A PROBLEM IN GEOMETRICAL PROBABILITY

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This paper is dedicated to Professor W. Randolph Church, 1904–1969.

Introduction. In 1955, Frank Hawthorne proposed the following problem in the *American Mathematical Monthly* [1]: *If three points are selected at random in a rectangle* $A \times 2A$, *what is the probability that the triangle so determined is obtuse?*

Although the problem does not at first glance seem to be especially difficult, no solutions were received. In 1962, C. S. Ogilvy, in his book *Tomorrow's Math* [2], mentioned Hawthorne's problem and noted that it was still at that time unsolved. To this day, the editors of the MONTHLY have failed to receive any solutions.

The following more general problem is immediately suggested: *Let there be given three points at random in an arbitrary rectangle. What is the probability that the triangle thus formed is obtuse?*

This more general problem was first posed to me several years ago by Professor Roger Pinkham; Professor Pinkham has informed me that he obtained a partial solution to it (including a solution to Hawthorne's problem) around 1956, but that the solution was never published.

In this paper we shall present a complete solution to the obtuse triangle problem for an arbitrary rectangle.

Analysis. Let $P(L)$ be the probability that three points chosen at random in a rectangle with dimensions $1 \times L$ form an obtuse triangle. The desired probability is invariant under change of scale; thus once we have determined $P(L)$, we have solved the general problem.

To state the problem precisely, let R denote the rectangle $\{(x, y): 0 \leqq x \leqq 1$ and $0 \leqq y \leqq L\}$ and let X_1, X_2, X_3, Y_1, Y_2, and Y_3 be independent random variables where each X_i is distributed uniformly on the unit interval $[0, 1]$ and where each Y_i is distributed uniformly on the interval $[0, L]$. Consider the following three random points in R which they determine: $P_1 = (X_1, Y_1)$, $P_2 = (X_2, Y_2)$, and $P_3 = (X_3, Y_3)$. With probability one, these points form a nondegenerate triangle $P_1P_2P_3$. Defining $P(L)$ as above to be the probability that triangle $P_1P_2P_3$ is obtuse, we see that $P(L) = Pr\{\text{Angle } P_1 \text{ is obtuse}\} + Pr\{\text{Angle } P_2 \text{ is obtuse}\} + Pr\{\text{Angle } P_3 \text{ is obtuse}\}$, since a triangle can have at most one obtuse angle. By symmetry, it follows that

$$P(L) = 3Pr\{\text{Angle } P_1 \text{ is obtuse}\}.$$

If we let P_1P_2 be the vector from P_1 to P_2 and P_1P_3 be the vector from P_1 to P_3, then

$$\cos P_1 = \frac{(P_1P_2)\cdot(P_1P_3)}{|P_1P_2|\ |P_1P_3|}.$$

Now angle P_1 is obtuse iff $\cos P_1$ is negative, and evidently $\cos P_1$ is negative iff the dot product

$$(P_1P_2)\cdot(P_1P_3) = (X_2 - X_1)(X_3 - X_1) + (Y_2 - Y_1)(Y_3 - Y_1)$$

is negative. If we define new random variables

$$X = (X_2 - X_1)(X_3 - X_1) \quad \text{and} \quad Y = (Y_2 - Y_1)(Y_3 - Y_1),$$

then we have

$$P(L) = 3Pr\{X + Y < 0\}.$$

Let $F(x)$ denote the cumulative distribution function (CDF) of X; then by a simple change of scale, the CDF of Y will be given by $G(y) = F(y/L^2)$; thus it is sufficient to determine $F(x)$. Once this is done, we see that X and Y are independent so that $P(L)$ can be expressed as the following Riemann-Stieltjes integral:

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)dF(x). \tag{1}$$

(We note that for all values of L, the range of integration is actually finite.)

Computation of $F(x)$. We shall compute first the conditional CDF $F_1(x, a) = Pr\{X \leqq x \text{ given that } X_1 = a\} = Pr\{(X_2-a)(X_3-a) \leqq x\}$. Evidently we need only consider $a \in [0, 1]$; $F(x)$ is then found by using the relationship

$$F(x) = \int_0^1 F_1(x, a)da.$$

We examine the case $x > 0$ first. Consider the unit square in the $X_2 - X_3$ plane. For fixed $x > 0$, the graph of the equation $(X_2-a)(X_3-a) = x$ is a hyperbola with asymptotes $X_2 = a$ and $X_3 = a$ as shown in Figure 1. The region between the two branches of the hyperbola will determine the set where $(X_2-a)(X_3-a) \leqq x$; since the joint distribution of X_2 and X_3 is uniform over the square, $F_1(x, a)$ is simply the shaded area in Figure 1.

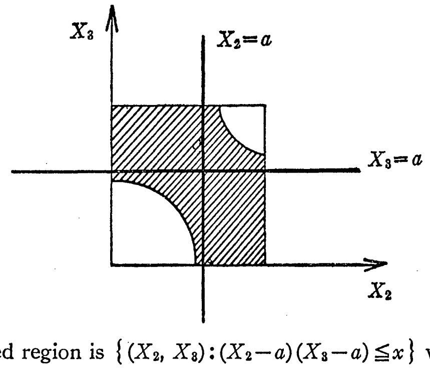

FIG. 1. The shaded region is $\{(X_2, X_3): (X_2-a)(X_3-a) \leqq x\}$ when x is positive.

Figure 1 is apt to be misleading, for it implies that there are always two unshaded areas: A_1 in the lower left corner and A_2 in the upper right corner.

This is not true in general, since depending on the relative magnitudes of x and a, one or both of these regions can disappear. The intersection of the lower branch of the hyperbola with either the X_2- or X_3-axis is seen to be at $a-(x/a)$, so that for the region A_1 to be present we need that $a-(x/a)\geqq 0$, i.e., that $a^2\geqq x$. By a similar argument it is seen that we must have $(1-a)^2\geqq x$ in order for the region A_2 to be present.

Let us define $A_1(x, a)$ to be the area of A_1 if A_1 is present, and 0 otherwise; thus if $a^2\geqq x$, we have that

$$A_1(x, a) = \int_0^{a-x/a}\left(a + \frac{x}{X_3 - a}\right)dX_3 \tag{2}$$
$$= x \log x - 2x \log a + a^2 - x.$$

Defining $A_2(x, a)$ analogously, we see that if $(1-a)^2\geqq x$, then

$$A_2(x, a) = \int_{a+x/(1-a)}^{1}\left(1 - a - \frac{x}{X_3 - a}\right)dX_3 \tag{3}$$
$$= x \log x - 2x \log(1 - a) + (1 - a)^2 - x.$$

With this notation, evidently,

$$F_1(x, a) = 1 - A_1(x, a) - A_2(x, a),$$

so that for $x>0$,

$$F(x) = 1 - \int_0^1 A_1(x, a)da - \int_0^1 A_2(x, a)da.$$

Certainly $F(x)=1$ if $x\geqq 1$; in the case $0<x<1$ we note that $A_1(x, a)=0$ whenever $a\leqq\sqrt{x}$ and that $A_2(x, a)=0$ whenever $(1-a)\leqq\sqrt{x}$, so that

$$F(x) = 1 - \int_{\sqrt{x}}^1 A_1(x, a)da - \int_0^{1-\sqrt{x}} A_2(x, a)da.$$

Substituting for $A_1(x, a)$ and $A_2(x, a)$ from (2) and (3) and evaluating the integrals yields the following expression for $F(x)$ when $x>0$;

$$F(x) = \begin{cases} -2x(\log x + 1) + \frac{1}{3}(1 + 8x^{3/2}) & \text{for } 0 < x < 1 \\ 1 & \text{for } x \geqq 1. \end{cases}$$

As $x\to 0^+$, it is seen that $F(x)\to 1/3$, so that by continuity $F(0)=1/3$. It is interesting to note that this can be obtained also by a combinatorial argument: There are $3!=6$ ways in which the three random variables X_1, X_2, and X_3 can be ordered. Since these random variables are independent and identically distributed, the six orderings are equally likely. But only two of the orderings are favorable to the event $(X_2-X_1)(X_3-X_1)\leqq 0$, namely the orderings $X_2\leqq X_1\leqq X_3$ and $X_3\leqq X_1\leqq X_2$. Therefore $F(0)=2/6=1/3$.

Let us now examine the case $x<0$. Again we consider the unit square in the

X_2-X_3 plane. As in the case of positive x, $F_1(x, a)$ is the total area of the shaded regions in Figure 2; note that the areas of the two regions are equal by symmetry. Also as in the case $x>0$, these regions can disappear depending on the relative magnitudes of x and a. However, in contrast to the case of positive x, by symmetry both regions must be present or both must be absent.

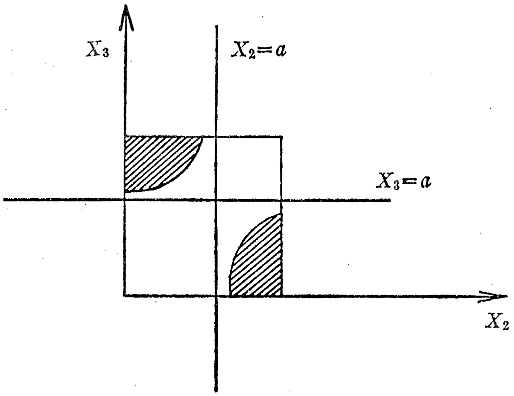

FIG. 2. The shaded region is $\{(X_2, X_3):(X_2-a)(x_3-a)\leqq x\}$ when x is negative.

Consider the upper region. The intersection of the upper branch of the hyperbola with the top of the square is at $X_2=a+x/(1-a)$ and its intersection with the X_3-axis is at $X_3=a-x/a$. This region (hence both regions) will disappear when either $a-x/a>1$ or $a+x/(1-a)<0$. It is not hard to see that these conditions are, in fact, equivalent: the regions will disappear whenever $x<a(a-1)$. Hence $F_1(x, a)=0$ if $x<a(a-1)$; if $x\geqq a(a-1)$, then

$$(4)\qquad \begin{aligned} F_1(x, a) &= 2\int_{a-x/a}^{1}\left(a+\frac{x}{X_3-a}\right)dX_3 \\ &= 2(-x\log(-x)+x\log(a(1-a))+a(1-a)+x). \end{aligned}$$

Now a is restricted to the interval $[0, 1]$, and on this interval $a(a-1)$ has the minimum value of $-1/4$; thus if $x<-1/4$, the inequality $x<a(a-1)$ is automatically satisfied so that $F_1(x, a)=0$ and so $F(x)=0$. Assume then that $x\geqq-1/4$. In this case, the inequality $x\geqq a(a-1)$ is equivalent to the following inequality:

$$\tfrac{1}{2}(1-\sqrt{1+4x})\leqq a\leqq\tfrac{1}{2}(1+\sqrt{1+4x}).$$

Therefore if $-1/4\leqq x<0$, we have that

$$F(x)=\int_0^1 F_1(x, a)da=\int_{\frac{1}{2}(1-\sqrt{1+4x})}^{\frac{1}{2}(1+\sqrt{1+4x})}F_1(x, a)da.$$

Substituting in our expression for $F_1(x, a)$ from (4) and performing the integration yields the following expression for $F(x)$ when $-1/4\leqq x<0$:

$$F(x)=2x\log\left(\frac{1+\sqrt{1+4x}}{1-\sqrt{1+4x}}\right)+\frac{1}{3}(1-8x)\sqrt{1+4x}.$$

(Observe that $F(x) \to 1/3$ as $x \to 0^-$.) Putting everything together, we have our final expression for $F(x)$:

$$(5)\quad F(x) = \begin{cases} 0 & \text{if } x \leqq -1/4 \\ 2x \log\left(\dfrac{1+\sqrt{1+4x}}{1-\sqrt{1+4x}}\right) + \dfrac{1}{3}(1-8x)\sqrt{1+4x} & \text{if } -1/4 \leqq x < 0 \\ \dfrac{1}{3} & \text{if } x = 0 \\ -2x(\log x + 1) + \dfrac{1}{3}(1+8x^{3/2}) & \text{if } 0 < x \leqq 1 \\ 1 & \text{if } x \geqq 1. \end{cases}$$

In Figure 3, we have sketched a graph of $F(x)$. Note the vertical tangent at $x=0$.

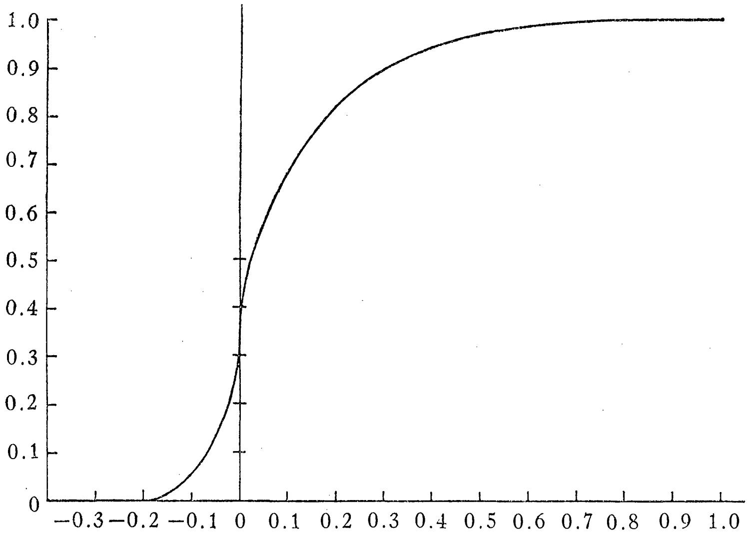

FIG. 3. Graph of $F(x)$, the CDF of $X=(X_2-X_1)(X_3-X_1)$.

Computation of $P(L)$. Obviously $P(L)=P(1/L)$ so that we can assume without loss of generality that $L \geqq 1$. As noted earlier, it is necessary to evaluate the following Riemann-Stieltjes integral:

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)dF(x).$$

The integrator function $F(x)$ has a vertical tangent at $x=0$, but is otherwise well behaved, i.e., it has a continuous derivative. We can, therefore, write $P(L)$ as an [improper] Riemann integral

$$P(L) = 3\int_{-\infty}^{\infty} F(-x/L^2)F'(x)dx,$$

where the existence of this integral can be verified, even though the integrand is not finite at $x=0$.

The lower limit of integration is always determined by the fact that $F'(x)=0$ if $x<-1/4$. However, the upper limit of integration is determined by the fact that $F'(x)=0$ if $x>1$ in the case $L\geqq 2$, and by the fact that $F(-x/L^2)=0$ if $x>L^2/4$ in the case $1\leqq L\leqq 2$. Thus in general

$$P(L) = 3\int_{-1/4}^{\alpha} F(-x/L^2)F'(x)dx,$$

where $\alpha=\min(1, L^2/4)$.

To evaluate this integral, we shall first split the range of integration into two parts: $-1/4$ to 0 and 0 to α, so that

$$P(L) = 3\int_{-1/4}^{0} F(-x/L^2)F'(x)dx + 3\int_{0}^{\alpha} F(-x/L^2)F'(x)dx. \tag{6}$$

In the first of these integrals, let $y=-x$ and integrate by parts as follows:

$$\int_{-1/4}^{0} F(-x/L^2)F'(x)dx = \int_{0}^{1/4} F(y/L^2)F'(-y)dy$$
$$=\frac{1}{9} + M^2\int_{0}^{1/4} F(-y)F'(M^2y)dy,$$

where we have written $M=1/L$. In the second of the integrals in (6), let $y=x/L^2$ so that

$$\int_{0}^{\alpha} F(-x/L^2)F'(x)dx = L^2\int_{0}^{\beta} F(-y)F'(L^2y)dy,$$

where $\beta=\alpha/L^2=\min(1/L^2, 1/4)$. The problem is thus reduced to the evaluation of the following integral:

$$I(x; A) = \int_{0}^{x} F(-y)F'(A^2y)dy,$$

where $A>0$ and where $x=\min(1/4, 1/A^2)$. In terms of this integral, we see that in general

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2I(\beta; L),$$

where $\beta=\min(1/L^2, 1/4)$.

The evaluation of $I(x; A)$ is straightforward, but long and tedious; we omit the details. We note the special case, though, of $x=1/4$:

$$I(1/4; A) = \frac{\pi A}{240} + \frac{47}{900} - \frac{\log A}{15}\,. \tag{7}$$

In the case $1\leqq L\leqq 2$, we see that $\beta=1/4$, so that $P(L)$ can be written

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2I(1/4; L).$$

Substituting in this equation the value of $I(1/4; A)$ from (7), we have the solution for $P(L)$ in the case $1 \leqq L \leqq 2$:

$$(8)\quad P(L) = \frac{1}{3} + \frac{47}{300}(L^2 + 1/L^2) + \frac{\pi}{80}(L^3 + 1/L^3) - \frac{\log L}{5}(L^2 - 1/L^2).$$

We note two special cases. If $L = 1$, so that the rectangle is a square, we have that

$$P(1) = \frac{97}{150} + \frac{\pi}{40} = 0.72520648 \cdots;$$

the case $L = 2$ of this equation yields the solution to Hawthorne's problem:

$$P(2) = \frac{1199}{1200} + \frac{13\pi}{128} - \frac{3}{4}\log 2 = 0.79837429 \cdots.$$

In the case $L \geqq 2$, we see that $\beta = 1/L^2$, so that

$$P(L) = \tfrac{1}{3} + (3/L^2)I(1/4; 1/L) + 3L^2 I(1/L^2; L).$$

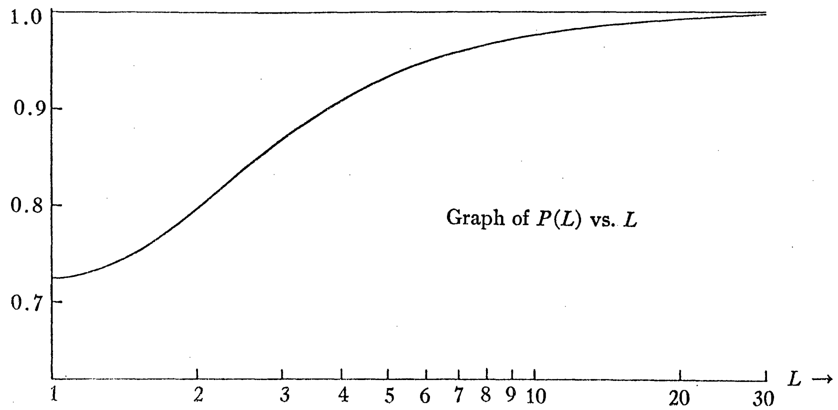

FIG. 4.

Substituting into this the general value of $I(x; A)$ we have our answer for $P(L)$ in the case $L \geqq 2$:

$$P(L) = \frac{1}{3} + \frac{1}{L^2}\left(\frac{\pi}{80L} + \frac{47}{300} + \frac{\log L}{5}\right) + \frac{47L^2}{300} - \frac{L^2 \log L}{5}$$

$$(9)\quad + \frac{L^3}{40}\operatorname{Arcsin}\left(\frac{2}{L}\right) + \left(\frac{L^2}{10} - \frac{3}{5L^2}\right)\log\left\{\frac{L + \sqrt{L^2 - 4}}{L - \sqrt{L^2 - 4}}\right\}$$

$$+ \frac{L\sqrt{L^2 - 4}}{150}\left(-31 + \frac{63}{L^2} + \frac{64}{L^4}\right).$$

Note that the last three terms make sense only if $L \geqq 2$; note also that if $L = 2$, then the last two terms vanish, and that $\operatorname{Arcsin}(2/L)=\pi/2$. Thus if we replace L by $\max(L, 2)$ in the last two terms of (9) and in the argument of the Arcsin, then (9) becomes valid even when $1 \leqq L \leqq 2$.

In Figure 4 we have graphed $P(L)$ versus L (on a logarithmic scale).

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A greatly abbreviated version of this paper has appeared in Biometrika [3] under the title, *The probability that a random triangle is obtuse.*

The author is now at the University of Maine, Orono, Maine.

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THE EXISTENCE OF FINITE BOLYAI-LOBACHEVSKY PLANES

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In the study of finite Projective and finite Euclidean geometries one is led to study more general finite geometries. Of recent interest are systems called finite Bolyai-Lobachevsky geometries, that is, finite geometries in which there is more than one line which is parallel (two lines which do not intersect will be called parallel) to a given line through a point not on that line. L. M. Graves [1] has exhibited a particular example of such a system and has raised the question of the existence of others.

As a basis for our discussion we will exhibit systems which satisfy the following axioms, which are somewhat more restrictive than those of Graves, but which have the desired property of being very similar to the axioms generally used to define finite Projective or finite Euclidean geometries.

Axiom 1. If P and Q are two points there is exactly one line containing P and Q.

Axiom 2. If l is any line there is a point P which does not lie on l.

Axiom 3. There are at least two points on every line.

Axiom 4. There exists at least one line.

Axiom 5. Given a point P not on a line l, there are exactly k lines which are parallel to l and pass through P.

From these postulates we can easily prove the following basic theorem:

THEOREM 1. *If there exists one line which contains exactly n points, then*

a) *Every line contains exactly n points.*

b) *There are exactly $n+k$ lines which pass through each point.*

c) *Space contains exactly $(n+k)(n-1)+1$ points.*

d) *Space contains exactly $[(n+k)(n-1)+1][n+k]/n$ lines.*