Applications of the Universal Surjectivity of the Cantor Set

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Our first encounter with the Cantor set is usually in a basic real analysis course. Its striking combination of unusual and seemingly counter-intuitive properties makes it the perfect example for illustrating the new notions introduced in the course. It is only much later that the student learns to appreciate that the Cantor set plays an important role in many branches of mathematics, and is not just an artificial construct, especially designed to exhibit the possible pathologies that can arise in the systematic development of real analysis.

In this article we discuss one of the basic properties of the Cantor set, its surjective universality in the class of compact metric spaces:

Theorem 1. *Every compact metric space is a continuous image of the Cantor set, i.e., for each compact metric space K there is a continuous map from the Cantor set Δ* onto K.

This classical theorem is due to Alexandroff [**1**] and Hausdorff [**7**, p. 226]. It is a standard theorem that appears in many books on real analysis and topology, e.g., [**5**, p. 363] and [**8**, p. 127].

We show how this theorem can be applied to a variety of seemingly unrelated problems in topology, geometry, and analysis. When each of the results is considered separately, the Alexandroff–Hausdorff Theorem seems to appear as an ad hoc trick. Put together, however, we soon realize that there is a common thread in all these applications, and that they actually represent a *method*. Phrased heuristically, the theorem gives a systematic way to "continuously encode" compact sets of data.

The Cantor set is the unique infinite, perfect, totally disconnected, compact metric space. Using this characterization, we can use any of its representations whenever it is convenient to do so. For example, the standard proof of the Alexandroff–Hausdorff Theorem uses the representation as an infinite product $\prod_{n=1}^{\infty} F_n$, where each F_n is a finite set. All the Cantor sets that we encounter here are, however, closed subsets of the real line.

Most of the results in this article are known. The only (possibly) novel parts are in Sections 4 and 5. It was the result in Section 4, and the reactions of several people to the proof, that prompted me to write this article.

1. SPACE FILLING CURVES. We start by constructing a space filling curve, i.e., a continuous function that maps the unit interval $[0, 1]$ onto the unit cube $[0, 1]^d$ in the d-dimensional space $\mathbf{R^d}$.

By the Alexandroff–Hausdorff Theorem, there is a continuous function ϕ from the Cantor set Δ onto the compact metric space $[0, 1]^d$.

Consider Δ as the classical Cantor set in the unit interval $[0, 1]$, and extend ϕ to a continuous function defined on the whole interval by linear interpolation: The complement of Δ is a countable union of open intervals. If (a, b) is one of these intervals, represent its points in the form $ta + (1 - t)b$ for $0 < t < 1$, and define $\phi(ta + (1 - t)b) = t\phi(a) + (1 - t)\phi(b)$.

The extended function takes its values in $[0,1]^d$ because the cube is convex, and it is easy to check that it is continuous on $[0,1]$ as required.

The extension of the Alexandroff–Hausdorff map from Δ to $[0,1]$ is a common step in almost all the proofs we present. Note that the only property of the unit cube that we used in the extension was its convexity. We thus obtain

Corollary 2. *Let K be a convex, compact, and metrizable subset of a linear topological vector space V. Then there is a continuous surjective map from* $[0,1]$ *onto K. More generally, if K is not assumed to be convex, then there is a continuous map from* $[0,1]$ *into V whose image contains K.*

Corollary 2 is a special case of the Hahn–Mazurkiewicz Theorem [**8**, p. 129], which characterizes the continuous images of the interval $[0,1]$ as the connected and locally connected compact metric spaces.

Remark. The extension procedure only used the fact that the Cantor set is closed. If f is a continuous real-valued function defined on any closed subset A of the real line, the same procedure can be used to extend it to all of $\mathbf{R}$. If A is bounded from above, define the extension for $x > b = \max\{t : t \in A\}$ by $f(x) = f(b)$; use a similar formula when A is bounded from below. Alternatively, we could just use the Tietze Extension Theorem.

2. A UNIVERSAL CONVEX SET. Here is a question in geometry: Does there exist a three-dimensional compact convex set B with the property that every compact convex two-dimensional subset of the unit square is congruent to one of its faces?

Recall that two sets in $\mathbf{R}^d$ are *congruent* if there is an affine isometry of $\mathbf{R}^d$ that takes one set onto the other. A hyperplane H in $\mathbf{R}^d$ is said to *support* a compact convex set B if B is contained in one of the closed half-spaces determined by H, and B touches H. If H is represented in the form $H = \{x \in \mathbf{R}^d : f(x) = \alpha\}$, where f is a linear functional on $\mathbf{R}^d$ and α is a real number, then H supports B if either $\max\{f(x) : x \in B\} = \alpha$ or $\min\{f(x) : x \in B\} = \alpha$. In this case we say that H supports B in the set F where this maximum (or minimum) is attained, i.e., $F = B \cap H$; such sets F are called the *faces* of B.

The answer to the question is no. The interior of each two-dimensional face of B is relatively open in the two-dimensional boundary of B, hence B can have at most countably many two-dimensional faces. But there are uncountably many noncongruent two-dimensional compact convex subsets of the unit square. For example, the square contains uncountably many noncongruent triangles.

This topological argument fails if we look for a four-dimensional set that is "universal" for two-dimensional sets. In this case the boundary is three-dimensional, and there is no topological obstruction to the existence of uncountably many two-dimensional faces. But is this topological argument the only obstruction? R. Grzaslewicz [**6**] proved the striking geometric fact that this is indeed the case, and that such a "universal" four-dimensional set exists! More generally, he proved:

Theorem 3. *For each $d \geq 1$ there is a compact convex set B in $\mathbf{R}^{d+2}$ with the property that each compact convex subset of the d-dimensional unit cube is congruent to a face of B.*

The case $d = 1$ is elementary, and it is instructive to visualize it since the higher dimensional construction follows the same ideas, and is more difficult to visualize.

The one-dimensional unit cube is just the interval $[0, 1]$, and its convex subsets are intervals of length l with $0 \le l \le 1$; by an interval of length 0 we mean a single point.

Represent $\mathbf{R}^3$ as $\mathbf{R}^2 \times \mathbf{R}$, and write the points in $\mathbf{R}^3$ as pairs (t, x), where $t \in \mathbf{R}^2$ and $x \in \mathbf{R}$. Let $\mathbf{T}$ be the unit circle in $\mathbf{R}^2$, and let f be a continuous function from $\mathbf{T}$ onto $[0, 1]$. Put

$$G = \{(t, x) : t \in \mathbf{T} \quad \text{and} \quad 0 \le x \le f(t)\}.$$

The set G is compact (since f is continuous), and we take B to be its convex hull. Since the convex hull of a compact set in a finite-dimensional space is compact, [**4**, p. 22], B is compact.

Fix any $l \in [0, 1]$, and choose a point $t_0 \in \mathbf{T}$ so that $f(t_0) = l$. Then $F = \{(t_0, y) : 0 \le y \le f(t_0)\}$ is a face of B, which is an interval of length l.

The proof of Theorem 3 for $d > 1$ uses the Alexandroff-Hausdorff Theorem. There is a standard preparatory step that we always need to take before we can apply Theorem 1: we first need to introduce a metric on the set of "data" that makes this set a compact metric space.

To this end we introduce the Hausdorff metric d_H on the set of all compact subsets of $\mathbf{R}^d$. For a bounded set A and any $\varepsilon > 0$, we denote the ε-neighborhood of A by $A_\varepsilon = \{x \in \mathbf{R}^d : \text{dist}(x, A) < \varepsilon\}$. The *Hausdorff distance* between two bounded sets A and B is then

$$d_H(A, B) = \inf\{\varepsilon : B \subseteq A_\varepsilon \text{ and } A \subseteq B_\varepsilon\}.$$

Intuitively, $d_H(A, B)$ measures how much each of the sets A or B needs to be "blown up" so that it covers the other.

We need the classical Blaschke Selection Theorem: *The set of all compact convex subsets of a fixed compact subset of* $\mathbf{R}^d$ *is compact under the Hausdorff metric* [**4**, *p*. 64].

Proof of Grzaslewicz's Theorem for $d > 1$: Consider $\mathbf{R}^{d+2}$ to be the product $\mathbf{R}^2 \times \mathbf{R}^d$, and write the points in $\mathbf{R}^{d+2}$ as pairs (t, x), where $t \in \mathbf{R}^2$ and $x \in \mathbf{R}^d$. By the Blaschke Selection Theorem, the space K of all compact convex subsets of the unit cube in $\mathbf{R}^d$ is a compact metric space. It follows that there is a continuous map ϕ from the Cantor set Δ onto K.

Let $\mathbf{T}$ be the unit circle in $\mathbf{R}^2$, and consider Δ to be a closed subset of $\mathbf{T}$. The graph of ϕ can be visualized as a subset G of $\mathbf{R}^{d+2}$:

$$G = \{(t, x) : t \in \Delta \text{ and } x \in \phi(t)\}.$$

It follows from the continuity of ϕ that G is compact. The desired set B is the convex hull of G, which is compact as the convex hull of the compact set G.

Fix any compact convex subset A of the d-dimensional cube. Since ϕ is surjective, there is a point $t_0 \in \Delta$ such that $\phi(t_0) = A$. The set $F = \{(t_0, x) : x \in A\}$ is clearly congruent to A. To see that F is a face of B, let $L \subset \mathbf{R}^2$ be the line tangent to the circle $\mathbf{T}$ at t_0, and consider the $(d + 1)$-dimensional hyperplane $H = L \times \mathbf{R}^d$. This hyperplane supports the cylinder $\mathbf{T} \times \mathbf{R}^d$ in $t_0 \times \mathbf{R}^d$. Since G is contained in this cylinder, H supports its convex hull B. Moreover, F is exactly the set of points in G whose first coordinate is t_0 and it is closed and convex. It follows that $H \cap B = F$.

3. A THEOREM OF BANACH AND MAZUR. In this section we present one of the early and basic results on the structure of Banach spaces, due to S. Banach and S. Mazur ([**3**] or [**2**, p. 185]).

Let K be a compact metric space, and denote by $C(K)$ the Banach space of all continuous real-valued functions on K (with the supremum norm). A Banach space X is said to be *linearly isometric* to a subspace of a Banach space Y if there is a linear isometry from X into Y, i.e., a linear operator $T: X \to Y$ such that $\|Tx\|_Y = \|x\|_X$ for every $x \in X$.

Theorem 4. *Every separable Banach space is linearly isometric to a subspace of* $C[0,1]$.

The proof of the Banach-Mazur theorem has two steps:

Step 1. Every separable Banach space is linearly isometric to a subspace of $C(K)$ for some convex, compact, and metrizable subset K of a linear topological vector space.

Step 2. $C(K)$ is linearly isometric to a subspace of $C[0,1]$.

The Alexandroff–Hausdorff Theorem is used in the second step, but we also sketch the proof of the first step, which is actually a combination of some standard facts in functional analysis.

Proof of step 1: Let X be a separable Banach space, and let X^* be its dual. Every element $x \in X$ can be considered to be a function on X^* by the formula

$$x(x^*) = x^*(x) \tag{1}$$

for $x^* \in X$.

Of the several topologies that make X^* into a linear topological vector space, we use the $\sigma(X^*, X)$ (or weak*) topology. It is the weakest topology on X^* under which all the elements of X are continuous when considered as functions on X^* by the identification (1) [**9**, p. 66].

The closed unit ball K of X^* is convex, and it is compact and metrizable in the weak* topology:

- Compactness is the celebrated theorem of Alaoglu [**9**, p. 66].
- Metrizability is an easy consequence of the separability of X [**9**, p. 68].

Using this K we now define an isometry J of X into $C(K)$ by

$$(J(x))(k) = k(x) \quad \text{for every } x \in X \quad \text{and } k \in K.$$

That $J(x)$ is a continuous function on K for each x, follows from the definition of the weak* topology. The operator J is clearly linear, and we now check that it is an isometry. For each $k \in K$ and $x \in X$

$$|(J(x))(k)| = |k(x)| \le \|k\|_{X^*} \|x\|_X \le \|x\|_X,$$

where the first inequality follows from the definition of the norm on X^*, and the second from the fact that $\|k\|_{X^*} \le 1$ for k in the unit ball K of X^*. It follows that

$$\|J(x)\|_{C(K)} = \sup\{|(J(x))(k)| : k \in K\} \le \|x\|_X$$

for every $x \in X$. The reverse inequality follows from the Hahn–Banach Theorem:

For every $x \in X$ it ensures that there is a point $k_x \in K$ such that $k_x(x) = \|x\|_X$. It follows that

$$\|J(x)\|_{C(K)} \geq (J(x))(k_x) = k_x(x) = \|x\|_X.$$

Proof of step 2: Since K is a convex, compact, and metrizable space, the Alexandroff–Hausdorff Theorem and Corollary 2 yields a continuous surjective map $\phi: [0,1] \to K$. The operator S of composition with this ϕ, given by

$$Sf(t) = f(\phi(t)) \quad \text{for every } t \in [0,1]$$

is a linear operator from $C(K)$ into $C[0,1]$, and it is an isometry because

$$\|Sf\|_{C[0,1]} = \sup\{|f(\phi(t))| : t \in [0,1]\} = \sup\{|f(k)| : k \in K\} = \|f\|_{C(K)},$$

where the second equality follows from the surjectivity of ϕ.

4. A CONTINUOUS FUNCTION THAT INTERPOLATES EVERY BOUNDED SEQUENCE. The following theorem answers a question that was posed to me by Dr. Moshe Leshno and by Professors Allan Pinkus and Vladimir Lin. It was motivated by Dr. Leshno's work on neural nets. We denote the set of all integers by $\mathbf{Z}$.

Theorem 5. *There is a real-valued, bounded, and continuous function f on the real line $\mathbf{R}$ with the property that for each doubly infinite sequence $\mathbf{y} = (y_n)_{n \in \mathbf{Z}}$ of real numbers satisfying $|y_n| \leq 1$ for all n, there is a point $t \in \mathbf{R}$ such that*

$$y_n = f(t+n) \quad \textit{for all } n \in \mathbf{Z}.$$

Proof: Consider the infinite product $K = [-1,1]^{\mathbf{Z}}$ of all doubly infinite sequences of real numbers $\mathbf{z} = (z_n)_{n \in \mathbf{Z}}$ such that $|z_n| \leq 1$ for all n. By Tychonoff's Theorem, K is compact when equipped with the product topology, and it is metrizable as a product of a countable number of metric spaces. (An explicit metric on K can be defined by $d(\mathbf{y}, \mathbf{z}) = \Sigma 2^{-|n|}|y_n - z_n|$, and the compactness can then be proved directly by a standard diagonal subsequence argument.)

Let ϕ be a surjective continuous mapping from the Cantor set Δ onto K. The topology on K is defined in such a way that for each fixed n, the nth coordinate function $(\phi(\cdot))_n$ of ϕ is a continuous real-valued function on Δ, and it is clearly bounded in absolute value by one.

We identify Δ as a closed subset of $[0, 1/2]$. It follows that $\Delta + n$ and $\Delta + m$ are disjoint for $n \neq m$, and we first define the function f on the closed subset $A = \bigcup\{\Delta + n : n \in \mathbf{Z}\}$ of $\mathbf{R}$ by

$$f(t+n) = (\phi(t))_n \quad \text{for } t \in \Delta \quad \text{and } n \in \mathbf{Z}.$$

The function f is well defined and continuous on A, and we extend it to a bounded continuous function on all of $\mathbf{R}$ by linear interpolation (or by Tietze's Extension Theorem).

The extended function (which we continue to denote by f) is the required function. Indeed, given any $\mathbf{y} = (y_n) \in K$, there is a point $t_0 \in \Delta$ such that $\phi(t_0) = \mathbf{y}$, i.e., $(\phi(t_0))_n = y_n$ for all n. Then the definition of f ensures that $f(t_0 + n) = y_n$ for all n.

5. VARIATIONS ON SECTION 4. The specific bound 1 on the sequences in the previous theorem can be replaced by any other fixed bound. Some common bound is, however, necessary, and it is impossible to find one continuous function that interpolates *all* bounded doubly infinite sequences. It is even impossible to find a continuous function that interpolates all constant sequences, i.e., a function f with the property that for every real number α there is a point t for which $f(t+n)=\alpha$ for every n. Indeed, such a function would have to take every real α as a value at some point in the compact interval $[0,1]$, which is impossible for a continuous function.

On the other hand, the same proof shows that if $\{M_n\}_{n\in\mathbf{Z}}$ are arbitrary positive numbers, then there is a continuous function f on $\mathbf{R}$ such that if $\mathbf{y}=(y_n)_{n\in\mathbf{Z}}$ is a sequence of real numbers satisfying $|y_n|\le M_n$ for all n, then there is point $t\in\mathbf{R}$ such that $y_n=f(t+n)$ for all $n\in\mathbf{Z}$. (Just replace the product $[-1,1]^{\mathbf{Z}}$ in the proof by the product $\prod_{n=-\infty}^{\infty}[-M_n,M_n]$.)

In particular it follows that it is possible to interpolate all one-sided bounded sequences of real numbers. More precisely,

There is a continuous real-valued function f on the real line $\mathbf{R}$, such that for each bounded sequence of real numbers $\mathbf{y}=(y_n)_{n\ge 0}$ there is a point $t\in\mathbf{R}$ such that $f(t+n)=y_n$ for all $n\ge 0$.

Indeed, let f be the function constructed above with $M_n=n$. Given any bounded sequence $(y_n)_{n\ge 0}$, choose a positive integer k such that $|y_n|\le k$ for all $n\ge 0$, and find a point $s\in\mathbf{R}$ such that $f(s+m)=0$ for $m<k$, and such that $f(s+m)=y_{m-k}$ for all $m\ge k$. Then take $t=s+k$.

In the next variation we interpolate continuous functions rather than sequences, and we consider only one of many results of this type. Let $\mathscr{F}$ be a family of continuous real-valued functions on the unit interval $[0,1]$. Under what conditions on $\mathscr{F}$ can one find a continuous real-valued function g on the unit square $[0,1]\times[0,1]$, such that each $f\in\mathscr{F}$ can be realized as a horizontal section of g? More precisely, we look for a function g such that for each $f\in\mathscr{F}$ there is a point $s\in[0,1]$ with

$$f(t)=g(t,s)\quad \text{for all } 0\le t\le 1.$$

Recall that the *modulus of continuity* $\omega_f(\varepsilon)$ of a real-valued uniformly continuous function f, defined on some metric space, is given by

$$\omega_f(\varepsilon)=\sup\{|f(x)-f(y)|: d(x,y)\le\varepsilon\}$$

and $\omega_f(\varepsilon)\to 0$ as $\varepsilon\to 0$. A family $\mathscr{F}$ of uniformly continuous functions is called *equicontinuous* if there is a positive function $\omega(\varepsilon)$, with $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$, such that $\omega_f(\varepsilon)\le\omega(\varepsilon)$ for all $f\in\mathscr{F}$. Since a continuous function on the unit square is bounded and uniformly continuous, it follows that its set of horizontal sections is necessarily uniformly bounded (i.e., they are bounded by a common bound), and equicontinuous. It turns out that these conditions are sufficient as well:

Theorem 6. *Let $\mathscr{F}$ be a uniformly bounded and equicontinuous set of continuous real-valued functions on the unit interval. Then there is a continuous function g on the unit square, such that each function in $\mathscr{F}$ can be realized as a horizontal section of g.*

Proof: By the Ascoli–Arzelà Theorem [**9**, p. 369], the closure K (in the supremum norm) of the set $\mathscr{F}$ is compact. By the Alexandroff–Hausdorff Theorem and Corollary 2, there is a continuous map ϕ from the interval $[0,1]$ into $C[0,1]$, whose image contains K. Consider this interval to be the interval $[0,1]$ on the y-axis, and define a function g on $[0,1] \times [0,1]$ by $g(t,s) = \phi(s)(t)$ for $t, s \in [0,1]$. It follows from the continuity of ϕ that g is continuous, and the horizontal sections of g contain K, hence also $\mathscr{F}$.

6. A VERY SLOWLY CONVERGENT SEQUENCE OF CONTINUOUS FUNCTIONS. W. Rudin constructed a sequence of continuous real-valued functions on the unit interval that converges pointwise to zero, but does so at an arbitrarily slow rate at the different points of the interval [**10**]. More precisely,

Theorem 7. *There is a uniformly bounded sequence of strictly positive continuous functions* $(f_n)_{n=1}^{\infty}$ *on* $[0,1]$ *with the property that*

(i) $f_n(x) \to 0$ *for every* $x \in [0,1]$.
(ii) *For each unbounded sequence* (λ_n) *of positive numbers there is a point* $x \in [0,1]$ *at which* $\limsup_{n\to\infty} \lambda_n f_n(x) = \infty$.

In Rudin's original proof, the f_n's are first defined on the classical Cantor set in $[0,1]$ by explicit formulas that use the ternary representation of the points in the set. They are then extended to all of $[0,1]$ by linear interpolation. We use the Cantor set in a different way, namely, by applying the Alexandroff–Hausdorff Theorem.

The functions in our proof satisfy the stronger condition that the series $\Sigma f_n(x)$ converges for every x. Simple variations of the proof could give even stronger conditions that would make the convergence of the sequence $f_n(x)$ to zero seem even faster.

Proof: Consider the set K of numerical sequences $\mathbf{a} = (\alpha_n)$ given by

$$K = \left\{\mathbf{a} : 4^{-n} \le \alpha_n \le 1 \text{ for all } n, \text{ and } \sum \alpha_n \le 1\right\}.$$

One checks easily that K is a closed convex subset of the compact metric space $[0,1]^{\mathbf{N}}$, and hence K is compact. By the Alexandroff–Hausdorff Theorem and Corollary 2, there is a surjective map ϕ from $[0,1]$ onto K. We then define

$$f_n(x) = (\phi(x))_n,$$

the nth coordinate of $\phi(x)$. The functions f_n are continuous, and they satisfy $f_n(x) \ge 4^{-n}$ and $\Sigma f_n(x) \le 1$ for every $x \in [0,1]$.

Let (λ_n) be any unbounded sequence of positive numbers, and choose a subsequence (n_j) with the property that $\lambda_{n_j} 4^{-j} \ge j$ for each j. The sequence

$$\alpha_n = \begin{cases} 4^{-j} & \text{if } n = n_j \\ 4^{-n} & \text{otherwise} \end{cases}$$

satisfies $\alpha_n \ge 4^{-n}$ for all n (because $n_j \ge j$), and also $\Sigma \alpha_n \le 1$ (because $\Sigma 4^{-n} + \Sigma 4^{-j} = 2\Sigma 4^{-n} < 1$). Thus $(\alpha_n) \in K$.

Since ϕ is surjective, there is an $x \in [0,1]$ such that $f_n(x) = (\phi(x))_n = \alpha_n$. Hence $\lambda_{n_j} \cdot f_{n_j}(x) = \lambda_{n_j} \cdot (\phi(x))_{n_j} = \lambda_{n_j} 4^{-j} \ge j \to \infty$.

Remark. As observed by Rudin, the functions in Theorem 7 can be chosen to be polynomials. Indeed, use Weierstrass' Theorem to approximate each f_n by a polynomial p_n up to $\frac{1}{2}\min\{f_n(x) : x \in [0,1]\}$ (which is strictly positive). Then $f_n/2 \le p_n \le 3f_n/2$.

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