A DIVISION ALGEBRA FOR SEQUENCES AND ITS ASSOCIATED OPERATIONAL CALCULUS

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1. Introduction. Mikusiński [1] has developed an operational calculus which is essentially a division algebra in which the elements are the set of continuous, real or complex valued functions over the interval $0 \leqq t < \infty$. This set forms a commutative ring [2] with the operations

$$\textit{Addition}: \quad (a+b)(t) = a(t) + b(t).$$

$$\textit{Convolution}: \quad (ab)(t) = \int_0^t a(\tau)b(t-\tau)d\tau.$$

Titchmarsh's theorem [3] shows that there are no divisors of zero in this algebra. The ring may therefore be extended into a division algebra whose elements a/b are called *operators* [4]. Mikusiński then applies this algebra to the solution of both ordinary and partial differential equations. The calculations are formally very similar to those in which the Laplace transform is used; but the method is more general since it is free from convergence considerations and applies to equations such as $x' - x = 2(t-1)e^{t^2}$ [5] in which the right member is not transformable.

In applying operational methods to the solution of difference equations [6], however, Mikusiński uses a method which, in his words, "is not connected in any essential way with the operational calculus" [7]. The purpose of this paper is to set up a commutative ring in which the elements are sequences and the operations are the addition and convolution (Cauchy product) of sequences $\{f(t)\}$, over the integers $0, 1, 2, \cdots$.

$$\textit{Addition}: \quad (a+b)(t) = a(t) + b(t)$$

$$\textit{Convolution}: \quad (ab)(t) = \sum_{\tau=0}^{t} a(\tau)b(t-\tau).$$

This ring proves to have no divisors of zero for the analog of Titchmarsh's theorem is readily proved. The ring may therefore be extended into a division algebra with unique identity elements. This algebra is then applied to the solution of difference equations in much the same way as Mikusiński's algebra is applied to differential equations.

This was first done by Josef Eliáš [8] who defined the convolution of sequences and constructed an operational calculus for sequences that closely parallels that of Mikusiński. His definition of convolution differs from that above in that the initial term is zero; his formulas are given in terms of the operator $1/\{1\}$ (the inverse of the sum operator) and are applied to solve difference equations in the Δ-form.

T. A. Newton [9] used the convolution of sequences in the form $(ab)(t)$ given above to obtain recurrence relations for the power series solution of linear difference equations. Finally D. H. Moore [10] showed that this could be done by purely algebraic operations by defining the algebraic derivative of a sequence as $D\{f(t)\}=\{(t+1)f(t+1)\}$ and expressing some basic sequences in terms of the shift operator $s=\{0, 1, 0, 0, \cdots\}$.

The transition from differential equation to recurrence relation is facilitated when the equation is expressed in terms of the operator $\vartheta=xD$. Since $\vartheta(\vartheta-1)=x^2D^2$, the equation

$$(1+x^2)y''+2xy'-2y=0,$$

considered by both Newton and Moore, becomes

$$\vartheta(\vartheta-1)y+x^2(\vartheta+2)(\vartheta-1)y=0.$$

This discloses the indices $\lambda_1=0$, $\lambda_2=1$ and the solutions

$$y_1=\sum_{n=0}^{\infty}c_nx^{2n},$$

where $(2n-1)c_n+(2n-3)c_{n-1}=0$ and $y_2=x$.

We shall also express sequences in terms of the shift operator s. In a formula such as $\{a(t)\}=A(s)$, $A(s)$ is precisely the generating function of the sequence in the sense of Laplace,

$$A(s)=\sum_{t=0}^{\infty}a(t)s^t,$$

but in which convergence no longer poses a germane question. Thus this operational calculus gives the well-known formulas for generating functions a new interpretation just as the calculus of Mikusiński reinterprets Laplace transforms —in both cases without reference to convergence.

2. Commutative ring. Let $\mathcal{S}$ be the set of real or complex valued sequences defined over the integers $t=0, 1, 2, \cdots$. With operations of addition and convolution the elements form a commutative ring satisfying the six postulates. (We parallel Mikusiński's notation in using single letters to denote the elements (sequences) of our algebra):

A_1: $a+b=b+a$;

A_2: $(a+b)+c=a+(b+c)$;

A_3: *Given* a, b, *there exists an element* x *such that* $a+x=b$;

M_1: $ab=ba$;

M_2: $(ab)c=a(bc)$;

D: $a(b+c)=ab+bc$.

M_1 and D are obvious and M_2 is easily proved. From A_1, A_2, A_3 the unicity of x and the existence and unicity of the identity element for addition may be deduced. For the set $\mathcal{S}$ it is the zero sequence $\{0, 0, 0, \cdots\}$.

We now show that this ring has no divisors of zero.

THEOREM. *The convolution of two sequences is zero when and only when one of the sequences is zero.*

Proof. Let $c=ab=0$. If $a(t)\equiv 0$ there is nothing to prove. Therefore let $a(n)\neq 0$ be the first nonzero element of $\{a(t)\}$. Then

$$c_n = \sum_{\tau=0}^{n} a(\tau)b(n-\tau) = a(n)b(0) = 0 \qquad b(0) = 0,$$

$$c_{n+1} = \sum_{\tau=0}^{n+1} a(\tau)b(n+1-\tau) = a(n)b(1) = 0 \qquad b(1) = 0,$$

and in general, by induction, $b(t)\equiv 0$.

3. Quotient field. From the commutative ring we now construct a division algebra whose elements are ordered pairs (a, b) with $b\neq 0$ and having the equivalence relation

$$(a, b) = (c, d) \quad \text{iff} \quad ad = bc. \tag{1}$$

In particular for any nonzero sequence f,

$$(a, b) = (af, bf), \qquad f \neq 0. \tag{2}$$

Addition and multiplication are defined as for fractions $(a, b)=a/b$:

$$(a, b) + (c, d) = (ad + bc, bd), \tag{3}$$

$$(a, b) \cdot (c, d) = (ac, bd). \tag{4}$$

Since $b, d\neq 0$, $bd\neq 0$ by the theorem above, and hence, the sum and product of the ordered pairs are elements of the field. The zero element is now $(0, f), f\neq 0$; for from (3) and (2)

$$(a, b) + (0, f) = (af, bf) = (a, b).$$

These ordered pairs satisfy all six postulates of a commutative ring. Moreover they satisfy the postulate

M_3. *Given (a, b), (c, d) with $a\neq 0$ (as well as b, d, $\neq 0$), there exists a unique pair (x, y) such that $(a, b)\cdot(x, y)=(c, d)$.*

Proof. By hypothesis $ab\neq 0$, $ad\neq 0$. Hence

$$(a, b)\cdot(bc, ad) = (abc, bad) = (c, d)$$

by (2); thus we may take $(x, y)=(bc, ad)$. Moreover (x, y) is unique; for if $(a, b)\cdot(x, y)=(a, b)\cdot(u, v)$ or $(ax, by)=(au, bv)$ then $ax\, bv=by\, au$ from (1). Since $ab\neq 0$, $xv=yu$ or $(x, y)=(u, v)$.

If f is any nonzero sequence, (f, f) is the multiplicative identity.

4. Numerical operators. The sequences $\mathbf{k} = \{k, 0, 0, \cdots\}$ are isomorphic with the numbers k; for

$$\mathbf{k}_1 + \mathbf{k}_2 = \{k_1 + k_2, 0, 0, \cdots\},$$
$$\mathbf{k}_1\mathbf{k}_2 = \{k_1k_2, 0, 0, \cdots\}.$$

On occasion we may use boldface $\mathbf{k}$ to denote the above sequence; but ordinarily italic k will serve. Thus k times a sequence in the usual sense is the same as $\mathbf{k}$ times the sequence in the sense of our algebra.

We shall also write

$$k\{a\} \quad \text{for} \quad \{k, 0, 0, \cdots\}\{a(0), a(1), a(2), \cdots\}$$
$$k\{1\} \quad \text{for} \quad \{k, 0, 0, \cdots\}\{1, 1, 1, \cdots\}.$$

All powers of the sequence $\mathbf{1} = \{1, 0, 0, \cdots\}$ are $\mathbf{1}$ and $\mathbf{1}\{a\} = \{a\}$. From (3) and (4)

$$(a, 1) + (b, 1) = (a + b, 1), \qquad (a, 1)\cdot(b, 1) = (ab, 1);$$

thus the pair $(f, 1)$ may be regarded as another notation for the sequence f; and we shall write $(f, 1) = f$. In particular the multiplicative identity $(f, f) = (1, 1) = \mathbf{1}$.

5. Fraction notation. If we write $(a, b) = a/b$ $(b \neq 0)$ equations (1) to (4) are the familiar rules for numerical fractions when multiplication is replaced by convolution. Any sequence $f(t)$ may be written as $f/1$; and the identity elements for addition and convolution are simply $0/1 = 0$ and $1/1 = \mathbf{1}$.

If $a = bc$ $(b \neq 0)$ we have $a/b = c/1 = c$, a sequence. But if a and b are arbitrary, a/b may not be a sequence; for example, if $a(0) \neq 0$, $b(0) = 0$, a/b is not a sequence c for $b(0)\ c(0) = 0$ cannot equal $a(0)$. We call the elements of our division algebra *operators*; they include, in particular, numbers and sequences.

6. Shift operator. The sequence

$$s = \{0, 1, 0, 0, \cdots\} \tag{5}$$

is called the *shift operator* on account of the convolution

$$s\{f(0), f(1), \cdots\} = \{0, f(0), f(1), \cdots\}$$

in which each element of f is shifted one step to the right. We have $s^2 = \{0, 0, 1, 0, 0, \cdots\}$, and in general

$$s^n = \{\underbrace{0, 0, \cdots, 0}_{n}, 1, 0, 0, 0, \cdots\}. \tag{6}$$

When $n = 0$ we define $s^0 = \{1, 0, 0, \cdots\}$ in agreement with (6). Thus any sequence $\{f(t)\}$ may be written as an infinite series

$$\{f(t)\} = \sum_{t=0}^{\infty} f(t)s^t = F(s). \tag{7}$$

The question of convergence is not germane to this notation, for (7) merely states that the term $f(n)$ of the sequence occupies the same place as the 1 in s^n.

From (7) we have

$$\{r^t f(t)\} = \sum_{t=0}^{\infty} f(t)(rs)^t = F(rs). \tag{8}$$

Moreover $\{t f(t)\} = \sum_{t=0}^{\infty} t f(t)\, s^t = s \sum_{t=0}^{\infty} f(t)\, ts^{t-1}$, or since the last sum is the formal derivative of the series $F(s)$ in (7),

$$\{tf(t)\} = sF'(s). \tag{9}$$

Formulas (8) and (9) have been proved when $F(s)$ is a simple series as in (7). These formulas also hold when $F(s) = A(s)/B(s)$, the quotient of two simple series. For if

$$\{f(t)\} = \frac{\{a(t)\}}{\{b(t)\}} = \frac{A(s)}{B(s)} = F(s)$$

we have the convolution $\sum_{\tau=0}^{t} f(\tau)b(t-\tau) = a(t)$. The identity

$$\sum_{\tau=0}^{t} r^\tau f(\tau) r^{t-\tau} b(t-\tau) = r^t a(t) \quad \text{or} \quad \{r^t f(t)\}\{r^t b(t)\} = \{r^t a(t)\}$$

now implies that

$$\{r^t f(t)\} = \frac{\{r^t a(t)\}}{\{r^t b(t)\}} = \frac{A(rs)}{B(rs)} = F(rs). \tag{8'}$$

Moreover the identity

$$\sum_{\tau=0}^{t} \tau f(\tau) b(t-\tau) + \sum_{\tau=0}^{t} f(\tau)(t-\tau)b(t-\tau) = ta(t)$$

implies that $\{tf(t)\}\{b(t)\} + \{f(t)\}\{tb(t)\} = \{ta(t)\}$. Therefore

$$\{tf(t)\}B(s) + \frac{A(s)}{B(s)} sB'(s) = sA'(s),$$

$$\{tf(t)\} = s\,\frac{BA' - AB'}{B^2} = s\left(\frac{A}{B}\right)' = sF'(s). \tag{9'}$$

7. Sum and difference operators. The sequence

$$\sigma = \{1, 1, 1, \cdots\} \tag{10}$$

is called the *sum operator* since the convolution

$$\sigma f = \left\{ \sum_{\tau=0}^{t} f(\tau) \right\}. \tag{11}$$

The reciprocal of σ, namely

$$\delta = \frac{1}{\sigma} = \{1, -1, 0, 0, \cdots\} = 1 - s \tag{12}$$

is called the *difference operator* from the property

$$\delta f = \{f(0), \Delta f(0), \Delta f(1), \cdots\}.$$

Writing this $(1-s)\{f(t)\} = f(0) + s\{\Delta f(t)\}$, and putting $\Delta f(t) = f(t+1) - f(t)$, we get the important result

$$s\{f(t+1)\} = \{f(t)\} - f(0) \tag{13}$$

in which $f(0)$ is a *number*. In particular when $f(t) = 1$, $s\{1\} = \{1\} - 1$, or

$$\{1\} = \frac{1}{1-s}; \tag{14}$$

since $\{1\} = \sigma$, this also follows from (12). Again with $f(t) = r^t$, we have $sr\{r^t\} = \{r^t\} - 1$, or

$$\{r^t\} = \frac{1}{1-rs} \tag{15}$$

which also follows from (8) and (14).

From (13), $s\{f(t+2)\} = \{f(t+1)\} - f(1)$; hence

$$s^2\{f(t+2)\} = \{f(t)\} - f(0) - sf(1), \tag{16}$$

and in general

$$s^n\{f(t+n)\} = \{f(t)\} - f(0) - sf(1) - \cdots - s^{n-1}f(n-1). \tag{17}$$

The powers of σ are readily computed:

$$\sigma^2 = \sigma\{1\} = \sum_{\tau=0}^{t} 1 = \tau \Big|_{\tau=0}^{t+1} = \{t+1\},$$

$$\sigma^3 = \sigma\{t+1\} = \sum_{\tau=0}^{t} (\tau+1) = \frac{(\tau+1)^{(2)}}{2}\Bigg|_{\tau=0}^{t+1} = \frac{\{(t+2)^{(2)}\}}{2!},$$

and in general, by induction (cf. [11] for notation), $\sigma^{n+1} = \{(t+n)^{(n)}\}/n!$; or since, $\sigma = (1-s)^{-1}$

$$\{(t+n)^{(n)}\} = \frac{n!}{(1-s)^{n+1}}. \tag{18}$$

If we put $f(t)=t^{(n)}$ in (17) and note that $f(0)=f(1)=\cdots=f(n-1)=0$, we have

$$\{t^{(n)}\} = s^n\{(t+n)^{(n)}\} = \frac{n!s^n}{(1-s)^{n+1}}. \tag{19}$$

From (18) and (8) we have the useful result:

$$\{r^t(t+n)^{(n)}\} = \frac{n!}{(1-rs)^{n+1}}. \tag{20}$$

8. Operational formulas. Starting with $\{1\}=(1-s)^{-1}$ and using (9) we find successively

$$\{t\} = \frac{s}{(1-s)^2}, \qquad \{t^2\} = \frac{s(s+1)}{(1-s)^3}, \qquad \{t^3\} = \frac{s(s^2+4s+1)}{(1-s)^4}. \tag{21}$$

Again from these formulas and (8),

$$\{r^t\} = \frac{1}{1-rs}, \qquad \{tr^t\} = \frac{rs}{(1-rs)^2}, \cdots. \tag{22}$$

From (22)

$$\{e^{i\alpha t}\} = \frac{1}{1-se^{i\alpha}} = \frac{1-se^{-i\alpha}}{1-2s\cos\alpha+s^2},$$

and on taking real and imaginary parts,

$$\{\cos\alpha t\} = \frac{1-s\cos\alpha}{1-2s\cos\alpha+s^2}, \tag{23}$$

$$\{\sin\alpha t\} = \frac{s\sin\alpha}{1-2s\cos\alpha+s^2}. \tag{24}$$

When $\alpha=\pi/2$ these become

$$\left\{\cos\frac{\pi}{2}t\right\} = \frac{1}{1+s^2}, \tag{25}$$

$$\left\{\sin\frac{\pi}{2}t\right\} = \frac{s}{1+s^2}. \tag{26}$$

Also from (9)

$$\left\{t\cos\frac{\pi}{2}t\right\} = \frac{-2s^2}{(1+s^2)^2}, \tag{27}$$

$$\left\{t\sin\frac{\pi}{2}t\right\} = \frac{s(1-s^2)}{(1+s^2)^2}, \tag{28}$$

and hence

$$\left\{\frac{t}{2}\cos\frac{\pi}{2}t+\cos\frac{\pi}{2}t\right\}=\frac{1}{(1+s^2)^2},\tag{29}$$

$$\left\{\frac{t}{2}\sin\frac{\pi}{2}t+\frac{1}{2}\sin\frac{\pi}{2}t\right\}=\frac{s}{(1+s^2)^2}.\tag{30}$$

When the left members of equations (23) to (30) are multiplied by r^t, we must replace s by rs in the right members.

If s were a real or complex variable in equation (7), $F(s)$ would be the generating function of the sequence $\{f(t)\}$. Apart from considerations of convergence, formulas (8) and (9) are formal consequences of (7) and show that $F(rs)$ and $sF'(s)$ are generating functions of the sequences $\{r^tf(t)\}$ and $\{tf(t)\}$ respectively. With the aid of these results all the operational formulas were deduced from the generating function $(1-s)^{-1}$ of the sequence $\{1\}$. Consequently the formulas above have a double meaning: 1°, when s is the shift operator they express a sequence $\{f(t)\}$ as a rational function $F(s)$ in a division algebra; 2°, when s is a numerical variable, $F(s)$ is the generating function of $\{f(t)\}$.

9. Difference equations. To solve a difference equation of order n with constant coefficients

$$a_0y(t+n)+a_1y(t+n-1)+\cdots+a_ny(t)=f(t),$$

multiply by s^n and use (17) to put the left member in the form

$$(a_0+a_1s+\cdots+a_ns^n)y+G(s),$$

where $G(s)$ is a polynomial of degree $n-1$ at most and whose coefficients depend upon the first n terms of the sequence y given as initial conditions. Express $\{f(t)\}$ as $F(s)$ by means of the appropriate operational formulas, solve the equation for

$$y=\frac{s^nF(s)-G(s)}{a_0+a_1s+\cdots+a_ns^n},$$

decompose into partial fractions and interpret them as sequences in t. The method is illustrated in the following examples in which $sy(t+1)=y-y_0$, $s^2y(t+2)=y-y_0-sy_1$, are used to transform the left members and $y_n=y(n)$.

Example 1. $y(t+2)+y(t)=\sin(\pi/2)t$, $y_0=1$, $y_1=0$. Multiply by s^2; then

$$y-1+s^2y=\frac{s^3}{1+s^2},$$

$$\begin{aligned}y&=\frac{1+s}{1+s^2}-\frac{s}{(1+s^2)^2}\\&=\left\{\cos(\pi/2)\,t+\sin(\pi/2)t-\tfrac{1}{2}(1+t)\sin(\pi/2)t\right\}\end{aligned}$$

on using (25), (26) and (30).

Example 2. $\Delta y+3y-2\sigma y=(-1)^t, y_0=1$; $\Delta y(t)=y(t+1)-y(t)$ and $\sigma=(1-s)^{-1}$ is the sum operator of equation (10). Multiply by s; then

$$y-1+2sy-\frac{2s}{1-s}y=\frac{s}{1+s},$$

$$\begin{aligned} y &= \frac{(1+2s)(1-s)}{(1-2s)(1+s)^2} \\ &= \frac{4/9}{1-2s}+\frac{11/9}{1+s}-\frac{2/3}{(1+s)^2} \\ &= \frac{4}{9}\{2^t\}+\frac{11}{9}\{(-1)^t\}-\frac{2}{3}\{(-1)^t(t+1)\} \end{aligned}$$

on using (15) and (20).

Example 3. $\Delta^2y+y=\sin(\pi/2)t, y_0=1, y_1=0$; $\Delta^2y(t)=y(t+2)-2y(t+1)+y(t)$. Multiply by s^2; then

$$s^2\{y(t+2)-2y(t+1)+2y(t)\}=\frac{s^3}{s^2+1},$$

$$\begin{aligned} y-1-2s(y-1)+2s^2y &= \frac{s^3}{s^2+1} \\ (1-2s+2s^2)y &= \frac{1-2s+s^2-s^3}{s^2+1} \\ y &= \frac{1-2s+s^2-s^3}{(s^2+1)(1-2s+2s^2)} \\ &= \frac{1}{5}\frac{s+2}{s^2+1}-\frac{1}{5}\frac{7s-3}{1-2s+2s^2}. \end{aligned}$$

Now $\frac{1}{5}(s+2)/(s^2+1)=\frac{1}{5}\{\sin(\pi/2)t\}+\frac{2}{5}\{\cos(\pi/2)t\}$. As to the second fraction, we make use of the formulas

$$\{r^t\cos\alpha t\}=\frac{1-rs\cos\alpha}{1-2rs\cos\alpha+r^2s^2}, \qquad \{r^t\sin\alpha t\}=\frac{rs\sin\alpha}{1-2rs\cos\alpha+r^2s^2}$$

derived from (8) applied to (23) and (24). Then

$$\begin{aligned} \frac{7s-3}{1-2s+2s^2} &= -3\,\frac{1-s-4s/3}{1-2s+2s^2} \\ &= -3\{2^{t/2}\cos(\pi/4)t\}+4\{2^{t/2}\sin(\pi/4)t\} \end{aligned}$$

observing that $r=\sqrt{2}$, $r\sin\alpha=r\cos\alpha=1$ and hence $\alpha=\pi/4$. The solution is therefore

$$y = \tfrac{1}{5}\{\sin(\pi/2)t\} + \tfrac{2}{5}\{\cos(\pi/2)t\} + \tfrac{3}{5}\{2^{t/2}\cos(\pi/4)t\} - \tfrac{4}{5}\{2^{t/2}\sin(\pi/4)t\},$$

$$y(t) = \tfrac{1}{5}(\sin(\pi/2)t + 2\cos(\pi/2)t) + (\tfrac{1}{5}2^{t/2})(3\cos(\pi/4)t - 4\sin(\pi/4)t).$$

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LAPLACE TRANSFORMS AND CANONICAL MATRICES

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I. The purpose of this article is to demonstrate the intimate connection between systems of differential equations and canonical matrices. In particular it will be shown, using only some basic concepts of differential equations and linear algebra, that for every matrix a similarity transformation can be found which will put the given matrix into the Jordan canonical form. Subsequently some special cases will be treated, in particular normal matrices. The method leads to a canonical form naturally and is also a constructive method. Although the basic results are not new, it is believed that the treatment is original.

II. This section will summarize some of the basic facts of differential equations, Laplace transforms and linear algebra which will be used. The class of matrices to be discussed will be square $n\times n$ matrices, whose entries are complex numbers; such a matrix will be denoted by A. Its hermitian transpose or adjoint will be denoted by A^*, and is defined as the complex conjugate of the transpose of A, that is

$$A^* = \overline{A}^T.$$

The inner product of two vectors X and Z, with entries x_i and z_i will be given by

$$(Z, X) = \sum_{i=1}^{n} \bar{z}_i x_i.$$