z^n$ $(n>3)$. Let $A=x^n$, $B=y^n$, and $C=z^n$, so $A+B=C$. Then the product of the primes dividing ABC is at most $xyz < C^{3/n}$, which is much less than C. If we could show that this is impossible (when phrased more rigorously) then we could prove Fermat's Last Theorem.

The *ABC* Conjecture is basically just this: if A, B, and C are relatively prime, and $A+B=C$, then the product of primes dividing ABC should be much greater than $C^{1-\varepsilon}$. Put very loosely, if a little carelessly, in examples when C is very large, the product must be at least of the same order of magnitude as C.

The *ABC* Conjecture, altered slightly, is known to be true if A, B, and C are polynomials instead of integers. The proof of this result is very similar to the solution to the following problem from the *Sixteenth William Lowell Putnam Mathematical Competition* in 1956:

> The polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z)\equiv Q(z)$, i.e., that $P(z)$ and $Q(z)$ are the same polynomials.

In between *ABC* and Fermat is a new conjecture that has recently been in the news, the $50,000 Beal Conjecture.

Noam has also suggested another unsolved problem that you might wish to investigate. The problem is a more general form of the following question first formulated by Sam Loyd (1841–1911), a renowned American puzzlist.

What is the greatest number of line segments which can be drawn through 16 points so that each line segment contains 4 points?

The Beal Conjecture

The equation $A^x+B^y=C^z$ has no solution in positive integers A, B, C, x, y and z with x, y and z at least 3 and A, B and C coprime.

Andrew Beal is a 44-year-old number theory enthusiast and founder/chairman/owner of Beal Bank, Dallas' largest locally owned bank. He is also the founder/CEO/owner of Beal Aerospace. Beal is offering a $5,000 prize for a proof or counterexample to his conjecture. The value of the prize will increase by $5,000 per year (up to a maximum of $50,000) until it is solved.

For more see "Move Over Fermat, Now It's Time for Beal's Problem" in the February 1998 *Math Horizons*.

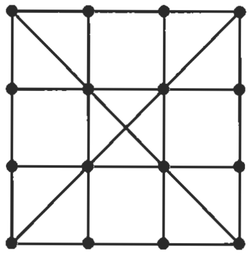


Figure 1

In Figure 1, there are sixteen dots and ten line segments, with four dots on each line segment. We will write this as "$16 \to 10 \otimes 4$".

However, in Figure 2, there are sixteen dots and fifteen lines, with four dots on each line. (We write this as "$16 \to 15 \otimes 4$".) This is the best-known result for 16 dots and 4 dots per line segment.

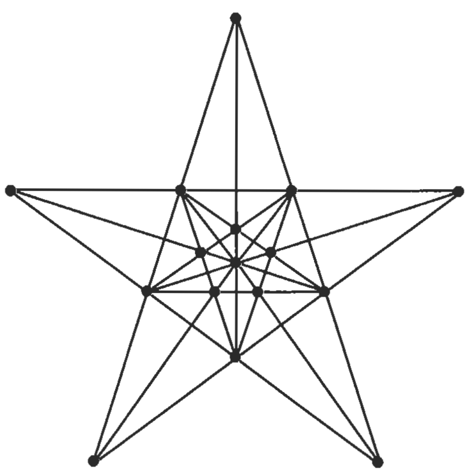


Figure 2

This idea can be generalized to get $n^2 \to (3n+3) \otimes n$, where n is even (and at least 4). Can you see how? Try it with $n = 6$. Is this the best that can be done?

Figure 3 shows the best result for $n = 3$:

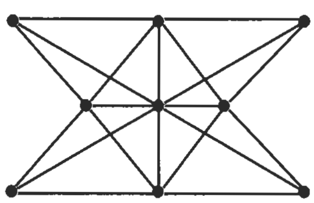


Figure 3. $9 \to 10 \otimes 3$

In 1981, Noam discovered the following example (Figure 4) that is the best known for $n = 5$:

It isn't known if $n^2 \to (3n+3) \otimes n$ is possible for n odd and greater than 5.

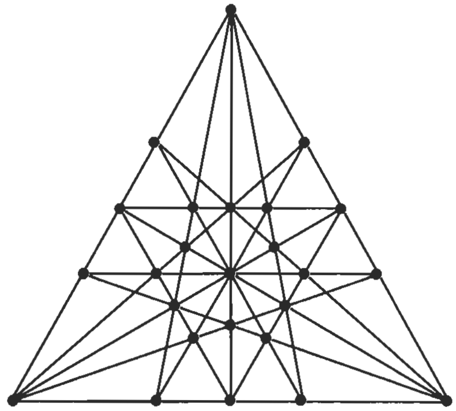


Figure 4. $25 \to 18 \otimes 5$

Can you find an example that shows that $7^2 \to 24 \otimes 7$? How close can you get? Also, can you prove that one can never get $n^2 \to m \otimes n$ for $m > 3n+3$?

Here is a Sam Loyd "meta-puzzle": Loyd died in 1911, yet his autobiography was written in 1928. How is this possible? For the answer, read Simon Singh's excellent book, *Fermat's Enigma*. Noam turns up in the book as a character in a notorious Fermat hoax.

World Chess Champion

Noam finds another source of aesthetic entertainment in the world of chess. He started competing in chess tournaments in grade school, and reached the rank of Master during his brief visit to graduate school. But Noam's real joy is chess problem-solving and composing, which he finds closer to both art and mathematics.

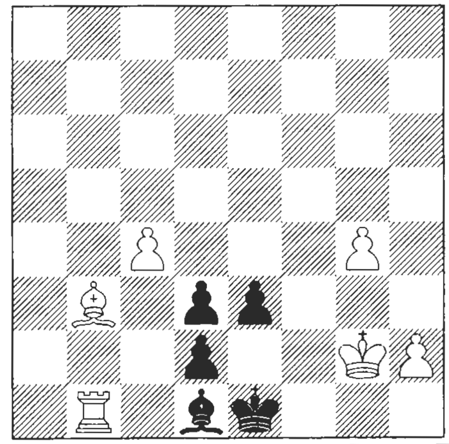


Figure 5. *E. Delpy,* Deutsches Wochenschach, *1908. White to play and mate in 5 (solution appears on p. 27).*

Chess problems are problems such as that given in Figure 5, where the solver has to find and prove a winning (or, in some cases, drawing) strategy. Studies are longer problems with more depth of analysis required. If beautiful chess problems are akin to gems, then studies can be likened to little works of art. Noam started composing studies at a young age, and began winning prizes for his creations when he was 17.

In October 1996, Noam happened to be visiting Israel for a cousin's bar mitzvah when the *World Championship of*

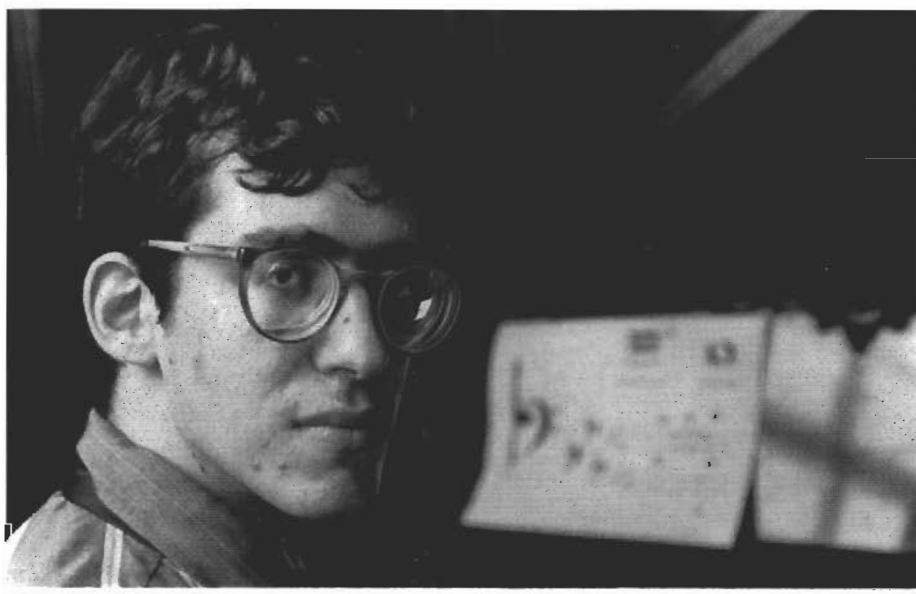


Photo by Laura Wulf. Courtesy of Harvard University News Office.

Elkies the composer.

Mathematics, music, chess—what is Elkies thinking about?

Chess Problem-Solving took place. He decided to try it out, and did well enough in the "Open" part of the competition that he was invited to take part in the Championship, as one of the few competitors not on a team. The competition involved solving 18 problems in 6 hours, spread over 2 days, a grueling marathon similar in length and intensity to the *Putnam Mathematical Competition*. When the event was over, everyone was astonished to learn that Noam, a last-minute entrant, had won the World Championship!

Not surprisingly, the next year Noam was invited back, as part of a three-person team. This time he won third prize, and his team placed first in the team competition. (The problem in Figure 5 appeared on the 1996 Open competition. Can you solve it?)

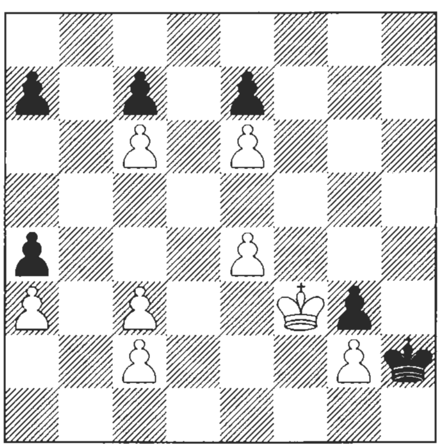


Figure 6. $1 - 2 + 4 = 3 > 0$

Chess has long had close links to mathematics. In recent years, a serious theory of "combinatorial games" has been built up under the leadership of Elwyn Berlekamp of Berkeley, John H. Conway of Princeton, Richard Guy of University of Calgary, and many others. Games such as Connect 4 have been completely analyzed, and perfect strategies found. For various reasons, chess is too complicated to be tractable by current methods, but it provides an interesting showcase for game-theoretic ideas. For example, people often analyze a chess game and say things such as "white has a good chance of winning." However, it is a fundamental result of game theory that if both players play perfectly, this is nonsense: it is a guaranteed win for white, or a guaranteed win for black, or a guaranteed tie. There is no uncertainty or chance involved. Can you see why? (You'll need to assume that the game must end in a finite number of moves; so assume that if the same position comes up three times, the game is drawn.) If this theorem seems mysterious, or you feel nervous thinking about chess, try showing the same fact about a simpler game, such as Tic Tac Toe or Connect 4.

Trébuchet

Here is one game-theoretic idea that has been illustrated in chess. (All of these examples are Noam's, and appeared in [ONAE].) Consider Figure 6. If the 10 pawns on the left five files (i.e., columns) weren't there, then the situation would be easy to analyze. Whoever had to move first would lose: she would have to move her king and leave her pawn vulnerable. (This is known in chess literature as the "trébuchet".) With the ten pawns on the board, White has the advantage. White can "kill time" with five pawn moves, while Black can stall with 2, so White has 3 moves to spare, and Black will be forced to move her king (and hence lose the game). In game-theoretic parlance, the value of Figure 6 for White is 3.

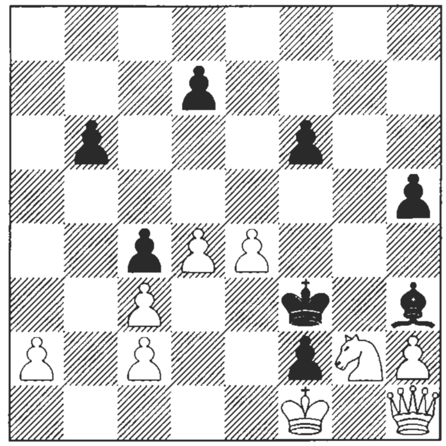
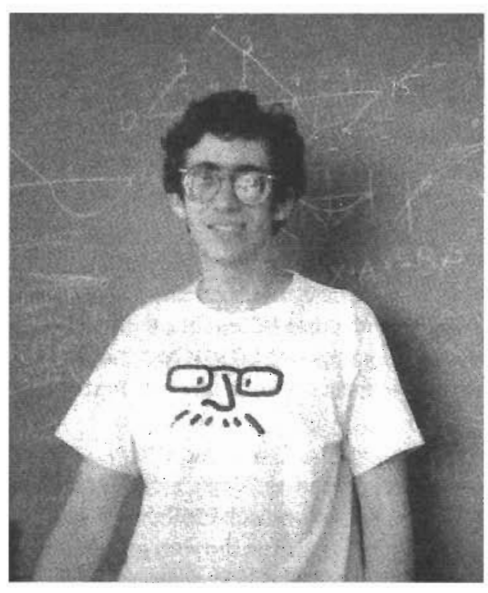


Figure 7. *Whoever moves loses:* $1/2+1/2-1=0$.

Game theory has a way of assigning a value to many positions, and mysteriously these values are not always integers. In Figure 7, the total value of the board is zero, but fractions make an appearance. The free black pawn in the rightmost column gives Black a 1 move advantage, but White has a 1/2-move advantage from the left three files, and another 1/2-move advantage from the pawns in the next three files, exactly cancelling Black's advantage.

Even more mysterious are "infinitesimal values," which come up in positions such as Figure 8.

Such analysis won't revolutionize chess in the near future. But there *have* been recent ideas coming out of the academic world that have surprised and



Professor Elkies

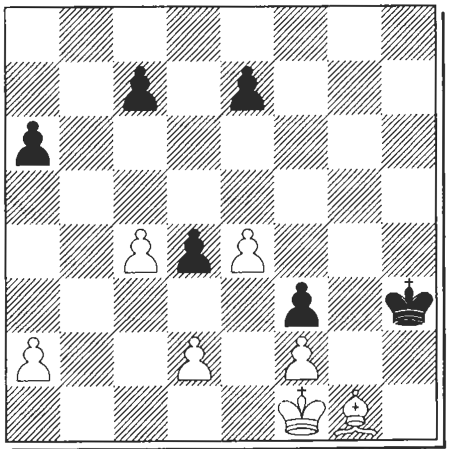


Figure 8. *Whoever moves loses:*
$\{0,* \mid \uparrow\} - \uparrow * = 0$

shocked the chess community. For example, several years ago, Lewis Stiller proved (with Noam's assistance) that a king, rook, and bishop will always be able to checkmate a king and two knights (unless the position is obviously exceptional), although the strategy may require over 200 moves. Such a situation was always thought to be drawn. This was the first of many surprising endgame results from Stiller that were made possible by ingenuity and computing power.

In short, Noam Elkies' career, although still in its early stages, has allowed him to pursue beauty and elegance in many arenas, including numbers, games, chess, and music. We will have to see where it takes him next. ■

An earlier version of this article appeared in [MM].

References

[BC] Mauldin, R.D., "A generalization of Fermat's Last Theorem: The Beal Conjecture and Prize Problem", *Notices of the AMS,* Dec. 1997.

[FE] Singh, S., *Fermat's Enigma* , Walker and Company, New York: 1997.

[ONAE] Elkies, N.D., "On Numbers and Endgames", *Games of No Chance* (R. Nowakowski ed.), MSRI Publ. #29, 1996.

[MM] Vakil, R., *A Mathematical Mosaic: Patterns and Problem Solving,* Brendan Kelly Publ. Inc., Toronto, 1996.

[SN] Conway, J.H. and Guy, R., "Surreal Numbers", *Math Horizons,* Nov. 1996, p. 26–31.

[WW] Berlekamp, E., Conway, J.H., and Guy, R., *Winning Ways,* Academic Press, New York 1982.